Peak Power Reduction for Multicarrier Transmission

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Abstract—This paper proposes two efficient and distortionless schemes to significantly reduce the Peak to Average power Ratio (PAR) of DMT and OFDM signals. These methods are based on adding a symbol dependent, time domain signal to the original DMT/OFDM symbol to reduce its peaks. This time domain signal can be computed efficiently at the transmitter and can be easily stripped off the received signal. For the first method, called Tone Reservation (TR), the transmitter does not send data on a small subset of carriers, which are optimized for PAR reduction. This technique can achieve $3dB$ or $6dB$ of PAR reduction for a loss in data rate of less than .2% and 5% respectively. The second scheme, called Tone Injection (TI), is based on replicating the subsymbol constellation with known translating vectors. In this expanded constellation several points represent the same information, and therefore we can choose those points that minimize the transmitter PAR. This method achieves more than $6dB$ PAR reduction without data rate loss and with negligible increase in transmitter average power.

I. Introduction

Multicarrier modulation has many well-known advantages [1] and has recently made its way into many applications in both baseband and passband transmission. Some of the most well known are xDSL over wired media using Discrete MultiTone (DMT) and DAB or DVB for wireless communications using Orthogonal Frequency Division Multiplexing (OFDM).

The disadvantage of such schemes is that multicarrier signals exhibit Gaussian-like, time-domain waveforms with relatively high Peak to Average power Ratio (PAR). These large peaks require linear and consequently inefficient amplifiers. To avoid operating the amplifiers with extremely large backoffs, we must allow occasional saturation of the power amplifiers or clipping in the D/A, leading to intermodulation products and spectral regrowth.

Recently, there has been a variety of creative methods on how to generate multicarrier symbols with low PAR [2]–[8], but none of these proposed methods is able to achieve simultaneously a large reduction in PAR with low complexity, with low coding overhead, without performance degradation and without transmitter-receiver symbol handshake. This paper proposes a novel family of methods which can achieve all of these goals.

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II. Problem Formulation

In the following, we use lower case x for time domain values, and upper case X for frequency domain values. Vectors are denoted either in boldface or as a sequence $x[n]$, and its components in italic with subindex, e.g., $\mathbf{x} = x[n] = [x_0 \cdots x_{N-1}]$. The index n is used for time and the index k for frequency.

A DMT/OFDM transmit signal is the sum of N independent Quadrature Amplitude Modulated (QAM) sub-signals each with equal bandwidth and frequency separation $1/T$, where T is the time duration of the DMT symbol [1]. In practical applications, the DMT/OFDM symbol is generated by using an Inverse Discrete Fourier Transform (IDFT). This results in the T/N-spaced discrete-time representation, $\mathbf{x} = [x_0 \cdots x_{N-1}]^T$ of the complex-valued QAM vector $\mathbf{X} = [X_0 \cdots X_{N-1}]^T$. Thus, the transmit signal is generated with:

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad 0 \le n < N, \tag{1}$$

and can also be written in matrix form as $\mathbf{x} = Q\mathbf{X}$, where Q is the IFFT matrix with elements $q_{n,k} = \frac{1}{\sqrt{N}} e^{j2\pi kn/N}$.

From now on, we will assume baseband DMT/OFDM, i.e. x_n must be real, but all these results can be extended to passband DMT/OFDM. Assuming N is even, for x_n to be a real sequence, X_k must verify $X_k = X^*_{N-k+1}$ and X_0 and $X_{N/2}$ must be real. Similar constraints are necessary for N odd.

It is well known that when N is large, x_n can be accurately modeled by a Gaussian random process with zero mean. Our goal is to add structured time domain vectors to x_n, such that we reduce its peak excursions. If we add a frequency domain vector $\mathbf{C} = [C_0 \cdots C_{N-1}]^T$ to $\mathbf{X}$, the new time-domain vector is:

$$\mathbf{x} + \mathbf{c} = Q(\mathbf{X} + \mathbf{C}) \tag{2}$$

where $\mathbf{c} = Q\mathbf{C}$. Since $\mathbf{c}$ must be real also, $\mathbf{C}$ must satisfy the same symmetry properties as $\mathbf{X}$.

