STANFORD

Lecture 5

Statistical Energy Loading

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Announcements & Agenda

- Announcements
  - Problem Set #1 Solutions posted – see PS1 (also link at course page and connects to canvas where actual solution is stored)
  - Problem Set #2 Wed April 19 at 17:00
  - Sections **1.6**, 4.4
  - Problem Set #3 due April 26 at 17:00

- Agenda
  - Statistical/Ergodic Models (for time varying channels)
  - Loading with Statistical Channels
  - Ergodic Coded-OFDM Loading

- Problem Set 3 = PS3, due 4/26
  1. 4.13 basic C-OFDM design
  2. 4.14 ergodic water-fill
  3. 4.22 wireless spatial loading
  4. 4.16 estimating gain distribution
  5. 4.15 Simple Wi-Fi Loading

Statistical/Ergodic Models

(for time varying channels, Section 1.6)

The statistically parametrized channel model

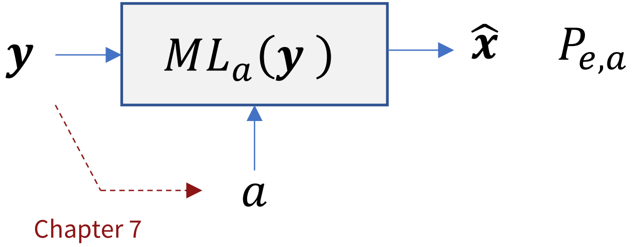
- The channel has a parameter a , so $p_{[\boldsymbol{y}\ a]/\boldsymbol{x}}$ where a is random
  - Deterministic-parameter examples are σ^2 for the AWGN or p for the BSC -- but now a can vary randomly
  - If a were just another channel output, then $[\boldsymbol{y}\ a] \rightarrow \boldsymbol{y}$ and all previous analysis applies

$$\boxed{p_{[\boldsymbol{y}\ a]/\boldsymbol{x}} = p_{\boldsymbol{y}/[\boldsymbol{x}\ a]} \cdot \underbrace{p_{a/\boldsymbol{x}}}_{p_a}}$$

Instead, $\boldsymbol{x}$ and a are independent

- The parameter a , is somewhat like an additional message to estimate, but not exactly
  - a can be a random process $a(t)$, whose probability density is stationary (or “quasi-stationary”)
  - The channel is “varying” with the random-variable selection $a = \alpha \in \mathcal{A}$ with distribution p_a
- The ML/MAP receiver is a function of a
  - Has error-probability distribution $P_{e,a}$

$$\langle P_e \rangle \triangleq \mathbb{E}\left[P_{e,a}\right]$$



Ergodic Analysis

- Mean value $$\mathbb{E}[a] = \sum_{\alpha \in \mathcal{A}} \alpha \cdot p_a(\alpha)$$

- **Sample Mean** $$\langle a \rangle_J = \frac{1}{J} \cdot \sum_{j=1}^{J} \alpha_j$$ Applies also to $f(a)$

- **Ergodic** if $$\langle a \rangle = \lim_{J \to \infty} \langle a \rangle_J = \mathbb{E}[a]$$

- Traditional deterministic analysis $a = constant$ so the channel represents an average over parameters

- Ergodic analysis averages $P_{e,a}$ over a so the performance is averaged over ML for each sample value

- **Monte Carlo Analysis** – pick a values from $p_a(\alpha)$, determine $P_{e,a}$ for each, and average results
  - Some $p_a(\alpha)$ admit a closed form expression for $\langle P_e \rangle$

AWGN Statistical model

- The channel-transfer amplitude h now becomes random (in addition to the noise)
  - Each dimension (real or almost always complex) has a random amplitude

$$y = \underset{\substack{\uparrow \\ \text{random}}}{h} \cdot x + \underset{\substack{\uparrow \\ \text{AWGN}}}{n}$$

- This equation omits the dimensional indices: time (k), frequency (n), or space (l)
  - This will be true in every dimension (the amplitude variable may have different $p_h(\alpha)$ in different dimensions)

- **Channel gain:** Remains the important (now random) quantity

$$\boxed{g = \frac{|h|^2}{\sigma^2}}$$

- **Channel gain distribution:** $p_g(v)$ derives from $p_h(\alpha)$ presuming fixed noise power spectral density.

See PS3.1 (Prob 4.13)

$$\boxed{p_g(v)}$$

- Many statistical models find use in wireless
- Code design can compensate for the uncertainty of h , as well as for n

Ergodic and Confidence Analysis

- Ergodic average error probability

$$\langle P_e \rangle = \int_{v=0}^{\infty} P_e(v) \cdot p_g(v) \cdot dv$$

- Ergodic average bits/dimension

$$\langle \bar{b} \rangle = \int_{u=0}^{\infty} \frac{1}{2} \cdot \log_2\left(1 + \frac{\mathcal{E}(v) \cdot v}{\Gamma}\right) \cdot p_g(v) \cdot dv$$

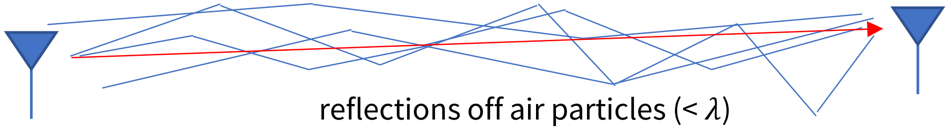
- Outage probability P_{out} (confidence-interval that SNR is high enough)

$$P_{out} = \int_{v=0}^{g_{out}} p_g\,(v) \cdot dv$$

- Above threshold $\int_{v=g_{out}}^{\infty} p_g(v) \cdot dv$ is 1- P_{out} and corresponds to the usual P_e during this time
  - And the outage is bad, presume “coin-flip” bit-error probability basically

Scattering leads to h variation

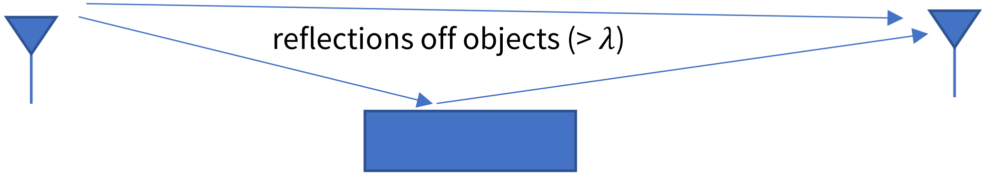
- micro



Central Limit adding on
inphase and independently
on quadrature at receiver
(each Gaussian, uniform phase)

