



STANFORD

Lecture 8

Vector MACs & Broadcast Precoders

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Announcements & Agenda

- Announcements
 - Problem Set #4 due next Tuesday.
 - Midterm in class Wednesday (April 29)

- Agenda (L8)
 - More MAC examples
 - `mu_mac.m`
 - Simultaneous Water-Filling for MAC max rate sum
 - `SWF.m` and `macmax.m`
 - **MAC**: Capacity region for frequency-indexed MACs
 - **BC**: Precoder Basics for the Matrix AWGN
 - Scalar Gaussian BC



MAC Examples

Sections 2.7.3-4

Matrix AWGN MAC Example 1 ($L_{x,u} \equiv \mathbf{1}$)

$$\tilde{H} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$$

```
>> Rf=H*H =
```

```
34 13
```

```
13 5
```

```
>> Rbinv=Rf+eye(2) =
```

```
35 13
```

```
13 6
```

```
>> Gbar=chol(Rbinv) =
```

```
5.9161 2.1974
```

```
0 1.0823
```

```
>> S0=diag(diag(Gbar))*diag(diag(Gbar)) =
```

```
35.0000 0
```

```
0 1.1714
```

```
>> G = inv(diag(diag(Gbar)))*Gbar =
```

```
1.0000 0.3714
```

```
0 1.0000
```

```
>>> b=0.5*log2(diag(S0)) =
```

```
2.5646
```

```
0.1141
```

```
>> sum(b) = 2.6788
```

REVERSE ORDER - same commands -other vertex

```
>> H=[ 2 5
```

```
1 3];
```

```
Rbinv =
```

```
6 13
```

```
13 35
```

```
Gbar =
```

```
2.4495 5.3072
```

```
0 2.6141
```

```
S0 =
```

```
6.0000 0
```

```
0 6.8333
```

```
G =
```

```
1.0000 2.1667
```

```
0 1.0000
```

```
b =
```

```
1.2925
```

```
1.3863
```

```
sum(b) = 2.6788
```

- These are the two vertices for dimension-share (pentagon outer face).
- Two receiver output dimensions for each one-dimensional input x_u (instead of 1 output dimension earlier)



Example 1 continued

- Receiver filters and bias are

Vertex 1

```
>> W=inv(S0)*inv(G') =
```

```
0.0286    0  
-0.3171  0.8537
```

```
>> Wunb=S0*inv(S0-eye(2))*W =
```

```
0.0294    0  
-2.1667  5.8333
```

```
>> MSWMFu=Wunb*H' =
```

```
0.1471  0.0882  
0.8333 -0.6667
```

```
>> Gunb=eye(2)+S0*inv(S0-eye(2))*(G-eye(2)) =
```

```
1.0000  0.3824  
0        1.0000
```

```
>> MSWMFu*H =
```

```
1.0000  0.3824  
2.1667  1.0000
```

Vertex 2

```
>> W=inv(S0)*inv(G') =
```

```
0.1667    0  
-0.3171  0.1463
```

```
>> Wunb=S0*inv(S0-eye(2))*W =
```

```
0.2000    0  
-0.3714  0.1714
```

```
>> MSWMFu=Wunb*H' =
```

```
0.4000  0.2000  
0.1143  0.1429
```

```
>> Gunb=eye(2)+S0*inv(S0-eye(2))*(G-eye(2)) =
```

```
1.0000  2.6000  
0        1.0000
```

```
>> MSWMFu*H =
```

```
1.0000  2.6000  
0.3714  1.0000
```

- Not really triangular, why?



Easier with mu_mac.m

```
function [b, GU, WU, S0, MSWMFU] = mu_mac(H, A, Lxu, cb)
```

channel Rxx1/2 \ 1 cplx, 2 real
#/user xmit
antennas

Per-tonal (temporal dimension) multiuser mac receiver and per-user bits

Inputs: H, A , Uind , cb

Outputs: b, GU, WU, S0, MSWMFU

H: noise-whitened channel matrix [HU ... H1] Ly x sum-Lxu

A: Block Diag sq-root sum-Lxu x sum-Lxu discrete modulators,
blkdiag([AU ... A1]); The Au entries derive from each MAC user's
Lxu x Lxu input autocorrelation matrix, where the trace is user
u's energy/symbol. This is per-tone.

Lxu: # of dimensions for each user U ... 1 in 1 x U row vector

cb: = 1 if complex baseband or 2 if real baseband channel

GU: unbiased feedback matrix sum-Lxu x sum-Lxu

WU: unbiased feedforward linear equalizer sum-Lxu x sum-Lxu

S0: sub-channel channel gains sum-Lxu x sum-Lxu

MSWMFU: unbiased mean-squared whitened matched filter, sum-Lxu x Ly
b - user u's bits/symbol 1 x U

the user should recompute b if there is a cyclic prefix

```
H=[5 2 ; 3 1];
```

```
[b, GU, WU, S0, MSWMFU] = mu_mac(H, eye(2), [1 1] , 2);
```

```
b = 2.5646 0.1141
```

```
GU =
```

```
1.0000 0.3824  
0 1.0000
```

```
WU =
```

```
0.0294 0  
-2.1667 5.8333
```

```
S0 =
```

```
35.0000 0  
0 1.1714
```

```
MSWMFU =
```

```
0.1471 0.0882  
0.8333 -0.6667
```

```
>> MSWMFU*H =
```

```
1.0000 0.3824  
2.1667 1.0000
```

```
>> SNR = 10*log10(diag(S0)) =
```

```
15.4407  
0.6872
```

```
>> sum(b) = 2.6788
```



Example 2: 2 x 3 MAC (secondary users)

```
H=[5 2 1
3 1 1]; basically added a 3rd user
[b, GU, WU, S0, MSWMFU] = mu_mac(H, eye(3), [1 1 1], 2)
```

```
b = 2.5646 0.1141 0.1137
GU = 1.0000 0.3824 0.2353
      0 1.0000 0.1667
      0 0 1.0000
```

```
WU =
0.0294 0 0
-2.1667 5.8333 0
-1.2857 -0.1429 5.8571
```

```
S0 =
35.0000 0 0
0 1.1714 0
0 0 1.1707
```

```
MSWMFU =
0.1471 0.0882
0.8333 -0.6667
-0.8571 1.8571
```

```
>> sum(b) = 2.7925
```

```
>> MSWMFU*H=
```

```
1.0000 0.3824 0.2353
2.1667 1.0000 0.1667
1.2857 0.1429 1.0000
```

```
>> SNR10*log10(diag(S0))=
15.4407
0.6872
0.6846
```