The modified transmit symbol, after adding $\mathbf{c}$ is $\mathbf{x}+\mathbf{c}$,and the new PAR is defined as:

$$PAR(\mathbf{x} + \mathbf{c}) = \frac{\|\mathbf{x} + \mathbf{c}\|^2_\infty}{\mathcal{E}[\|\mathbf{x}\|^2_2]/N} \tag{3}$$

where $\mathcal{E}[\cdot]$ denotes expectation. The two special structures for $\mathbf{c}$ and $\mathbf{C}$ are described in the following sections.

III. Peak Reduction by Tone Reservation

This first method[1] restricts the data vector $\mathbf{X}$, and the peak reduction vector $\mathbf{C}$ to lie in disjoint frequency subspaces, i.e $X_k = 0,\ k \in \{i_1, \ldots, i_L\}$ and $C_k = 0,\ k \notin \{i_1, \ldots, i_L\}$. This formulation is distortionless and leads to very simple decoding of the data subsymbols which are extracted from the received sequence by choosing the set of values $k \notin \{i_1, \ldots, i_L\}$ at the receiver DFT output. Moreover, it allows simple optimization techniques for the computation of the peak reduction vector $\mathbf{c}$. The L nonzero values in $\mathbf{C}$ will be called Peak Reduction Tones (PRT).

Calling $\hat{\mathbf{C}}$ the nonzero values of $\mathbf{C}$, i.e. $\hat{\mathbf{C}} = [C_{i_1} \cdots C_{i_L}]^T$, and $\hat{Q} = [q_{i_1} | \cdots | q_{i_L}]$ the submatrix of Q constructed by choosing its columns $\{i_1, \ldots, i_L\}$, then $\mathbf{c} = Q\mathbf{C} = \hat{Q}\hat{\mathbf{C}}$. To minimize the PAR of $\mathbf{x} + \mathbf{c}$ we must compute the vector $\mathbf{c}^\star$ that minimizes the maximum peak value, i.e.:

$$\min_{\mathbf{c}} \|\mathbf{x} + \mathbf{c}\|_\infty = \min_{\hat{\mathbf{C}}} \|\mathbf{x} + \hat{Q}\hat{\mathbf{C}}\|_\infty \quad (4)$$

This optimization problem is convex on the variables $\hat{C}_{i_k}$ and can be easily cast as a Linear Program (LP) [9]. Convex optimization problems and in particular Linear Programs have been studied extensively. In addition to having a unique solution, their convergence properties are well known and many algorithms provide lower bounds on the solution at every iteration [10]. This LP has L unknowns and $2N$ inequalities. For a general LP of this size, the complexity would be $O(LN^2)$. In our case, we have a very structured LP as Q is the IFFT matrix. Therefore, using this structure we can solve this LP with complexity O(NlogN). As we will explain in Section A, we can use gradient algorithms with $O(N)$ complexity that give good approximations to (4). The formulation in (4) can be extended without sacrificing convexity, to include additional constraints on the transmit symbol, such as limiting the maximum symbol power $\|\mathbf{X} + \mathbf{C}\|_2^2 < P_1$, or the maximum tone power $\|\mathbf{X} + \mathbf{C}\|_\infty^2 < P_2$.

By setting L values of X_k to zero, we cannot send information on these L tones. We call L/N, the Peak Reduction Tone Fraction (PRTF). Denoting b_k, the number of bits transmitted in tone k, the Data Rate Loss (DRL) would be

$$DRL = \frac{\sum_{k=1}^{L} b_{i_k}}{\sum_{k=0}^{N-1} b_k} \quad (5)$$

In many applications, the N tones in our DMT symbol do not have equal SNR_k and the values of b_k can be very different from tone to tone. By choosing the set $\{i_1, \ldots, i_L\}$ such that the values of b_{i_k} are small, we can get $DRL << PRTF$. In particular, if $b_{i_k} = 0$, then there would be no data rate loss on that tone. For applications where $b_k = const$, e.g. OFDM, we will have to minimize L to reduce the DRL.