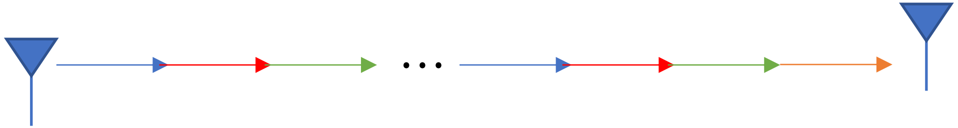
Amplitude is Rayleigh Dist'n
With line-of-sight (LoS) mean Ricean

- multi-path



Sum of Rayleighs/Riceans
each with own delay ~ T
Power-Delay Profile

- macro (shadow)



Product of several attenuations
Log-product has Central Limit Thm
applying to overall gain (dB)
Lognormal, so applies to LoS mean
"link budget"

Micro Scattering: Rayleigh Fading

- **Rayleigh Fading** (micro-scattering) model

$$h = \sqrt{h_I^2 + h_Q^2} \qquad \mathcal{E}_h = \mathbb{E}[|h|^2] \qquad p_h(u) = \frac{u}{\bar{\mathcal{E}}_h} \cdot e^{-\left(\frac{u^2}{\bar{\mathcal{E}}_h}\right)}$$

Channel SNR g then has χ-squared Distribution (2 degrees)

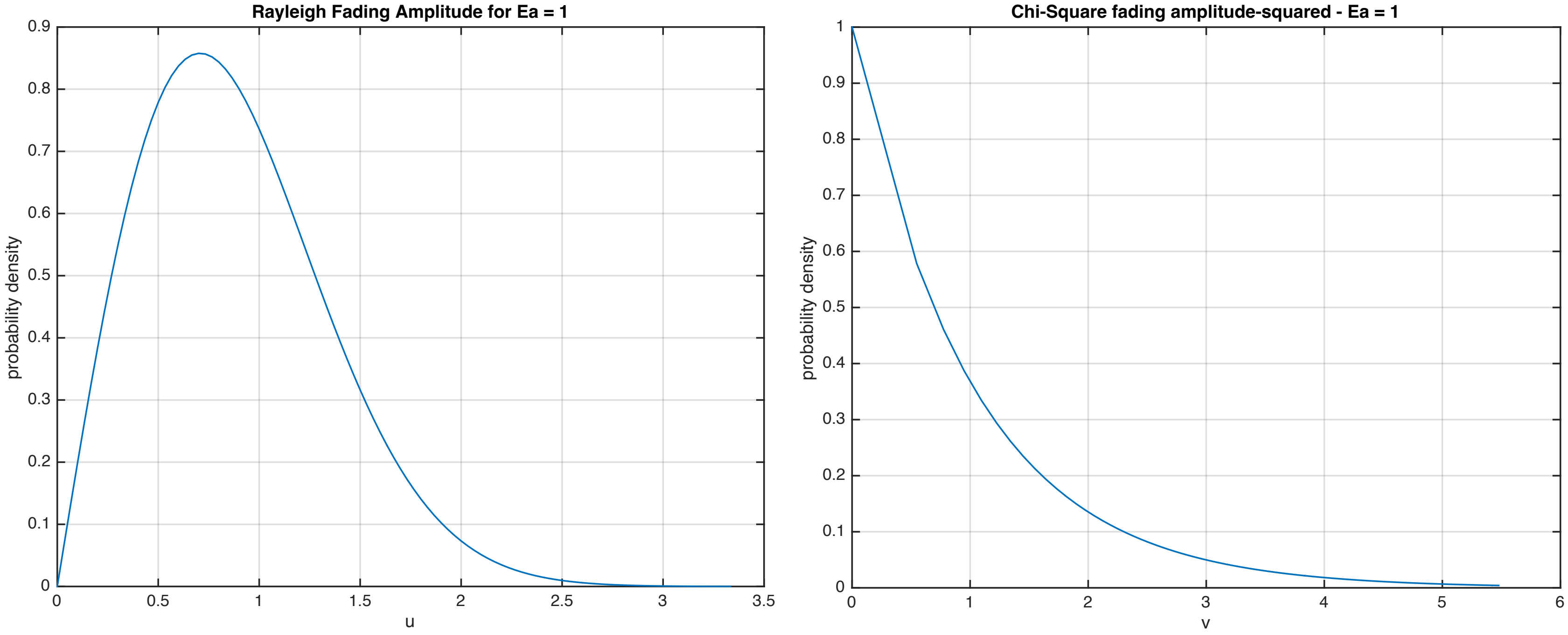
$$p_g(v) = \frac{1}{\mathcal{E}_g} \cdot e^{-\frac{v}{\mathcal{E}_g}}$$

Also called exponential

$$\mathbb{E}[g] = \mathcal{E}_g = \frac{\mathbb{E}[|h|^2]}{\sigma^2}\Big|_{\bar{\mathcal{E}}_x=1}$$

- P_e becomes random
  - Time average = statistical average (ergodic)

Distribution Plots



- Squaring h small value makes it smaller (when <1), forcing more probability to the left above

Ave Error Prob <P_e>

- For QPSK, $\kappa = 1$, the **average error probability** is

$$\langle P_e \rangle = \int_0^\infty Q\left(\sqrt{\kappa \cdot g}\right) \cdot p_g(g) \cdot dg$$

- For Rayleigh →

$$\langle P_e \rangle = \frac{1}{2}\left[1 - \sqrt{\frac{\kappa \cdot g}{\kappa \cdot g + 1}}\right] \simeq \frac{1}{4\kappa \cdot g}$$

$$\boxed{\kappa = \frac{3}{M-1} \text{ for square QAM}}$$

- $\langle P_e \rangle$ Only decays linearly with SNR
  - When $\langle P_e \rangle$ is low, it is possible that instantaneous P_e is unacceptable

- *Will need good codes (which spread redundancy over all dimensions)*
  - More important than increasing distance, although often (not always) the distance grows with spreading
  - That is, with **diversity**

$$\left[\frac{1}{4 \cdot \kappa \cdot g}\right]^{d/2}$$

The rcvr must make $d/2$ sample errors to cause a symbol error

Outage Probability (stability)

- The average $\langle P_e \rangle$ is less helpful in that the instantaneous values are more important
- There is a minimum SNR, and corresponding g, for which the system has too-high instantaneous P_e
  - This **outage probability** is

$$\boxed{P_{out}(\delta) = \Pr\{\, P_e > \delta \,\}}$$

- Outage is more important than single errors when its likelihood (P_{out}) is high
  - Outages are often not like single symbol errors or bit errors that are caused by isolated large noises
    - Outage is essentially a guarantee that ($100 \times P_{out}$) % of symbols in an s-b-s decoder will be unreliable, so “flip coins”
  - Something is lost - don’t want it too often. It measures user experience.
  - Most of the dimensions within the “**coherence** dimensions” will also be lost (see next slide)
- Typical outage probabilities
  - 5% - good enough for most internet traffic (translates to about a hour a day, depends on when)
  - 1% for video – only a few minutes a day when video does not work
  - FIVE 9’s (industrial or “carrier” grade – almost no faults) – .00001 -- less than 1 second per day.