- The channel rank is 2 so at least 1 secondary comp = 3-2.
- But secondary applies to energy-sum MAC (which this is not, yet).
- If original 2 units of energy is spread over 3 users?

```
>> [b, GU, WU, S0, MSWMFU] = mu_mac(H, (2/3)*eye(3), [1 1 1], 2)
```

```
b = 2.0050 0.1009 0.0696
GU =
1.0000 0.3824 0.2353
0 1.0000 0.3878
0 0 1.0000
```

```
WU =
0.0662 0 0
-2.3878 6.6582 0
-2.0000 -0.5000 9.8750
```

```
S0 =
16.1111 0 0
0 1.1502 0
0 0 1.1013
```

```
MSWMFU =
0.2206 0.1324
0.9184 -0.3367
-0.7500 2.2500
```

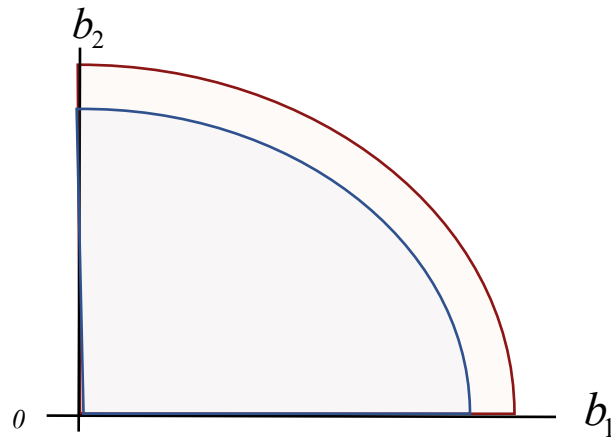
```
>> sum(b) = 2.1755 (lower than 2x2 value of 2.6788)
```

- Relatively more energy on secondary-user comp(s), bsum ↓.



Non-Zero Gap Achievable Region

- Construct $\mathcal{C}(\mathbf{b})$ with $\Gamma = 0$ dB.
- Reduce all rates by γ_b relative to boundary points.
- Inscribe smaller region $\mathcal{C}(\mathbf{b}) - (\gamma_b \odot \mathbf{1})$.
- Square constellations instead of spheres (AWGN) loss 1.53 dB in gap above (0.25 bit/dimension).



Simultaneous Water-Filling for MAC max rate sum

Sections 2.7.3-4

Revisit the rate-sum mutual information

$$b = \sum_{u=1}^U \tilde{b}_u \leq \mathbb{I}(\mathbf{x}; \mathbf{y}) = \log_2 \frac{|H \cdot R_{xx} \cdot H^* + R_{nn}|}{|R_{nn}|}$$

- Maximum rate-sum focuses on the numerator,
 - when optimizing over R_{xx} .

$$\max_{\{R_{xx}(u)\}} \left| H_u \cdot R_{xx}(u) \cdot H_u^* + \underbrace{\sum_{i \neq u} H_i \cdot R_{xx}(i) \cdot H_i^* + R_{nn}}_{R_{noise}(u)} \right|$$

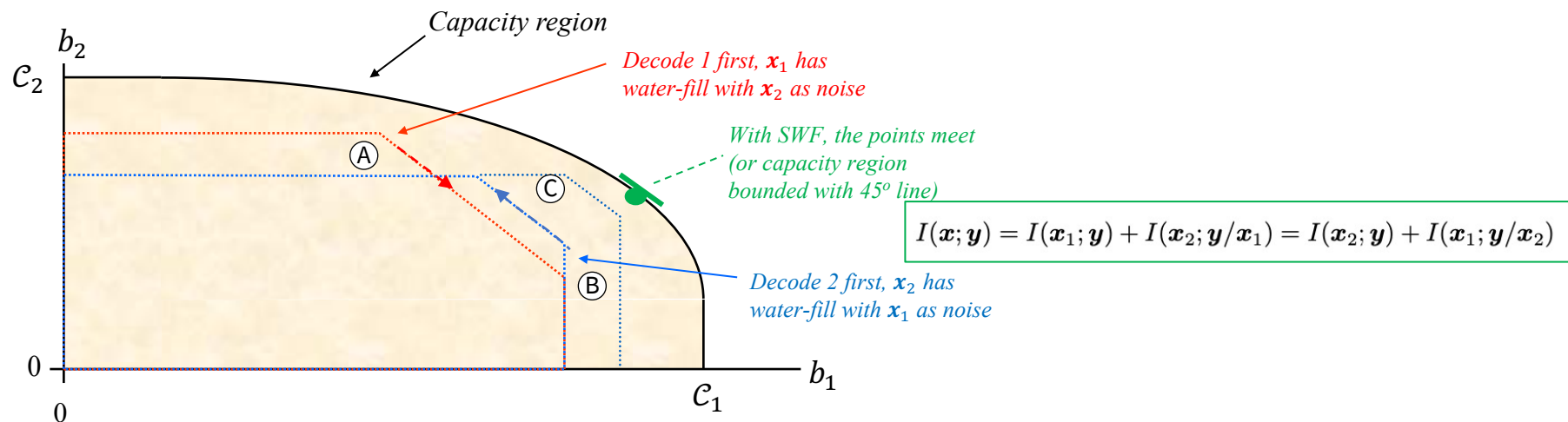
- Have we seen this problem before?
 - Yes, it is Vector Coding / Waterfilling, except with $\tilde{H}_u \rightarrow R_{noise}^{-1/2} \cdot H_u$ for each u
- But now it repeats U times in the same form for each user.
 - Optimum has each user **simultaneously water-fill** by treating all other users (water-fill) spectra as noise.

Simultaneous Waterfilling

$$\begin{aligned} \mathcal{E}_{u,l} + \frac{1}{g_{u,l}} &= K_u \quad \forall u = 1, \dots, U' \\ \sum_{l=1}^{L_x} \mathcal{E}_{u,l} &= \mathcal{E}_u \\ \mathcal{E}_{u,l} &\geq 0 \\ \mathbf{x}_u &= M_u \cdot \mathbf{v}_u \end{aligned}$$



Compute Using Iterative Water-filling



- SWC problem is convex, and each single-water-fill step is “gradient-like” in improving direction, swf.m
- E-Sum SWC is a saddle point with enlarged region.
 - 2nd optimization is on the allocation of $\mathcal{E}_{x,u} \rightarrow \sum_{u=1}^U \mathcal{E}_{x,u} = \mathcal{E}_x$.



SWF.m Program for MAC's max sum rate

```
function [Rxx, bsum , bsum_lin] = SWF(Eu, H, Lxu, Rnn, cb)
```

Simultaneous water-filling MAC max rate sum (linear and nonlinear GDFE)

The input is space-time domain h , and the user can specify a temporal block symbol size N (essentially an FFT size).