To evaluate the performance of our PAR reduction algorithms, we will plot the symbol clipping probability, i.e. the probability that our symbol exceeds a given PAR threshold, PAR_0, or equivalently, the complementary cumulative distribution function (CCDF) of the PAR. Thus, all figures plot $Prob(PAR(\mathbf{x}) > PAR_0)$ for different values of PAR_0. The solid line in Fig. 1 plots the clipping rate for the original DMT symbols ($\mathbf{c} = 0$). The dashed-dotted line plots the clipping probability after peak-reduction when $L/N = 5\%$. For this case, if we desire a clipping probability of less than 10^{-5}, we can reduce the PAR_0 from $15dB$ to $9dB$. Therefore, the statistical PAR is reduced by $6dB$ if we can only tolerate a clipping probability below 10^{-5}. The dashed line in Fig. 1 plots results for $L/N = 20\%$. In this case, PAR decreases from $15dB$ to $5dB$ for a clipping probability below 10^{-5}.

Adding $\mathbf{c}$ to $\mathbf{x}$ for PAR reduction results in a small increase in average transmit power. If we compute the optimal solution $\mathbf{c}^\star$, when $L/N = 5\%$, the mean power increase is $1dB$ and for $L/N = 20\%$ it is $0.5dB$. As we will show in the following sections, we can get most of this PAR reduction from low complexity iterative methods after a small number of iterations. Moreover, for these simplified solutions the power increase is much smaller.

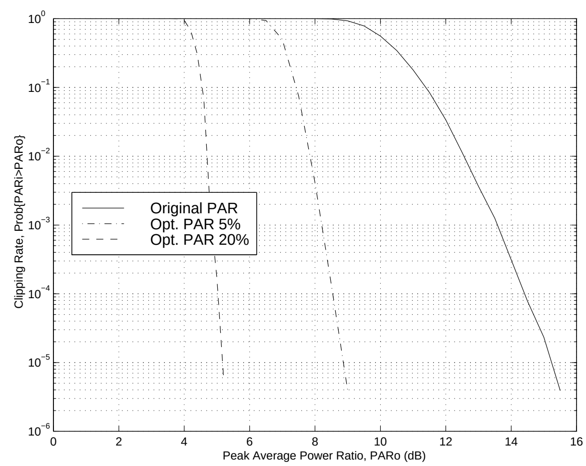

Fig. 1. *Probability that the PAR of a randomly generated 256-carrier baseband DMT/OFDM symbol exceeds PAP_0 for $\frac{L}{N} = 5\%$ and $\frac{L}{N} = 20\%$ with Randomly-Optimized set $\{i_k^\star\}$*

A. Gradient algorithm for computing $c^\star$

The previous section showed how to minimize the PAR exactly. Here, we show that by taking the gradient of the clipping noise Mean Square Error (MSE), we get simple iterative algorithms that achieve most of the PAR reduction promised by the optimal solution after a few steps[2].

[1] This formulation was first presented in [9]. Independently, Gatherer et al. [4] have developed a related method based on unused tones that achieves worse performance than the procedure described here.

[2] A related gradient technique based on *distortionless clipping* is described in [9], [11]

An ideal clipper is defined as:

$$clip_A(x_k) = \begin{cases} x_k, & |x_k| \le A \\ A\ sign(x_k), & |x_k| > A \end{cases} \quad (6)$$

then the Clipping Noise Power of the transmitted time sequence, $\mathbf{x}$, is:

$$\|\mathbf{x} - clip_A(\mathbf{x})\|_2^2 = \sum_{n=0}^{N-1} (x_n - clip_A(x_n))^2 \quad (7)$$

and the Signal to Clipping noise Ratio (SCR) will be:

$$SCR = \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{x} - clip_A(\mathbf{x})\|_2^2} \quad (8)$$

If we include the PRT to these equations, the transmitted time sequence is $\mathbf{x} + \mathbf{c}$, where $\mathbf{c} = \hat{Q}\hat{\mathbf{C}}$ and the SCR is

$$SCR = \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{x} + \mathbf{c} - clip_A(\mathbf{x} + \mathbf{c})\|_2^2} \quad (9)$$