Coherence

- Coherence Types:
  - **Time:** how correlated in time are the amplitude variables → the "**coherence time**"
  - **Frequency:** how correlated in frequency are the amplitude variables → the "**coherence bandwidth**"
  - **Space:** how correlated in space are the amplitude variables → "**spatial coherence**" (coherence length in optics)

Measured by the correlation coefficient $\rho = \frac{\mathbb{E}\left[x^* y\right]}{\sigma_x \cdot \sigma_y}$

- **Coherence Time:** T_Δ
  - 3dB point in phase shift for doppler frequency f_d of moving vehicle
  - Where $f_d = (v/c) \cdot f_c$ = ratio speed/light-speed times carrier-freq
  - If symbols are sent slower than this, then new h every message (not good)

$$T_\Delta < \frac{.125}{f_d \cdot (2^{2 \cdot \bar{b}} - 1)}$$

- **Coherence Bandwidth:** $W_\Delta = {}^{1}/_{\tau_{rms}}$
  - Where the rms delay spread τ_{rms} uses the power-delay profile as a probability distribution (normalizes it) to compute variance of delay around a nominal (mean) delay.

$$\tau_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} (t - \tau_0)^2 \cdot |h(t)|^2 \cdot dt}{\int_{-\infty}^{\infty} |h(t)|^2 \cdot dt}}$$

- **Spatial Coherence:** antenna spacing needs to be more than ½ wavelength for independence of noise
  - So signal (which is correlated) can increase amplitude coherently versus random noise
  - Often called "far field" of antenna

$c = 3 \times 10^8$ m/s

Macro Fading Model

- Lognormal is most common model for the macro fade
  - Essentially determines a multiplier for the Rayleigh (or Ricean) average value, so micro is about this Raleigh value
  - Cascade of transfer functions multiply, so their logs' add. Sum many random variables and get "normal" (Gaussian) by Central Limit Theorem

$$h_0 = e^{\mu_h + \sigma_h \cdot Z}$$

LOG NORMAL with mean and variance related to original h mean/variance

Mean and standard deviation usually specified in dB (so 10·log10)

$$\begin{aligned} \mu_h &= \ln\left(\frac{m_{h_0}}{\sqrt{1+\frac{\sigma_{h_0}^2}{m_{h_0}^2}}}\right) \quad \text{and} \\ \sigma_h^2 &= \ln\left(1+\frac{\sigma_{h_0}^2}{m_{h_0}^2}\right) . \end{aligned}$$

Typo, p. 140, Chap 1, $a_0 \rightarrow h_0$

Example – “simple” Wi-Fi model

- Multipath’s with Ricean/Rayleigh scattering

$$h = \sum_k h_k$$

u_k is unit-variance Gaussian

$$h_k = 10^{-L(d)/20} \cdot \sqrt{P_{h,k}} \cdot \left[\underbrace{\sqrt{\frac{K}{K+1}}}_{LOS} \cdot \delta_k + \underbrace{\sqrt{\frac{1}{K \cdot \delta_k + 1}} \cdot u_k}_{Rayleigh} \right]$$

K	delay (ns)	cluster 1 $P_{1,h,k}$ (dB)	cluster 2 (dB) $P_{2,h,k}$ (dB)
0	0	0	–
1	10	-5.4	-
2	20	-10.8	-3.2
3	30	-16.2	-6.3
4	40	-21.7	-9.4
5	50	–	-12.5
6	60	–	-15.6
7	70	–	-18.7
8	80	–	-21.8

- $K = 1$ is small home; $K = 4$ is big home/office
- $L(d)$ is the amplitude on overall (macro/shadow) fade/scattering = Path Loss (d in meters)

$$\frac{1}{\sqrt{2\pi\sigma_Z^2}} \cdot e^{-[L_{shadow}(d)]^2/(2\sigma_Z^2)}$$

$$\begin{aligned} L(d) &= L_{path}(d) + L_{shadow}(d) && \text{dB} \quad d \leq d_{bp} \\ &= L_{path}(d_{bp}) + L_{shadow}(d_{bp}) + 35\log_{10}\left(\frac{d}{d_{bp}}\right) && \text{dB} \quad d > d_{bp} \end{aligned}$$

$\sigma^2 = 3\ dB$ for L_{shadow}

(log normal)