Inputs:

Eu $U \times 1$ energy/SAMPLE vector. Single scalar equal energy all users
any $(N/N+nu)$ scaling should occur BEFORE input to this program.

H The **FREQUENCY-DOMAIN** $L_y \times \text{sum}(L_x(u)) \times N$ MIMO channel for all users.

N is determined from size(H) where $N = \#$ used tones

Lxu $1 \times U$ vector of each user's number of antennas

Rnn The $L_y \times L_y \times N$ noise-autocorrelation tensor (last index is **per tone**)

cb cb = 1 for complex, cb=2 for real baseband

cb=2 corresponds to a frequency range at an sampling rate $1/T'$ of $[0, 1/2T']$ while with cb=1, it is $[0, 1/T']$. The Rnn entered for these two situations may differ, depending on how H is computed.

Outputs:

Rxx A block-diagonal psd matrix with the input autocorrelation for each user on each tone. Rxx has size $(\text{sum}(L_x(u)) \times \text{sum}(L_x(u)) \times N$.

sum trace(Rxx) over tones and spatial dimensions equal the Eu

bsum the maximum rate sum.

bsum bsum_lin - the maximum sum rate with a linear receiver

b is an internal convergence sum rate value, not output

This program significantly modifies one originally supplied by student
Chris Baca

- Eu is each user's energy/**sample**.
- For now, $N = 1$, so time/freq are same:
 - $H=h$.
- Lxu is number of antennas for each user.
- Separate specification of Rnn removes need for noise whitening.
- cb=1 for complex, =2 for real.



Revisit Previous example (slides L8: 26-29)

```
H =  
 5 2 1  
 3 1 1  
>> [Rxx, bsum, bsum_lin] = SWF([1 1 1], H, [1 1 1], eye(2), 2)  
Rxx =  
 1 0 0  
 0 1 0  
 0 0 1  
bsum = 2.7925  
bsum_lin = 1.4349
```

- Same result as L8:7, so each user waterfills with all others as noise; this is trivial when each user has only 1 input dimension. (Why?)
- This is for input energy-vector constraint.
- Note linear solution (no feedback, so matrix MMSE-LE) loses roughly $\frac{1}{2}$ the data rate.
- SWF becomes more interesting when $N > 1$ tones or if $L_{x,u} > 1$ antennas.

For $L_{x,u} = 2$; $u = 1,2$?

```
>> H2=[4 3 2 1  
5 6 7 8];  
>> [Rxx, bsum, bsum_lin] = SWF([0.5 0.5], H2, [2 2], eye(2), 2)  
Rxx =  
 0.7121 0.4528 0 0  
 0.4528 0.2879 0 0  
 0 0 0.2876 0.4527  
 0 0 0.4527 0.7124  
bsum = 5.3434  
bsum_lin = 4.0920  
>> trace(Rxx) % = 2 (check)  
>> trace(Rxx(1:2,1:2)) % = 1  
>> trace(Rxx(3:4,3:4)) % = 1
```

**Energy input is per
trace{Rxx,u}
per sample!**

- Note block-diagonal Rxx.
- Linear-only loses about 25% in data rate (for this channel).



Or use Macmax.m for Esum MAC

```
function [Rxx, bsum , bsum_lin] = macmax(Eu, h, Lxu, N , cb)
```

Simultaneous water-filling Esum MAC max rate sum (linear & nonlinear GDFE)
The input is space-time domain h , and the user can specify a temporal block symbol size N (essentially an FFT size).

This program uses the CVX package

the inputs are:

E_u The sum-user energy/SAMPLE scalar.

This will be increased by the number of tones N by this program.

Each user energy should be scaled by $N/(N+nu)$ if there is cyclic prefix

This energy is the sum trace of the corresponding users' $R_{xx}(u)$.

The sum energy is computed as the sum of the E_u components internally.

h The **TIME-DOMAIN** $L_y \times \sum(L_x(u)) \times N$ channel for all users

L_{xu} The number of antennas for each user $1 \times U$

N The number of used tones (equally spaced over $(0,1/T)$ at N/T .

cb $cb = 1$ for complex, $cb=2$ for real baseband

the outputs are:

R_{xx} A block-diagonal psd matrix with the input autocorrelation for each user on each tone. R_{xx} has size $(\sum(L_x(u)) \times \sum(L_x(u)) \times N$.

sum trace(R_{xx}) over tones and spatial dimensions equal the E_u
 $bsum$ the maximum rate sum.

$bsum_lin$ - the maximum sum rate with a linear receiver

b is an internal convergence (vector, rms) value, but not sum rate

- ENERGY-SUM input (per sample)
 - L_{xu} = numbers of xmit antennas/user
- Time-domain (noise-whitened) h
- This is actually a double loop that:
 - water-fills each and every user for some current set of per-user energies and
 - adjusts energies so they sum to total but increase the rate sum.
- It corresponds to a saddle point.
 - It is not convex (although each sub loop is).
 - It has a solution and converges anyway.
- This will be easier understood later as a dual of a broadcast problem as to why this is true.



Back to Example

```
>> H3(:,:,1)=H  
  
H =  
 5  2  1  
 3  1  1  
>> [Rxx, bmacmax, bmaclin]=macmax(3/2, H, [1 1 1], 1, 2)  
Rxx =  
 3.0000  0  0  
 0  0.0000  0  
 0  0  0.0000  
bmacmax = 3.3432  
bmaclin = 3.3432
```

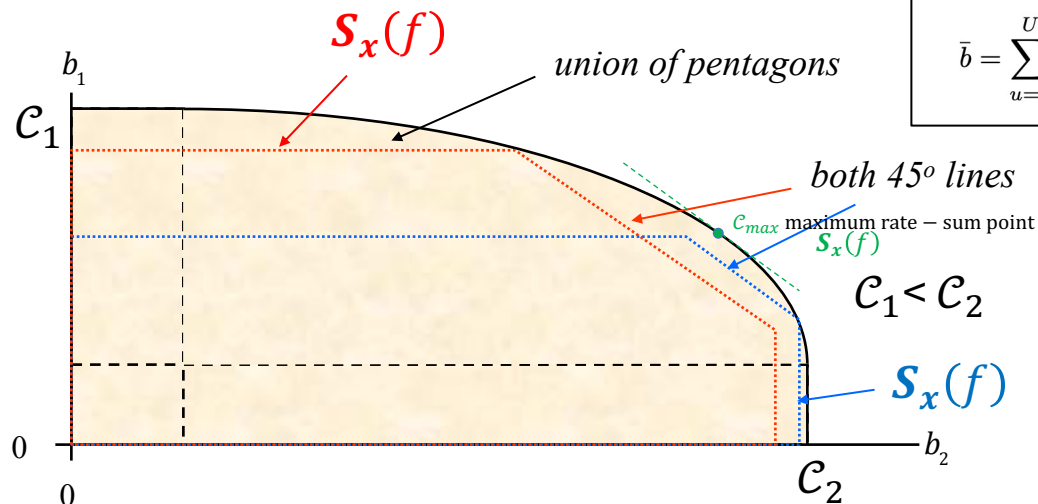
- This produces a larger data rate because there is less energy restriction.
- Rxx energizes just user 3! (It's all primary user component, and users 1 and 2 are secondary)
- Linear is the same. Why?