To maximize SCR, we must minimize the denominator of (9). This optimization problem is also convex (quadratic program) in the unknowns $\hat{\mathbf{C}}$ and can be optimized exactly using standard software. Instead of solving for $\hat{\mathbf{C}}$ exactly, we will describe a gradient based iterative algorithm with complexity O(N). After some algebra we can show that the gradient with respect to $\hat{\mathbf{C}}$ of the denominator of (9) is:

$$\nabla_{\hat{\mathbf{C}}} \|\mathbf{x} + \mathbf{c} - clip_A(\mathbf{x} + \mathbf{c})\|_2^2 = \quad (10)$$

$$\sum_{|x_i + c_i| > A} sign(x_i + c_i)(|x_i + c_i| - A)\hat{Q}\hat{q}_{row}^i \quad (11)$$

where $\hat{q}_{row}^i$ denotes the i-th row of $\hat{Q}$. Using the notation $\alpha_i^{(k)} = sign(x_i + c_i^{(k)})(|x_i + c_i^{(k)}| - A)$ we have the following gradient based iterative algorithm to update $\mathbf{c}$:

$$\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} - \mu \sum_{|x_i + c_i^{(k)}| > A} \alpha_n^{(k)} \hat{Q}\hat{q}_{row}^n \quad (12)$$

To reduce the complexity in the update equation (12) we can restrict the number of terms in the summation to include only its largest terms, i.e. the largest values of $|x_i + c_i^{(k)}|$. This will also let us use a larger value for μ. Calling $\mathbf{p}^m = \hat{Q}\hat{q}_{row}^m$, it is easy to show that these N vectors are simply circularly shifted replicas of each other:

$$p^m[n] = p^0[((n-m))_N],\ 0 \le n, m < N \quad (13)$$

and that $p^m[m] = max_n\{p^m[n]\}$. Intuitively, this SCR gradient algorithm searches for the largest terms in $\mathbf{x} + \mathbf{c}^{(k)}$ and subtracts scaled, circularly shifted replicas of the vector $\mathbf{p}^0$ to cancel these large peaks. The convergence rate of this SCR gradient algorithm depends on the properties of $\mathbf{p}^0 = \hat{Q}\hat{q}_{row}^0$, which is only a function of the PRT set $\{i_k\}$. Each time $\mathbf{p}^m$ reduces the peak at location m, it is possible to increase the value of other samples. Thus, we are interested in choosing the PRT set $\{i_k\}$ that reduces the values of $\mathbf{p}^m[n]$, for $n \ne m$, or in other words, that minimizes the secondary peaks. In mathematical form:

$$\{i_k^\star\} = \arg\min_{\{i_k\}} \|[p_1^0\ p_2^0 \cdots p_{N-1}^0]\|_\infty,\ \mathbf{p}^0 = \hat{Q}\hat{q}_{row}^0 \quad (14)$$

The solution to this problem is NP since we must optimize over the discrete set $\{i_k\}$. But, good sets are obtained by generating random sets and selecting the best. This method for tone selection will be called random set optimization. If the bits per tone, b_k, were not equal, we could also consider the DRL when choosing the PRT set. The importance of a good choice of $\{i_k\}$ can be seen in Fig. 2. The dashed line plots the optimized PAR for the choice $\{i_k\} = \{K_0 + k\}$, $1 \le k \le L$ (contiguous tones). Similar result was obtained for the choice equally spaced tones or other structured sets. For these choices, the PAR reduction for a 10^{-5} clipping rate is about $3.4dB$. On the other hand, $6.2dB$ PAR reduction is obtained by using the random set optimization method.

The complexity of the algorithm is O(N) since at each iteration, we must find the maximum of $\mathbf{x}+\mathbf{c}^{(k)}$, scale $p[((n-i))_N]$ by α_i and add them together. Therefore, we must do N real multiply/adds per iteration. Since $\mathbf{p}^m$ is fixed, we can precompute and store $\alpha_k\mathbf{p}$ for several values of α_k. This way, we can reduce the complexity to finding the maximum and N real additions per iteration.