$\sigma^2 = 4\ dB$ for L_{shadow}

Above the Break-point-distance value

where the break-point distance is $d_{bp} = 5$m for smaller homes and

$$L_{path} = 20 \cdot \log_{10}(d) + 20 \cdot \log_{10}(f) - 147.5 \text{ dB}$$

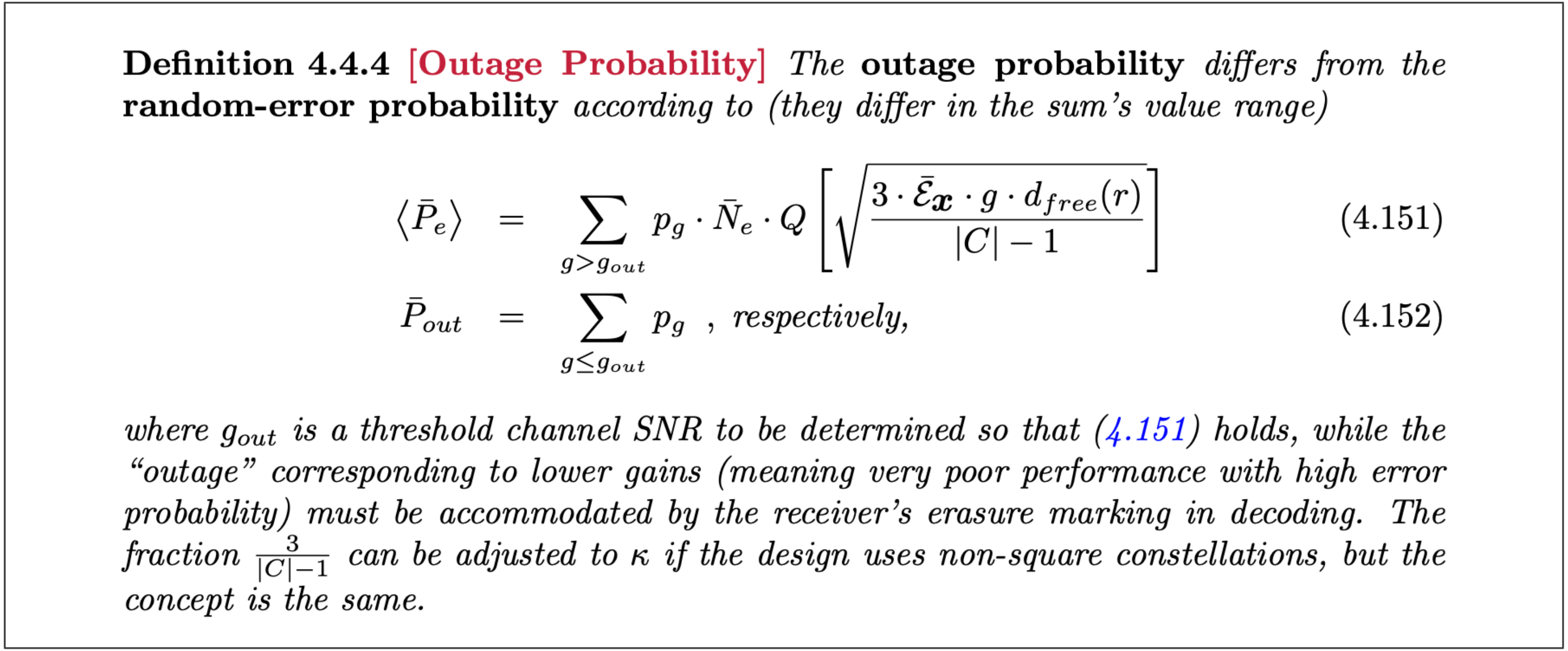
- Run Monte Carlo simulations on this

Loading with Statistical Channels

Subsection 4.4.2

PS3.1 (Prob 4.13), PS3.3 (Prob 4.22) and PS3.4 (Prob 4.16)

Calculation of average and outage error probs



Definition 4.4.4 [Outage Probability] *The* **outage probability** *differs from the* **random-error probability** *according to (they differ in the sum's value range)*

$$\langle \bar{P}_e \rangle = \sum_{g > g_{out}} p_g \cdot \bar{N}_e \cdot Q\left[\sqrt{\frac{3 \cdot \bar{\mathcal{E}}_{\boldsymbol{x}} \cdot g \cdot d_{free}(r)}{|C| - 1}}\right] \qquad (4.151)$$

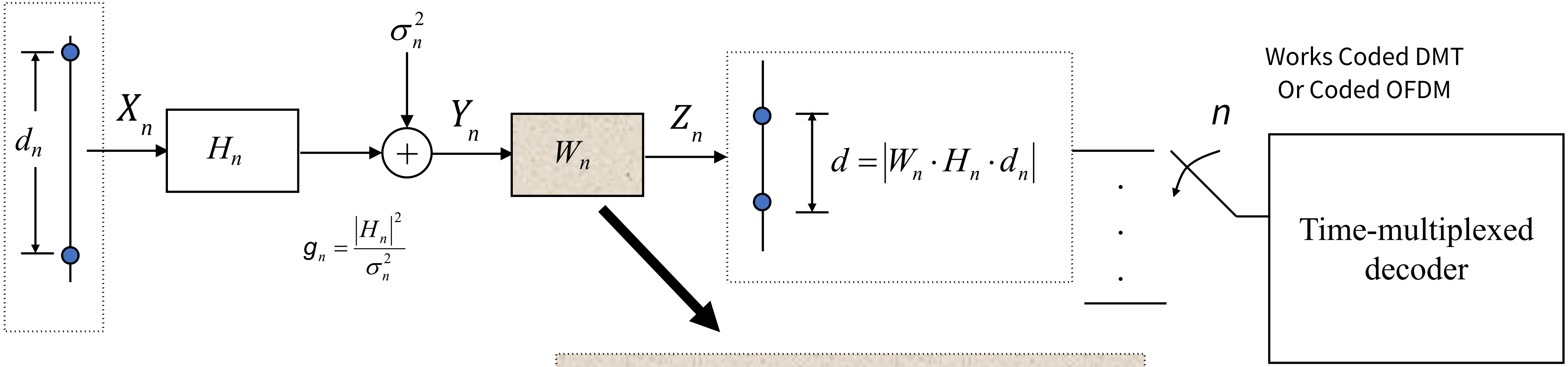
$$\bar{P}_{out} = \sum_{g \leq g_{out}} p_g \quad , \textit{respectively}, \qquad (4.152)$$

where g_{out} is a threshold channel SNR to be determined so that (4.151) holds, while the "outage" corresponding to lower gains (meaning very poor performance with high error probability) must be accommodated by the receiver's erasure marking in decoding. The fraction $\frac{3}{|C|-1}$ can be adjusted to κ if the design uses non-square constellations, but the concept is the same.

- These formulas depend on a channel-gain distribution p_g that is given or measured (by receiver)

- The average error probability is weighted by p_g for just those transmissions not in outage

Multichannel Normalizer (or "FEQ"), See L4

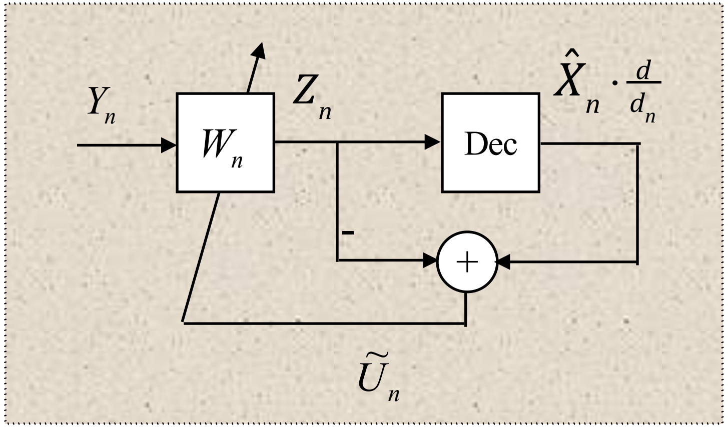


Works Coded DMT
Or Coded OFDM

Zero-Forcing Algorithm

$$W_{n,k+1} = W_{n,k} + \mu_n \cdot \tilde{U}_{n,k} \cdot \left(\frac{d}{d_n} \cdot \hat{X}^*_{n,k} \right)$$

$$\tilde{\sigma}^2_{n,k+1} = \left(1 - \mu'\right) \cdot \tilde{\sigma}^2_{n,k} + \mu' \cdot \left|\tilde{U}_{n,k}\right|^2$$