Capacity region for frequency-indexed MACs

Sections 2.7.4.1-2

$\mathcal{C}(b)$ is union of $\mathcal{S}_x(f)$ -indexed Pentagons



$$\bar{b} = \sum_{u=1}^U \bar{b}_u \leq \bar{I}(\mathbf{x}; \mathbf{y}) = \int_{-\infty}^{\infty} \frac{1}{2} \cdot \log_2 \left[1 + \frac{\sum_{u=1}^U S_{x,u}(f) \cdot |H_u(f)|^2}{S_n(f)} \right] df$$

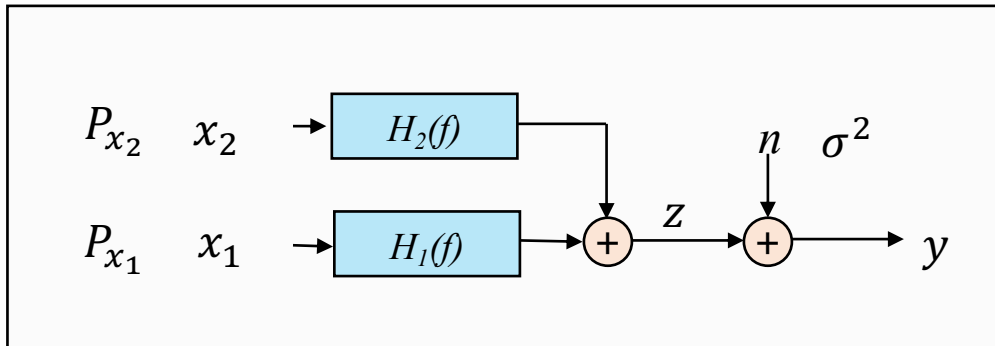
$$S_{x,u}(f) + \frac{\sigma^2 + \sum_{i \neq u} S_{x,i}(f) + |H_i(f)|^2}{|H_u(f)|^2} = K_u$$

Simultaneous water-filling
 → Maximum rate sum

- Each pentagon corresponds to an $\mathcal{S}_x(f)$ choice.
 - The pentagons become triangles for the sum-energy MAC.
- The union (convex hull is union when inputs are Gaussian) can dimension-share in frequency as $N \rightarrow \infty$.



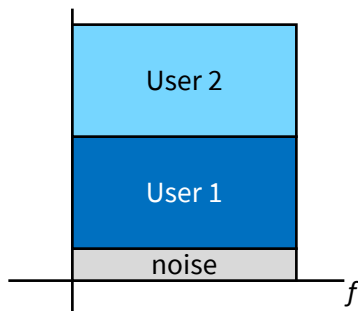
MT MAC



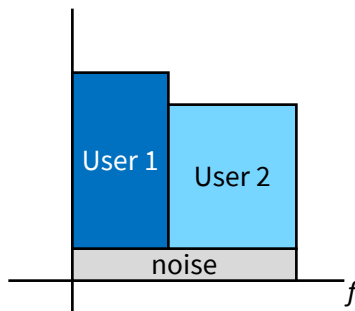
- The users have continuous-time/frequency channels \rightarrow use MT on each, theoretically.
- This really means dimensionality is infinite (or very large) so “dimension-sharing” may be inherent.
- SWF applies, but with some interpretation (like power instead of energy and power per dimension instead of power-spectral density, etc.).



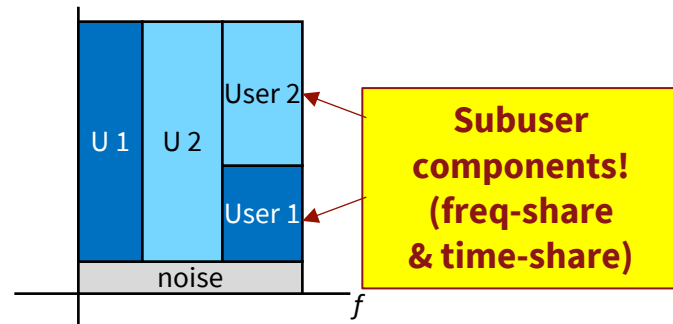
Decoders and SWF



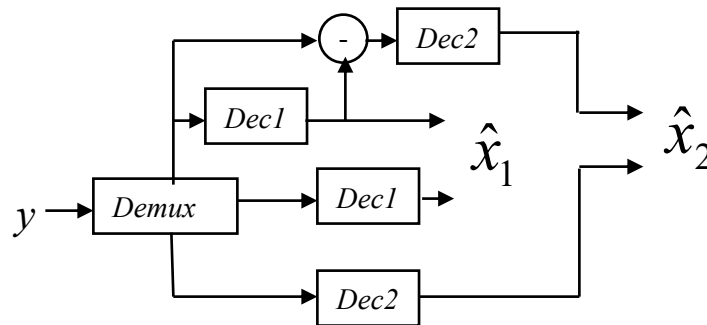
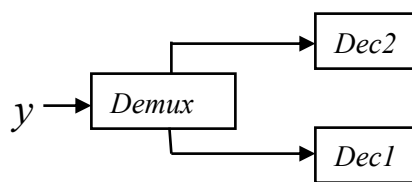
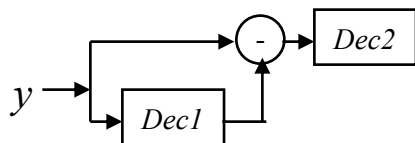
a). both flat