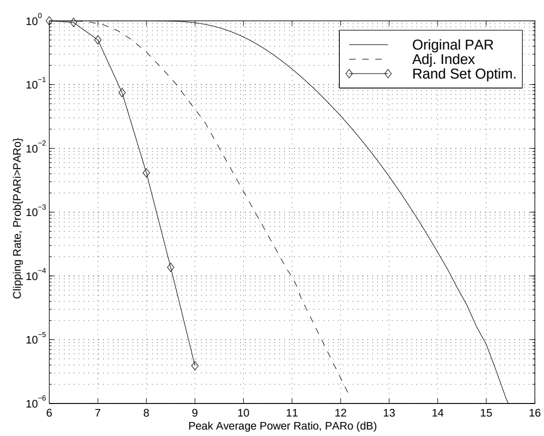

Fig. 2. *Probability that the PAR of a randomly generated 256-carrier baseband DMT/OFDM symbol exceeds* PAP_0 *for* $\frac{L}{N} = 5\%$ *for two PRT index set choices: contiguous set and randomly optimized set.*

B. Tone Reservation Results

Fig. 3 plots the PAR reduction from applying the low complexity, iterative algorithm described in Section A, for the case $L/N = 5\%$. The index set was chosen using the random set optimization procedure. This figure plots from right to left, the CCDF of $PAR(\mathbf{x} + \mathbf{c}^{(i)})$, $i = 0, 1, \ldots, 10, 40, \infty$. For a desired clipping probability of 10^{-7}, the PAR reduction is $2dB$ after one iteration, $3dB$ after two iterations, and $4.5dB$ after 6 iterations. The transmit power increase after 10 iterations, is $0.13dB$. This value is smaller than its counterpart in Section III, where $\mathbf{c}^\star$ was computed exactly.

For channels with very different values for $\{b_k\}$, we can get a very low DRL, that is $DRL << PRTF << 1$. If the $\{b_k\}$ values are very similar, then the $DRL \approx PRTF$ and will no longer be negligible. If our application can accommodate variable rate transmission, we can choose to use the PRT set $\{i_k\}$ for peak reduction purposes only if the $PAR(\mathbf{x})$ is above some desired value, otherwise, we can use these tones for data transmission. In this case:

$$DRL = Prob\{PAR(\mathbf{x}) > PAR_{desired}\} \frac{\sum_{k=1}^{L} b_{i_k}}{\sum_{k=0}^{N-1} b_k} \quad (15)$$

If we assume $b_k = const$ (worst case), then $DRL = PRTF = \frac{L}{N} Prob\{PAR_i > PAR_{desired}\}$. For example, if our target is $PAR_{desired} < 12dB$ with a clipping probability of less than 10^{-7}, from Fig. 3, we see that we can achieve this by correcting only 3% of DMT symbols after a maximum of 4 iterations of our algorithm. The DRL is $DRL = 5\% \times 3\% = .15\%$. One drawback of this technique is that the data rate will no longer be constant.

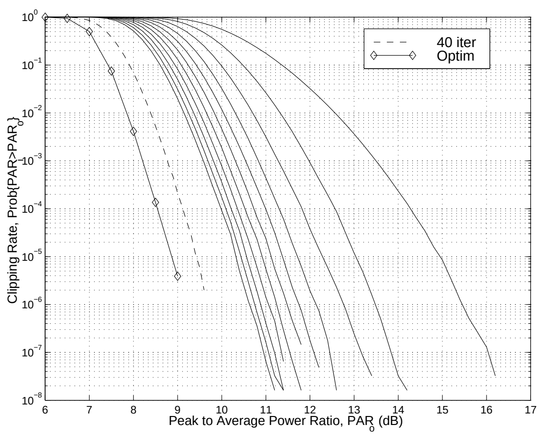

Fig. 3. $PAR(\mathbf{x} + \mathbf{c}^{(i)})$ *distribution for* $i = 0, \ldots, 10, 40, \infty$ *and* $\frac{L}{N} = 5\%$ *with Randomly Optimized set* $\{i_k^\star\}$

IV. PAR Reduction by Tone Injection

If X_k carries b_k bits, then it could take one of 2^{b_k} discrete values. Let's assume for simplicity a square QAM constellation and denote the minimum distance between constellation points d_k. Then, the real part of X_k, R_k and the imaginary part, I_k can take values $\{\pm d_k/2, \pm 3d_k/2, \ldots, \pm(M_k - 1)d_k/2\}$, where $M_k = 2^{b_k/2}$ is the number of levels per dimension. Fig. 4 shows a $16QAM$ constellation, for which $b_k = 4$, and $M_k = 4$.