Channel estimate

$$\hat{H}_n = \frac{d}{d_n \cdot W_n}$$

Noise Estimate

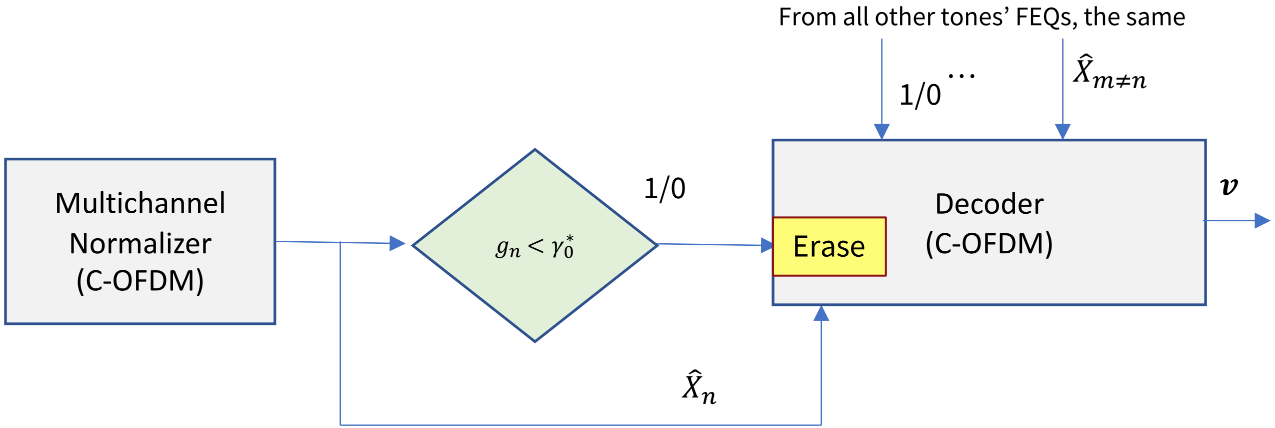
$$\hat{\sigma}^2_n = \frac{\tilde{\sigma}^2_n}{|W_n|^2}$$

can also estimate actual gain distribution p_g from this

Section 4.3.6.2

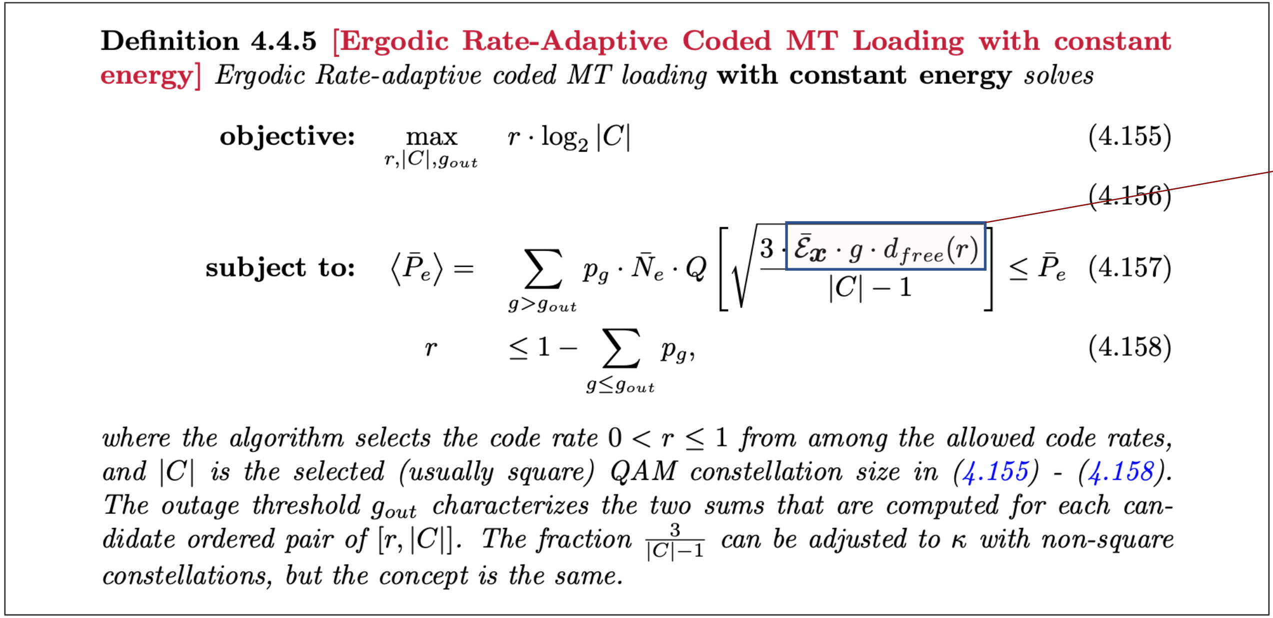
Erasures generally with FEQ & Decoder

- Erasure deletes the FEQ output from further consideration by the decoder



- Good codes can recover bits in erasures if their occurrence-fraction < 1 - *r*
- Upcoming Ergodic Water-Fill (both MA and RA) provide theoretical guidance, but not practical
- **Erasure avoids:**
  - "feedback delay that $g_n < \gamma_0$, so don't transmit" would take too long (issue largely only in wireless)
  - On/off or variable gain that frustrates power amplifiers (on/off transients) and receiver automatic gain-control circuits
- Adaptive power is often feasible for space (MIMO) to use/not use certain spatial channels, but not for time-frequency

Ergodic rate-adaptive loading



Definition 4.4.5 **[Ergodic Rate-Adaptive Coded MT Loading with constant energy]** *Ergodic Rate-adaptive coded MT loading* **with constant energy** *solves*

$$\textbf{objective:} \quad \max_{r,|C|,g_{out}} \quad r \cdot \log_2 |C| \tag{4.155}$$

$$\tag{4.156}$$

$$\textbf{subject to:} \quad \langle \bar{P}_e \rangle = \sum_{g > g_{out}} p_g \cdot \bar{N}_e \cdot Q\left[\sqrt{\frac{3 \cdot \boxed{\bar{\mathcal{E}}_{\boldsymbol{x}} \cdot g \cdot d_{free}(r)}}{|C|-1}}\right] \leq \bar{P}_e \tag{4.157}$$

$$r \quad \leq 1 - \sum_{g \leq g_{out}} p_g , \tag{4.158}$$

where the algorithm selects the code rate $0 < r \leq 1$ from among the allowed code rates, and $|C|$ is the selected (usually square) QAM constellation size in (4.155) - (4.158). The outage threshold g_{out} characterizes the two sums that are computed for each candidate ordered pair of $[r, |C|]$. The fraction $\frac{3}{|C|-1}$ can be adjusted to κ with non-square constellations, but the concept is the same.