b). FDM



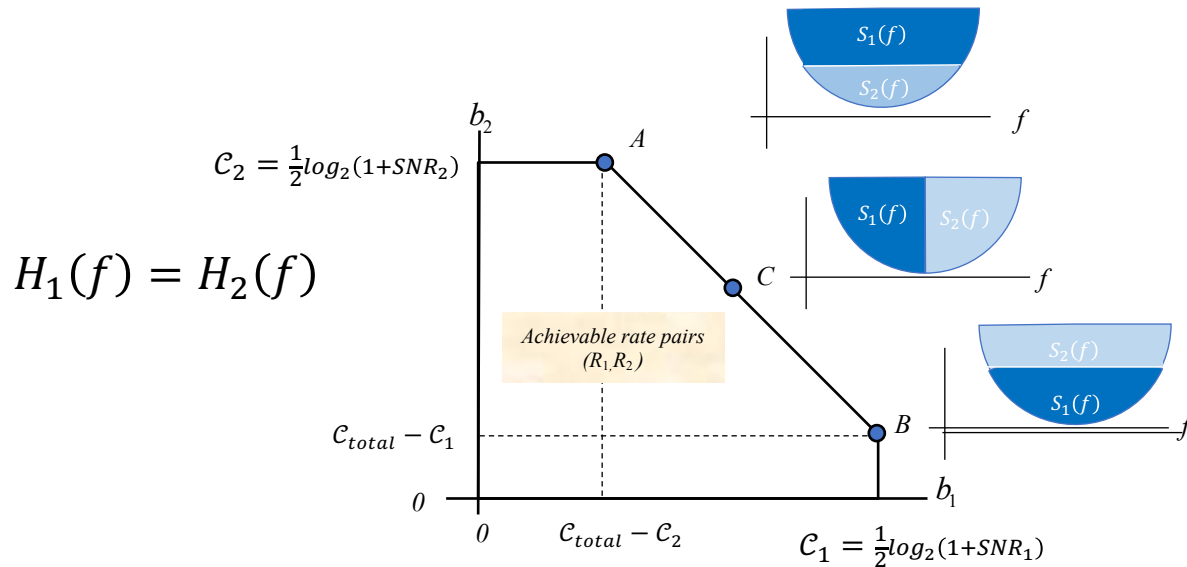
c). Mixed – sub users



- FDM is clearly simplest decoder for max rate-sum case.
- Both users (and all components in case c) are primary.



Symmetric 2-user channel and SWF



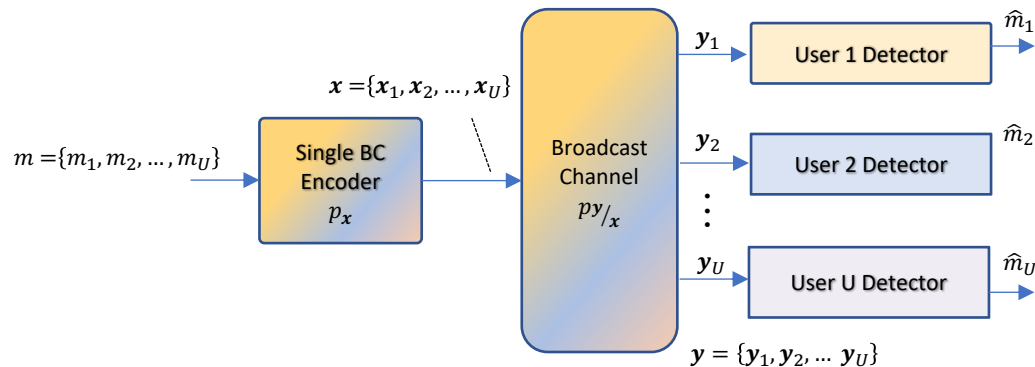
- Symmetric means $H_1(f) = H_2(f)$ (noise is one-dimensional and added to sum)
- Each of points A, B, and C have different SWF spectra – all have same (max) rate sum



Basic Precoders and the Matrix AWGN

PS5.1 - 2.28 modulo precoding function

Broadcast Channel (BC)

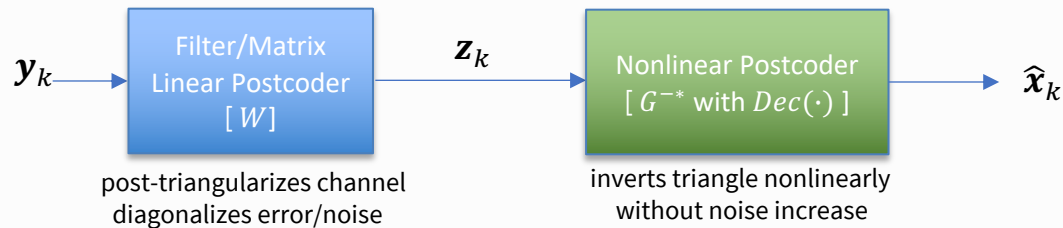


- The BC is the “Dual” of the MAC.
- Receivers are in different places and so cannot “co-process” $\{y_u\}$.
- Transmitter can co-encode/generate x , although input messages remain independent.
 - Who encodes first? (may be at disadvantage)
 - Who encodes last? (knowing other users’ signals is an advantage)
 - What then is the **order**?

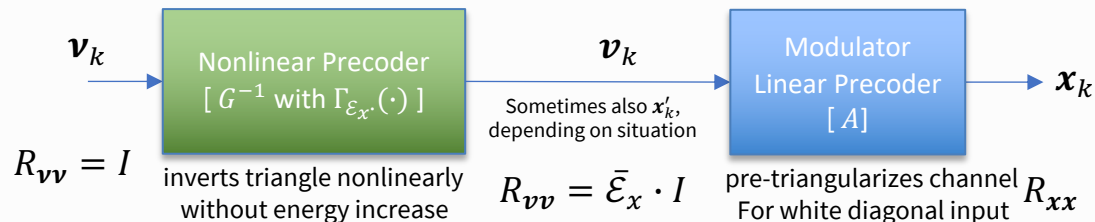


BC is “reversed” MAC

MAC receiver



BC transmitter



- The MAC’s uncoordinated user input is a kind of “worst case” transmitter, reducing data rate.
 - With only an energy-sum constraint, these worst-case inputs’ users best pass as primary user components; secondary components “freeload” on the primary’s passage.
- The BC similarly will effectively correspond to a worst-case noise for which receiver coordination is useless, reducing data rate.
 - With worst-case noise, the channel best passes the primary components’; secondary components freeload on the primary’s passage.

Triangular Matrices - Innovations and Prediction

- Prediction for some **user order** separates a modulated input to independent message components.

$$\mathbf{v}_u = \mathbf{x}_u - \hat{\mathbf{x}}_u / \{x_{u+1} \dots x_U\}$$

Innovations or predictions, but for BC \mathbf{v}_u become the independent-users' subsymbols, with normalization $R_{\mathbf{v}\mathbf{v}}(u) = I$.

- This is a triangular relationship (**inverse of upper triangular is also upper triangular**).