Let's assume that $X_k = A = d_k/2 + j3d_k/2$. Modifying the real and/or imaginary part of A, could reduce the PAR of the transmitted vector. However, if we want the receiver to decode A correctly without sending any side information, we must change A by an amount that can be estimated at the receiver. A simple special case would be to transmit $\hat{A} = A + pD + jqD$ where p and q are any integer values and D is a positive real number known at the receiver. For

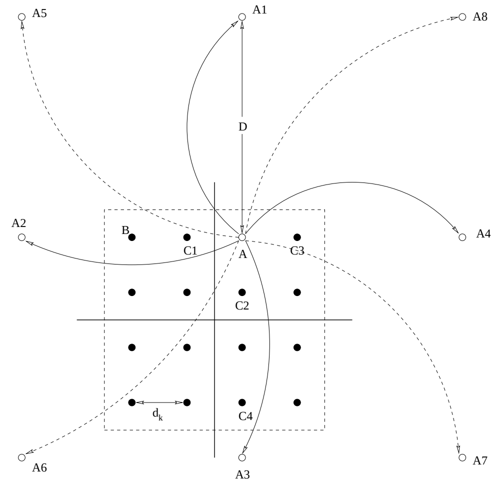

Fig. 4. *The constellation value A is the minimum energy point of the equivalent set* $A_i = A + p_i D + jq_i D$

example, the choice $p = -1$ and $q = 0$ will map A into $\hat{A} = A2$.

If we do not want to increase the complexity of the receiver we must choose the value of D carefully. For example, if $D_k = d_k M_k/2$, $A2$ would overlap with B, and the receiver would decode $\hat{A}$ erroneously. Other choices, such as $D_k = d_k M_k/2 + d_k/2$ would not overlap points in the transmitted constellation, but the minimum distance between possible transmit points would be only $d_k/2$ instead of the original d_k. On the other hand, if we choose $D \geq d_k M_k$ then the probability of decoding $\hat{A}$ erroneously for an uncoded system is roughly the same as the probability of decoding A erroneously, and therefore the Symbol Error Rate (SER) will not change. For the special case where $D = d_k M_k$, the generalized constellation becomes a lattice. We have therefore expanded the original 16QAM constellation to a bigger constellation where we can choose one of several values that carry the same information. These extra degrees of freedom can be used to generate DMT symbols with lower PAR[3].

From Fig. 4 it is clear that whenever $p \neq 0$ or $q \neq 0$, $\hat{A}$ has more energy than A, and therefore the new transmit vector will have more power. To minimize the power increase, we should choose lower power equivalents whenever possible. The importance of reducing this power increase is twofold. First, any average power increase results in a reduction in SNR margin. Second, unnecessary power increases can lead to higher secondary peaks, which will complicate iterative algorithms for computing $\hat{X}_k$.

FurthermoreGiven these more general constellations, we can now choose the transmitted vector from the following set:

$$\hat{x}_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (X_k + p_k D_k + jq_k D_k) e^{j2\pi kn/N}, \quad (16)$$

where p_k, q_k are integers and $D_k = \rho d_k M_k$, $\rho \geq 1$. Since

[3]This formulation was first presented in [12]. Independently, Motorola ISG [13] has developed a related method based on the lattice expansion of the standard constellation.

$D_k \geq d_k M_k$, the receiver can decode X_k from $\hat{X}_k$ by performing a $modulo - D_k$ operation. Moreover the uncoded SER will not increase. For trellis coded systems we can prevent BER increases by choosing an appropriate $\rho > 1$ or by taking the modulo of the symbol *after* the coset has been decoded [12], [11].

A. Maximum PAR reduction per D-shift

The following derivations and results will focus on baseband multicarrier systems, but these results can be easily replicated for the passband case. This section quantifies the maximum PAR reduction this algorithm can achieve per D-shift. Next, we describe a simple algorithm that can achieve most of this promised PAR reduction. There are many improvements/variations of these algorithms that tradeoff between complexity, PAR reduction and power increase.