Q-func is correct ;
because this is sample AWGN;
however

$$\mathcal{E}_x \cdot g \cdot d_{free}$$

is an approximation that is correct only for QPSK & BPSK. However, BICM spreads good binary codes' proportional distance increase to good γ_f also to

$$d_{min,SQ\ QAM}$$

Potential project –
find better function
of r for 64-state ½ code

- Finds the cut-off channel gain g_0 that maximizes data rate ($\tilde{b} = r \cdot |\mathcal{C}|$)
  - Of course, this is restricted by allowed constellations, code ($d_{free}(r)$), and choice of average error probability
  - Assumes $r \leq 1 - P_{out}$
- Assumes bit-interleaved coded modulation (multiplex xmit bits from multiple codewords, rcvr demux)
  - **BICM** spreads outage bit errors over the interleave "depth" so as to avoid an error burst overwhelming the system

Loading on Actual Wireless Channel with Flat Energy

- Compute gap-based data rate for both average and max (over all tones) with rough gap estimate

$$b_{flat-geo} = \frac{1}{N} \cdot \sum_{n=1}^{N} \log_2\left(1 + \frac{\bar{\mathcal{E}}_x \cdot g_n}{\Gamma}\right) \qquad b_{\max} = \max_n \left\{ \log_2\left(1 + \frac{\bar{\mathcal{E}}_x \cdot g_n}{\Gamma}\right) \right\} = \log_2 \text{ (max constellation size)}$$

- Compute code rate for "good code" (low gap) with this flat energy
  - Then code rate *r* then leads to a d_{free} for the selected code

$$r = \frac{b_{flat-geo}}{b_{max}} \leq 1$$

- Compute outage probability from *r*

$$\bar{P}_{out} = 1 - r$$

- Form/update p_g by binning or counting of *g* value range (6 dB – *10 log10* $(d_{free\text{-}new}/d_{free\text{-}old})$ for which *b* value would not change choice of constellation size (see S27-28)
  - Counts FEQ noise estimate values in each range (current symbol or over many symbols)
- Solve for g_0 -- tones with $g<g_0$ will be "erased"
- Indicate "erasure" (delete from sum is one way) in ML detector
  - Can be real erasure in Reed-Solomon for instance
  - Can be zero LLR in iterative decoder for bits corresponding to that tone

$$\bar{P}_{out} = \sum_{g<g_0} p_g$$

Error Correction Code (Section 2.2)

- Example:
  - Reed Solomon Block FEC byte wise (N <256)
    - Parity P is up to 32 bytes
    - Can correct up to $P/2$ bytes if in random codeword positions; up to P if “erasures” (locations of likely byte error) are determined
  - Suppose P_{out} = 5% and $N=200$
    - Then 10 to 20 parity bytes are needed

  - Suppose P_{out} = 50%, then $N=40$ and $P > 20$ might be needed (low rate code < ½)

- This type of corrective ability extends to convolutional coding, particularly when soft decoding (with iterative decoding and soft information metrics) are used with BICM

Wireless Example

- Extension of well-known code as example (64-state rate-1/2 code with puncturing)
  - >> r =   0.9   0.8   0.75   0.67   0.5   0.25   0.2
  - >> dfree =   2   4   6   7   10   20   25
- Given (measured) channel-gain distribution
  - >> g = 3   30   300   600   1200   2400   4800   10000
  - >> pg = 0.0500 0.0500 0.1000 0.1000 0.3500 0.2000 0.1000 0.0500

```
>> prob2=kron(ones(7,1),pg)
>> Pout=cumsum(prob2(1,1:8)) =
    0.0500    0.1000    0.2000    0.3000    0.6500    0.8500    0.9500    1.0000


>> ones(1,8)-Pout =
    0.9500    0.9000    0.8000    0.7000    0.3500    0.1500    0.0500         0
```

correctable   Not correctable

```
>> r =
    0.9000    0.8000    0.7500    0.6700    0.5000    0.2500    0.2000
```

PS3.1 (Prob 4.13)

Wireless Example continued for <Pe>

- The SNR (with $\tilde{\mathcal{E}}_x = 1$)

```
>> SNR=kron(dfree',g);
>> 10*log10(SNR) =  (in dB)
   7.7815  17.7815  27.7815  30.7918  33.8021  36.8124  39.8227  43.0103
  10.7918  20.7918  30.7918  33.8021  36.8124  39.8227  42.8330  46.0206
  11.7609  21.7609  31.7609  34.7712  37.7815  40.7918  43.8021  46.9897
  12.5527  22.5527  32.5527  35.5630  38.5733  41.5836  44.5939  47.7815
  14.7712  24.7712  34.7712  37.7815  40.7918  43.8021  46.8124  50.0000
  17.7815  27.7815  37.7815  40.7918  43.8021  46.8124  49.8227  53.0103
  18.7506  28.7506  38.7506  41.7609  44.7712  47.7815  50.7918  53.9794
```

$$\tilde{\mathcal{E}}_x \cdot d_{free} \cdot g$$

Only top 4 rows and above diag are eligible

- Compute $\langle P_e \rangle$ for several SNRs and SQ QAM Constellations
  - 4QAM -- >> prob1=q(sqrt(SNR(1:4,1:4)))
  - 16 QAM -- >>prob1=2*q(sqrt((3/15)*SNR(1:4,1:4)))
  - etc

Compute <Pe> tables

4SQ QAM

```
>> Pe=cumsum((prob2.*prob1_4)','reverse')' =
```

$g_0 \rightarrow$

```
 r    1.0e-03 *
.9      0.3576    0.0000    0.0000    0.0000
.8      0.0133    0.0000    0.0000         0
.75     0.0006    0.0000         0         0
.67     0.0001    0.0000         0         0
```

$$\bar{b} = \frac{2}{3} \cdot 1 = .67$$

16SQ QAM

```
>> Pe=cumsum((prob2.*prob1_16)','reverse')' =

  0.013692684189872  0.000026600275257
  0.006066810685775   0.000000048167850
  0.002888978654839   0.000000000098659
  0.002021198971409   0.000000000004564
```