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_U \end{bmatrix} = \begin{bmatrix} 1 & g_{1,2} & \dots & g_{1,U} \\ 0 & 1 & \dots & g_{2,U} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_U \end{bmatrix} = G^{-1} \cdot \mathbf{x}$$

- OR, $\mathbf{x} = G \cdot \mathbf{v}$ (G is also upper triangular).
- *Generating \mathbf{x} from \mathbf{v} can increase energy (~ enhance noise in MAC rcvr) if implemented directly (linearly).*
(order reversal is intentional)



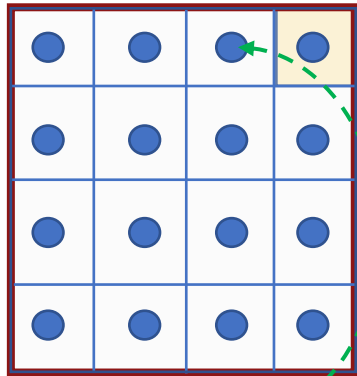
Voronoi Regions and Modulo Addition (Sec 2.1)

- A lattice is a (countable) group of vectors $\Lambda = \{\mathbf{x}\}$ that is closed under an operation addition, so that
 - If $\mathbf{x}_1 \in \Lambda$ and $\mathbf{x}_2 \in \Lambda$, then $\mathbf{x}_1 + \mathbf{x}_2 \in \Lambda$. (Section 2.2.1.1 and Appendix B.2)
 - A constellation is a finite subset of a lattice, plus a constant (coset) $C \subset \Lambda + \lambda_0$. (λ_0 ensures average value is zero.)

SQ rectangular or \mathbb{Z}^2

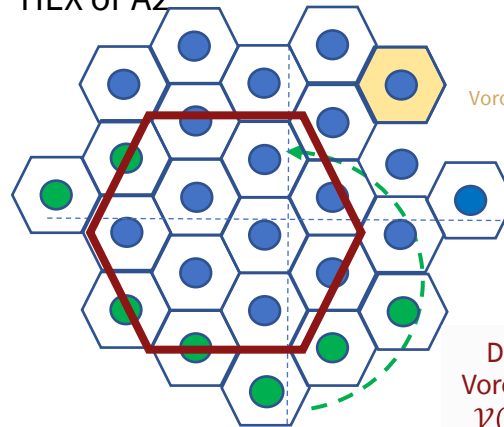
Decision or
Voronoi Region
 $\mathcal{V}(\Lambda' = 4\mathbb{Z}^2)$

Contains 16



Decision or
Voronoi Region
 $\mathcal{V}(\Lambda = \mathbb{Z}^2)$

HEX or A_2



Decision or
Voronoi Region $\mathcal{V} = A_2$

Decision or
Voronoi Region
 $\mathcal{V}(\Lambda' = 3A_2)$
contains 9

λ_0 is the "coset leader"

Green indicates
Modulo operation
(split ties equally)

- Voronoi Region of a lattice, $\mathcal{V}(\Lambda_c)$ is the decision region around any point with volume $V(\Lambda_c)$.
 - Λ_c is the "coding" lattice; codes try to pack more points into limited space (volume/area). – HEX is better than SQ.
- A constellation C typically selects points in one (coding-gain) lattice, Λ_c , within the $\mathcal{V}(\Lambda_s)$ of another (shaping-gain) lattice Λ_s that is larger (can be scaled versions of one another or possibly different). (Subtract any nonzero vector mean to save energy.)
 - All points in Λ_c outside of $\mathcal{V}(\Lambda_s)$ map into a point inside $\mathcal{V}(\Lambda_s)$ - disguised detector problem.



More general precoder (than Tomlinson/Laroia ...)

- Generalize **Modulo Operation**:

$$(\mathbf{v})_{\Lambda_S} = \mathbf{e} \ni \min_{\lambda \in \Lambda_S} \|\mathbf{e}\|^2 \text{ where } \mathbf{e} = \mathbf{v} - \lambda$$

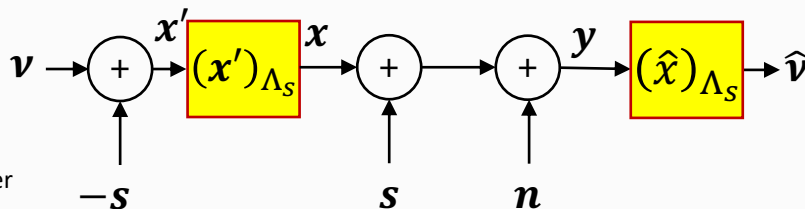
- \mathbf{e} does not necessarily need to be a point in Λ_C ; instead, it is a point in $\mathcal{V}(\Lambda_S)$.
 - It's essentially the error between input and output of decoder with decision region equal to $\mathcal{V}(\Lambda_S)$ - sbs in trivial cases of "uncoded" $\mathcal{V}(\Lambda_S) = \mathbb{Z}^2$.

Lemma 2.8.1 (distribution of modulo addition) *Modulo addition distributes as*

$$(\boldsymbol{\mu} + \boldsymbol{\nu})_{\Lambda} = (\boldsymbol{\mu})_{\Lambda} \oplus_{\Lambda} (\boldsymbol{\nu})_{\Lambda} . \quad (2.454)$$

- Side info is \mathbf{s} .

- \mathbf{s} is known (ISI for causal $G(D)$ = in Tomlinson/DFE)
- Pre-subtract (precode) and use modulo to set \mathcal{E}_x level (no energy increase for $-\mathbf{s}$).
- Any Λ_S shaping gain also applies here (8.5, $\mathbf{s}=0$).
 - \mathbf{v} then is any (e.g., SQ) input constellation.
 - In shaping case, $\hat{\mathbf{v}}$ also needs a subsequent detector yet for whatever code is used on \mathbf{v} .
- \mathbf{x} will effectively have continuous uniform distributions over Λ_S .