For the baseband case, assuming N is even, the transmitter IFFT output vector is[4]:

$$x[n] = \frac{2}{\sqrt{N}} \sum_{k=1}^{N/2-1} \left[R_k cos(2\pi kn/N) - I_k sin(2\pi kn/N) \right],$$

where $X_k = R_k + jI_k$. Let's assume the location of the maximum value of $|x[n]|$ is n_0. , w.l.o.g, we assume that $x_{n_0} > 0$ and that $cos(2\pi k_0 n_0/N) = l > 0$ for some frequency bin k_0. If we subtract D from R_{k_0}, the new transmit DMT symbol vector $\hat{x}[n]$ can be computed without repeating the IFFT because the algorithm has only modified one tone. The new symbol is simply

$$\hat{x}[n] = x[n] - \frac{2D}{\sqrt{N}} cos(2\pi k_0 n/N), \quad 0 \leq n < N, \qquad (17)$$

After replacing R_{k_0} for $R_{k_0} - D$, the reduced peak at $\hat{x}_{n_0}$ will satisfy the relationship:

$$\hat{x}_{n_0} = x_{n_0} - \frac{2lD}{\sqrt{N}} \geq x_{n_0} - \frac{2D}{\sqrt{N}} \qquad (18)$$

If all tones have unit energy, the maximum peak reduction of sample x_{n_0} that can be achieved, after replacing one dimension, occurs when $l = 1$:

$$\hat{x}_{n_0} \geq x_{n_0} - \sqrt{\frac{6M_{k_0}^2}{M_{k_0}^2 - 1}} \frac{2\rho}{\sqrt{N}} \qquad (19)$$

and the new DMT symbol will satisfy:

$$max|\hat{x}[n]| \geq max|x[n]| - \sqrt{\frac{6M_{k_0}^2}{M_{k_0}^2 - 1}} \frac{2\rho}{\sqrt{N}} \qquad (20)$$

A similar argument would follow for all other permutations. If $cos(2\pi k_0 n_0/N) = l < 0$ we must substitute R_{k_0} for $R_{k_0} + D$. These ideas can be also extended to the $I_k sin(2\pi k n_0/N)$ terms. In general, the single dimension *D-shift* update, Eq. (17) can take any of 4 options:

$$\hat{x}[n] = x[n] \pm \frac{2D}{\sqrt{N}} \{cos, sin\}(2\pi k_0 n/N), \qquad (21)$$

which justifies the term *Tone Injection* for this method. From (20) the maximum peak reduction per tone shift is:

$$\delta = \sqrt{\frac{6M_{k_0}^2}{M_{k_0}^2 - 1}} \frac{2\rho}{\sqrt{N}} \qquad (22)$$

If we apply this procedure on K real/imaginary dimensions simultaneously, the maximum peak reduction is $K\delta$. Since the peak reduction factor δ decreases as N increases, we must increase K to achieve a fixed PAR reduction for larger multicarrier symbols. Thus, for larger values of N increasing ρ is a good choice. For example, if $N = 512$, $M_k = 4$, $\forall k$, and the original symbol worst case PAR is 15.5 dB, a $5dB$ PAR reduction would require $K = 12$ D-shifts for $\rho = 1$, but only $K = 8$ if $\rho = 1.5$.

B. Algorithms for computing $\hat{X}_k$ and $\hat{x}[n]$

The previous section provided upper bounds on the PAR reduction that is possible from these generalized constellation methods. Here, we will describe some simple iterative algorithms that are close to achieving these upper bounds with small transmit power increase and low complexity. Finding the values of p_k and q_k that produce the lowest PAR for $\hat{x}[n]$ requires solving an integer programming problem, which has exponential complexity. Assuming L duplicate points per constellation, if K dimensions are to be modified we must search over all

$$\binom{N}{K} L^K \approx \frac{N^K}{K!} L^K \approx (NL)^K \qquad (23)$$

combinations for the vectors $\{p_k\}$ and $\{q_k\}$. From the argument above, a peak reduction of amplitude Δ requires a minimum of $K = \gamma\sqrt{N}$ dimension shifts, where γ is a function of M_k and ρ as given by Eq. (22) and the desired peak reduction Δ. For a PAR reduction of more than $5dB$, $M \geq 4$ and $c = 0$, we have $\gamma \approx 1/2$. Therefore, the number of combinations required for a fixed PAR reduction is lower bounded by:

$$\binom{N}{\gamma\sqrt{N}} L^{\gamma\sqrt{N}} \approx (NL)^{\gamma\sqrt{N}} \qquad (24)$$

which explains the exponential complexity of the optimal solution. Fortunately, good approximations to the optimal solution are possible with low complexity with an iterative algorithm. Instead of searching over all permutations for $\{p_k\}$ and $\{q_k\}$ we will search for a single tone D-shift that achieves good (best) PAR reduction, i.e. as close to δ as possible, and update $x[n]$ to $\hat{x}^{(1)}[n]$. A new single tone search can be repeated on the updated symbol. Two heuristics are used in the tone search, first, since a nonzero value for p_k or q_k increases the symbol energy, choosing the equivalent