$$\bar{b} = \frac{4}{5} \cdot 2 = 1.8$$

256SQ QAM

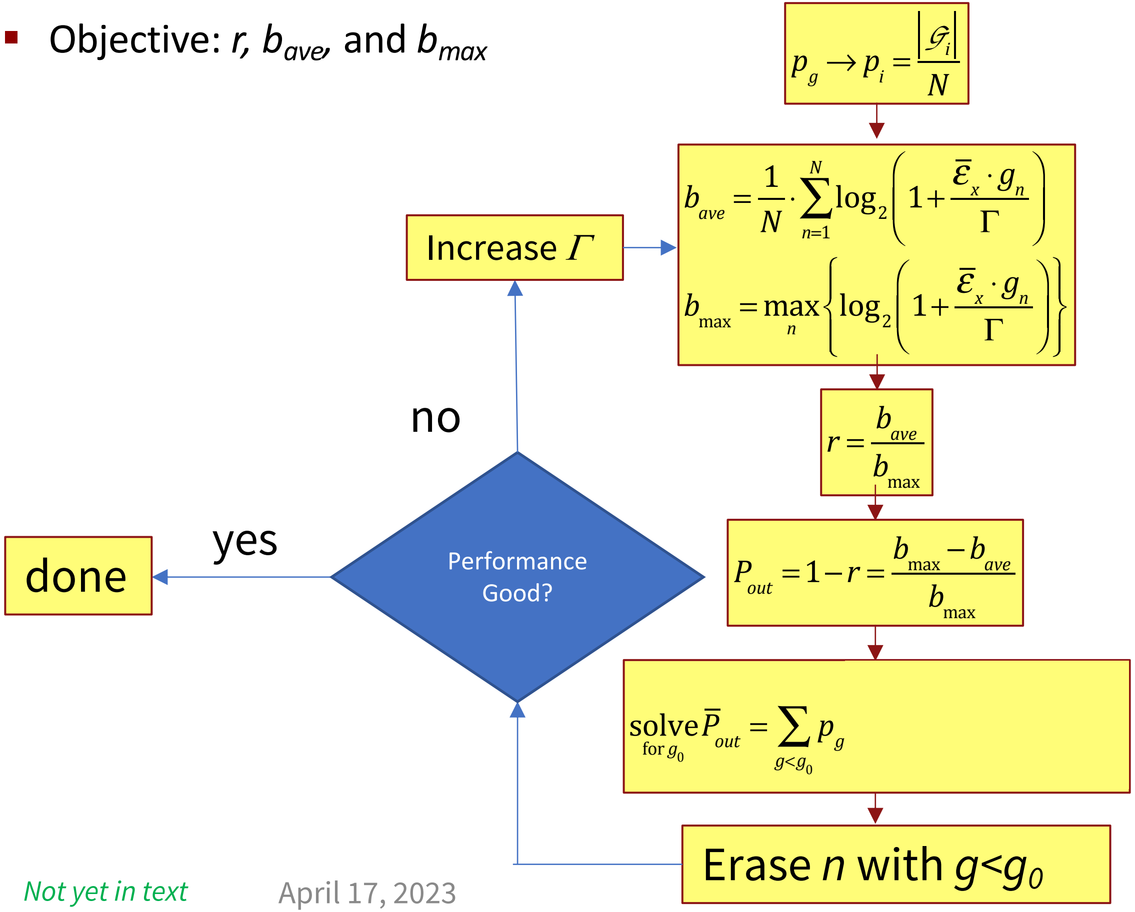
```
>> prob1=2*q(sqrt((3/255)*SNR(1:4,1:4)))
>> Pe=cumsum((prob2(1:4,1:4).*prob1)','reverse')' =

0.060370750183693  0.020846638776725  0.000805930307579
0.047111077964265  0.011755366399767  0.000017183258383
0.039550311046489  0.007280923653021  0.000000418883178
0.036757477446216  0.005799792111764  0.000000066780250
```

$$\bar{b} = \frac{2}{3} \cdot 4 = 2.67$$

- 256 QAM produces best solution at 2.67 (check 64 QAM for 2.0 bits/dim, or see Example 4.4.3)

Project: Flow Chart for C-OFDM Loading?

- Objective: *r*, b_{ave}, and b_{max}



$$p_g \rightarrow p_i = \frac{\left|\mathcal{G}_i\right|}{N}$$

Calculate Distribution

$$b_{ave} = \frac{1}{N} \cdot \sum_{n=1}^{N} \log_2\left(1 + \frac{\bar{\mathcal{E}}_x \cdot g_n}{\Gamma}\right)$$

$$b_{\max} = \max_n \left\{ \log_2\left(1 + \frac{\bar{\mathcal{E}}_x \cdot g_n}{\Gamma}\right) \right\}$$

Compute averate and max bits/tone

$$r = \frac{b_{ave}}{b_{\max}}$$

For the used code find *r* and d_{free}

$$P_{out} = 1 - r = \frac{b_{\max} - b_{ave}}{b_{\max}}$$

Code should be able to correct for P_{out} % erased

$$\underset{\text{for } g_0}{\text{solve}} \bar{P}_{out} = \sum_{g<g_0} p_g$$

Need to approximate Q function or store it for each *i* value

Erase *n* with $g<g_0$

Performance Good?

yes

done

no

Increase Γ

untested by instructor

Not yet in text

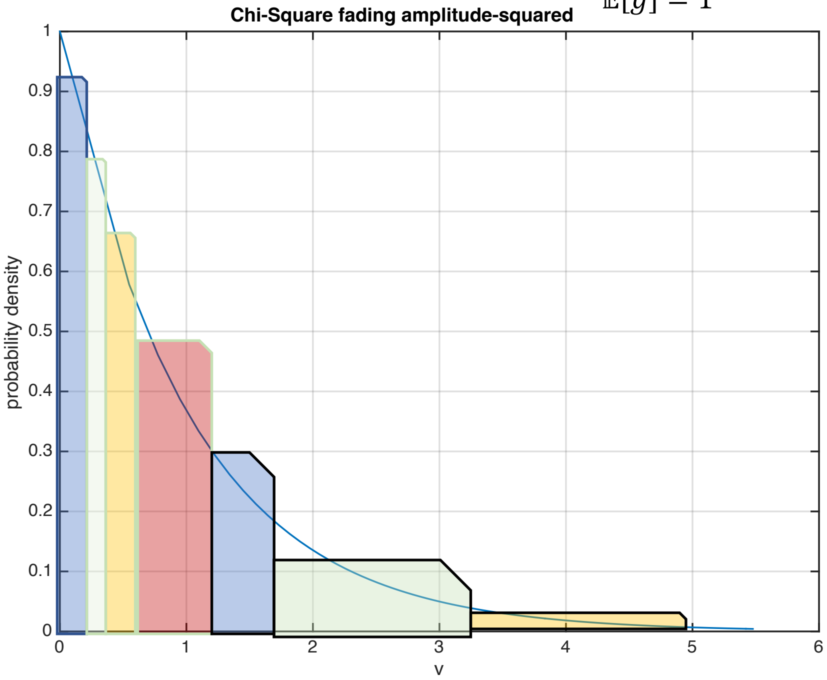
Ergodic Coded-OFDM Loading

Subsection 4.4.2

See PS3.2 (Prob 4.14)