Dirty paper
"noiseless"

(nonlinear)
precoder

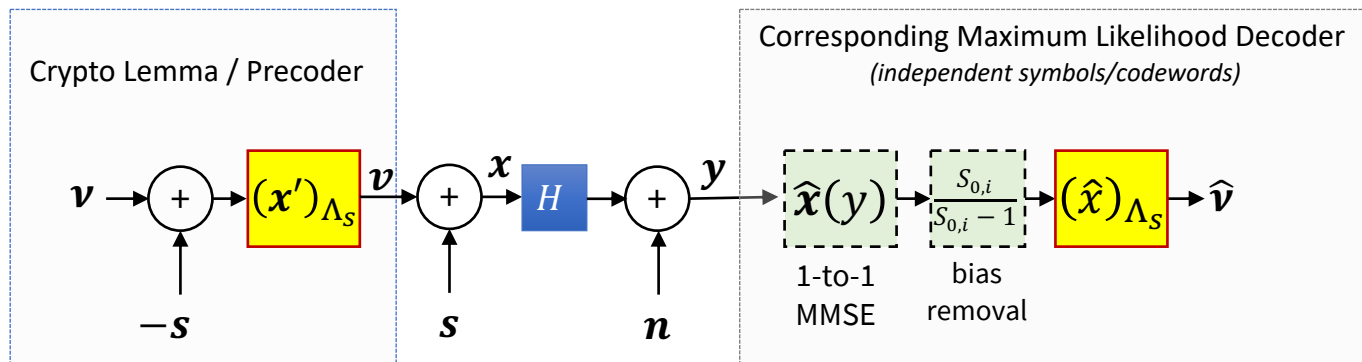
...

For BC: \mathbf{s} will be the earlier users (but their xtalk cancels) ~ **order**.



With nontrivial channel, need MMSE version

Forney's Crypto Lemma – 2003 (Section 2.8.1.2)



- The MMSE part can be important in non-trivial cases (often missed in most info-theory texts).
 - It's reshaping the channel crosstalk and/or ISI in MMSE (not zero-forcing) sense.
- When s is uniform over $\mathcal{V}(\Lambda_S)$, then so is v , **AND** v is independent of both s and v (like encryption), s is the "key"
 - Or "writing on dirty paper" (s is the dirt, v is the writing, and the second modulo cleans the paper).
- Sometimes the channel adds s (ISI/xtalk), sometimes the transmitter adds s (xmit case, s shares dimensions and energy with x).
 - The add-at-xmit case has s as other users, effectively (with a twist .. later).

**No xmit energy increase
Simplifies ML detection**



Non-Causal ?

- Subtly, the lattice Λ_s has a dimensionality N over which \mathbf{s} and \mathbf{x} are uniformly distributed.
- Wise dimension use with fixed energy \mathcal{E}_x suggests Λ_s has a hyper-spherical boundary, as $N \rightarrow \infty$.
 - This infinite-length precoder then also obtains full 1.53 shaping gain.
- Asymptotically, the modulo has infinite number of dimensions, so requires infinite delay for \mathbf{s} to be fully known in the formation of \mathbf{x} ; whence “non-causal.”
 - Approximated with finite delay in practice, \mathbf{s} becomes another user’s encoded signal known first (\sim non-causal) \rightarrow **order** .

$$(\cdot)_{\Lambda_s}$$



$$(\cdot)_{\mathcal{E}_x}$$

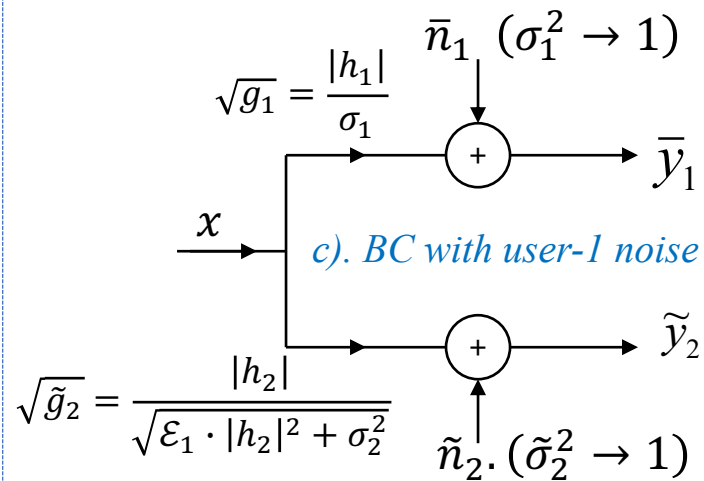
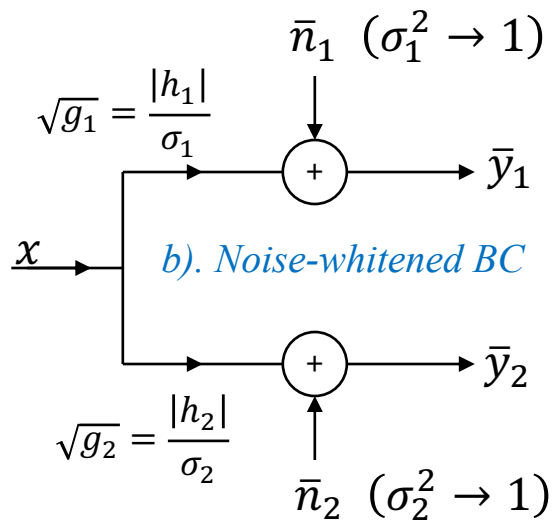
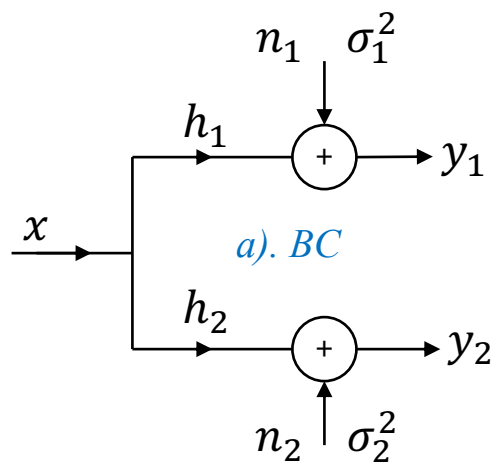
- Mod holds energy at \mathcal{E}_x (Gaussian in any finite number of dimensions, uniform in infinite dimensional hypersphere).
- If Λ_s is hypercube, Forney’s crypto still holds but with SNR loss of (up to) 1.53 dB (the maximum shaping gain).
 - So reuse code with $\Gamma \rightarrow 0$ dB, with QAM constellations and the (up to) 1.53 dB loss remains (greatly simplifies precoder implementation),
 - but everything else works the same. This then is essentially 379A’s Tomlinson Precoder over the user dimensions.



Scalar Gaussian BC

PS 5.2 - 2.29 scalar BC region

3 SCALAR-BC “scalings”



- They're all equivalent, but the 3 scalings are different.