[4] The DC term (R_0) and Nyquist term (I_0) have set to zero to simplify the discussion

points that reduce the power increase is important to facilitate subsequent iterations. For example if all tones transmit 16QAM constellations, by choosing the terms where $|R_k| = 3d_k/2$ or $|I_k| = 3d_k/2$, we minimize the transmitter power increase by selecting $sign(p_k) = -sign(R_k)$ or $sign(q_k) = -sign(I_k)$. Second, by choosing the tones where the sinusoid value, l in (18) is large at the peak locations, we get larger PAR reductions per step. Similarly, larger values of D will cancel the peak faster but at the expense of larger power increases.

The algorithm starts with the original DMT symbol ($p_k = 0, q_k = 0, \forall k$). After finding the location of the maximum magnitude is n_0, we find a tone k_0 such that either $|R_{k_0}|$ or $|I_{k_0}|$ is large, and such that the sinusoid is of the opposite sign and close to one in magnitude at n_0. After the desired tone, k_0, is found, $\hat{x}[n]$ is updated using (21). If more than one value of $x[n]$ is above the target PAR, we must find a tone k_0 that reduces as many peaks as possible. This procedure can be repeated several times until the desired PAR is achieved or the maximum number of iterations or maximum transmit power increase has been reached.

Fig. 5 plots the PAR symbol CCDF of 6 iterations of the described algorithm. The DFT has size 256 with 16QAM in each tone and $\rho = 1.5$. The proposed algorithm reduces PAR by more than $5dB$ at a clipping rate of 10^{-6}. This is less than $.5dB$ away from the theoretical upper bound on PAR reduction of 6δ. The average power increase is only 1.6%.

Larger reduction in PAR is possible by increasing the number of iterations or by using more complex algorithms. Other results not included here have reduced PAR up to $7dB$.

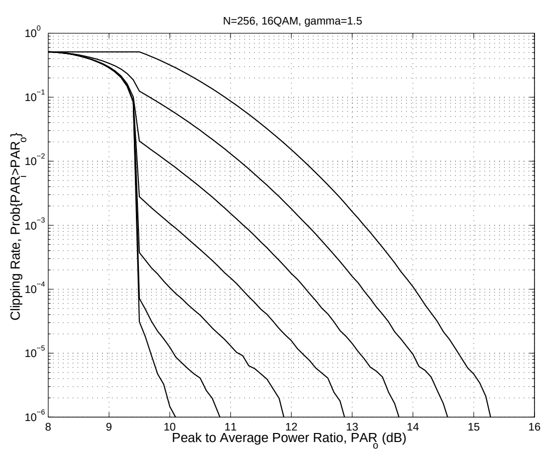

Fig. 5.

V. Conclusions

Two efficient and distortionless PAR reduction algorithms are proposed for reducing PAR of multicarrier signals. These methods are based on adding a symbol dependent time domain signal to the original DMT/OFDM symbol to reduce its peaks. This time domain signal can be computed efficiently at the transmitter and can be easily stripped off the received signal. For the first method, called Tone Reservation (TR), the transmitter does not send data on a small subset of carriers, which are optimized for PAR reduction. This technique can achieve $3dB$ of PAR reduction for a loss in data rate of less than .2%, or $6dB$ reduction for less than 5% rate loss. The added signal is subtracted at the receiver end by ignoring the PRT values. The second scheme, called Tone Injection (TI), is based on replicating the subsymbol constellation with known translating vectors. In this expanded constellation, several points represent the same information. This helps reshape the original time domain DMT symbol and achieves more than $6dB$ PAR reduction without data rate loss and with negligible increase in transmitter average power. These generalized constellations can be easily mapped into the original constellation with a modulo operation at the receiver.

These two methods described here can also be combined to obtain further reductions in PAR. Moreover these techniques, especially TI, can be applied to reduce the PAR of single carrier signals whenever the transmit filters are relatively long.

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