Approximate the distribution as discrete

$\mathbb{E}[g] = 1$



$$p_g(v) \cdot dv \rightarrow p_{g,n}$$

May care about
lower ranges more and
so more finely divide
distribution there into
samples

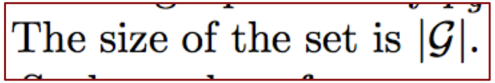
- This simplifies calculations for loading, and the original model was only approximate anyway

Discrete Distributions

- All distributions (Rayleigh, Ricean, Log-Normal, etc) are gross approximations in wireless
- Can approximate by discrete distributions (which can be learned) $p_g(v) \cdot dv \rightarrow p_{g,n}$
- May need to renormalize so probabilities sum to 1

$$p_{g,n} = \frac{p_g(n \cdot \Delta) \cdot \Delta}{\sum_n p_g(n \cdot \Delta) \cdot \Delta}$$

- The probability $p_{g,n}$ represents the fraction of time that $n\Delta - \frac{\Delta}{2} < g \leq n\Delta + \frac{\Delta}{2}$
  - Intervals do not need to be the same size as long as indexed and correct weighting (may want to align with constellation-size choices)
- Simplify so that *g* is discrete set of center values $p_g(n\Delta)$ and discrete index $g \in \mathcal{G}$


The size of the set is $|\mathcal{G}|$.

See PS3.4 (Prob 4.16) & PS3.5 (Prob 4.15)

- *g* takes place of dimension index, almost (weight not 1 as with integer index)

Average bit rate and ergodic capacity ($\Gamma = 0$ dB)

- Average bit rate

$$\langle b \rangle = \sum_{g \in \mathcal{G}} p_g \cdot \log_2\left(1 + \frac{\mathcal{E}_{x,g} \cdot g}{\Gamma}\right)$$

Ave Mutual Info has Γ=0 dB

***g* takes n's place in DMT** RA

- Average Energy

$$\mathcal{E}_x = \sum_{g \in \mathcal{G}} \mathcal{E}_{x,g} \cdot p_g$$

See PS3.2 (Prob 4.14)

- Rate Adaptive and Margin-Adaptive Water-fill $\mathcal{E}_g = K - \Gamma/g$

$$\mathcal{E}_{x,g} = K_{ra} - \frac{\Gamma}{g} \quad ,$$

$$\begin{aligned}
\mathcal{E}_{\boldsymbol{x}} &= \sum_{g \in \mathcal{G}^*} p_g \cdot \left(K_{ra} - \frac{\Gamma}{g}\right) \\
&= K_{ra} \cdot \sum_{g \in \mathcal{G}^*} p_g - \sum_{g \in \mathcal{G}^*} p_g \cdot \frac{\Gamma}{g} \\
K_{ra} &= \frac{\mathcal{E}_{\boldsymbol{x}} + \Gamma \cdot \sum_{g \in \mathcal{G}^*} \frac{p_g}{g}}{\sum_{g \in \mathcal{G}^*} p_g}
\end{aligned}$$

RA

$$g_{geo}(*) \triangleq \prod_{g \in \mathcal{G}^*} (g)^{\left[\frac{p_g}{\sum_{g \in \mathcal{G}^*} p_g}\right]}$$

$$\begin{aligned}
< b > &= \log_2 \prod_{g \in \mathcal{G}^*} \left(1 + \frac{\mathcal{E}_{x,g} \cdot g}{\Gamma}\right)^{p_g} \\
&= \log_2 \prod_{g \in \mathcal{G}^*} \left(\frac{K_{ma} \cdot g}{\Gamma}\right)^{p_g} \\
2^{<b>} &= \left(\frac{K_{ma}}{\Gamma}\right)^{\sum_{g \in \mathcal{G}^*} p_g} \cdot \prod_{g \in \mathcal{G}^*} g^{p_g} \\
K_{ma} &= \Gamma \cdot \left(\frac{2^{<b>}}{\prod_{g \in \mathcal{G}^*} g^{p_g}}\right)^{\frac{1}{\sum_{g \in \mathcal{G}^*} p_g}} \\
&= \Gamma \cdot \frac{\left(2^{<b>}\right)^{\left[\frac{1}{\sum_{g \in \mathcal{G}^*} p_g}\right]}}{g^*_{geo}}
\end{aligned}$$

MA

Section 4.4.2.1 and 4.4.2.3

Ergodic Water-filling - Goldsmith

- The amount of energy is either zero or a value given by the water-fill equation for $g \in \mathcal{G}^*$

$$\mathcal{E}_{x,g} = \begin{cases} \Gamma \cdot \frac{2^{<b>}}{g_{geo}} - \frac{\Gamma}{g} & g > \frac{\Gamma}{K_{ma}} \\ 0 & g \leq \frac{\Gamma}{K_{ma}} \end{cases}$$

$$\text{transmit if } g > g_0 = \frac{\Gamma}{K_{ma}}$$

- Direct water-fill calc from time-domain g values is non-causal; can't really be made causal with delay
- Instead, the stationary statistics have been exploited above
  - Those statistics (p_g) may need to be estimated by counting g values though over time
  - If the system is ergodic, this will get better and better
- Instantaneous transmit energy often has limit (less than the $\mathcal{E}_{x,g}$ value above) for small g

If so, reduce $\langle b \rangle$ -- this is a kind of margin also

not practical – use erasures instead, but provides bound

Average and Outage Capacities

- **Average Capacity:** $\langle \tilde{\mathcal{C}} \rangle = \sum_g p_g \cdot \log_2\left(1 + \mathcal{E}_{x,g} \cdot g\right)$ bits/complex-subsymbol
  - Depends on energy distribution, which would be ergodic water-fill for maximum value

$$\boxed{= \quad \log_2 \prod_{g \in \mathcal{G}^*} \left(\frac{K_{ma} \cdot g}{\Gamma} \right)^{p_g}}$$

- **Outage Capacity:** $\tilde{\mathcal{C}}_{out} \triangleq (1 - P_{out}) \cdot \langle \tilde{\mathcal{C}} \rangle$ bits/complex-subsymbol
  - Lose the data transmitted during outage (may need retransmission)

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End Lecture 5