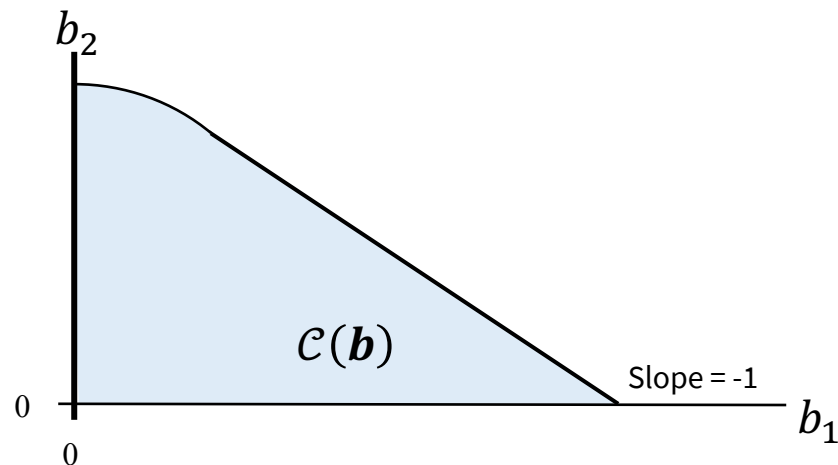


Rate region

$$g_1 > g_2$$

$$\bar{b}_1 \leq \mathbb{I}(x_1: y_1/x_2) = \frac{1}{2} \cdot \log_2(1 + \alpha \cdot \bar{\epsilon}_x \cdot g_1)$$

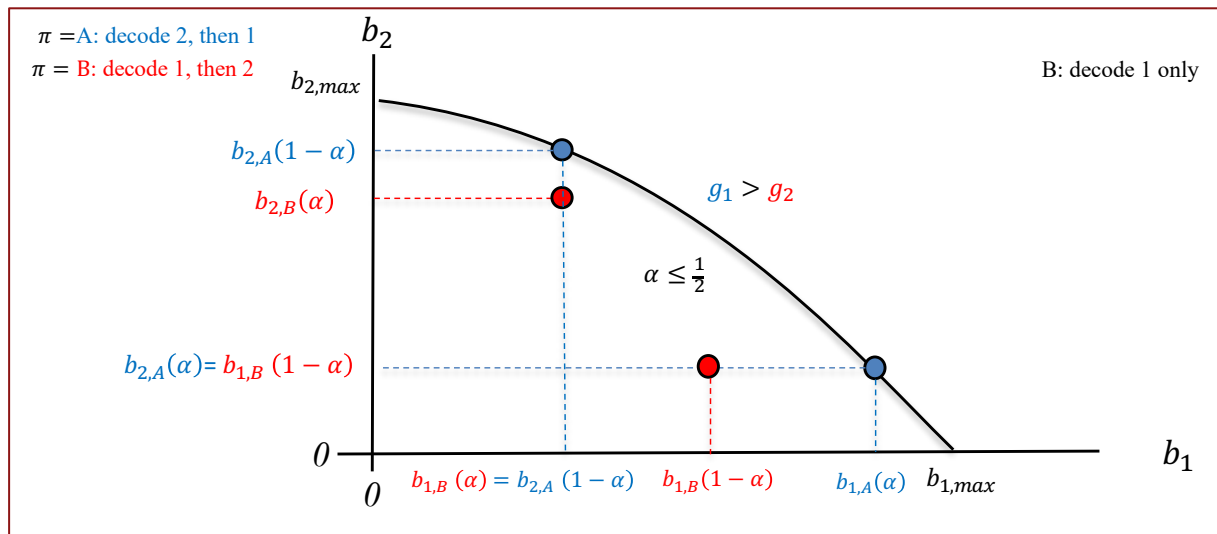
$$\bar{b}_2 \leq \mathbb{I}(x_2: y_2) = \frac{1}{2} \cdot \log_2\left(1 + \frac{(1-\alpha) \cdot \bar{\epsilon}_x \cdot g_2}{1 + \alpha \cdot \bar{\epsilon}_x \cdot g_2}\right)$$



- $\mathcal{C}(\mathbf{b})$'s calculation runs through all energy splits (this is single parameter α in 2-user scalar BC).
- Can also reverse order and take convex hull (not necessary though, see next slide).



Single best order for scalar BC, $g_1 > g_2$



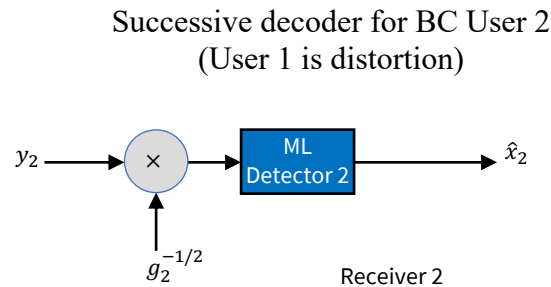
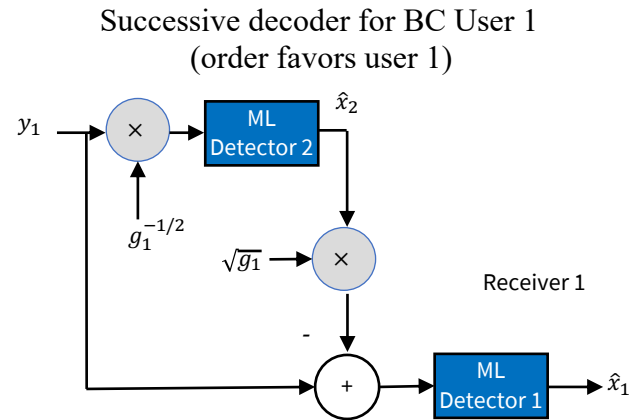
$$b_A = \begin{bmatrix} \frac{1}{2} \cdot \log_2(1 + \alpha \cdot \bar{\epsilon}_x \cdot g_1) \\ \frac{1}{2} \cdot \log_2\left(1 + \frac{(1-\alpha) \cdot \bar{\epsilon}_x \cdot g_2}{1 + \alpha \cdot \bar{\epsilon}_x \cdot g_2}\right) \end{bmatrix}$$

$$b_B = \begin{bmatrix} \frac{1}{2} \cdot \log_2\left(1 + \frac{\alpha \cdot \bar{\epsilon}_x \cdot g_1}{1 + (1-\alpha) \cdot \bar{\epsilon}_x \cdot g_1}\right) \\ \frac{1}{2} \cdot \log_2(1 + (1-\alpha) \cdot \bar{\epsilon}_x \cdot g_2) \end{bmatrix}$$

- Best order? $g_1 > g_2$ **both users'** data rates are on boundary if 2 is decoded first with 1 as noise.
 - Order reversal at RCVR2 decodes user 1 first; this then limits user 1 at RCVR 1 (even if 1 is last decoded at RCVR 1 because user 1 must be decodable at RCVR 2 also.)
 - See equations in text.
- Inductively, $g_1 > \dots > g_U$ is the single best order (no search needed on scalar Gaussian BC!).



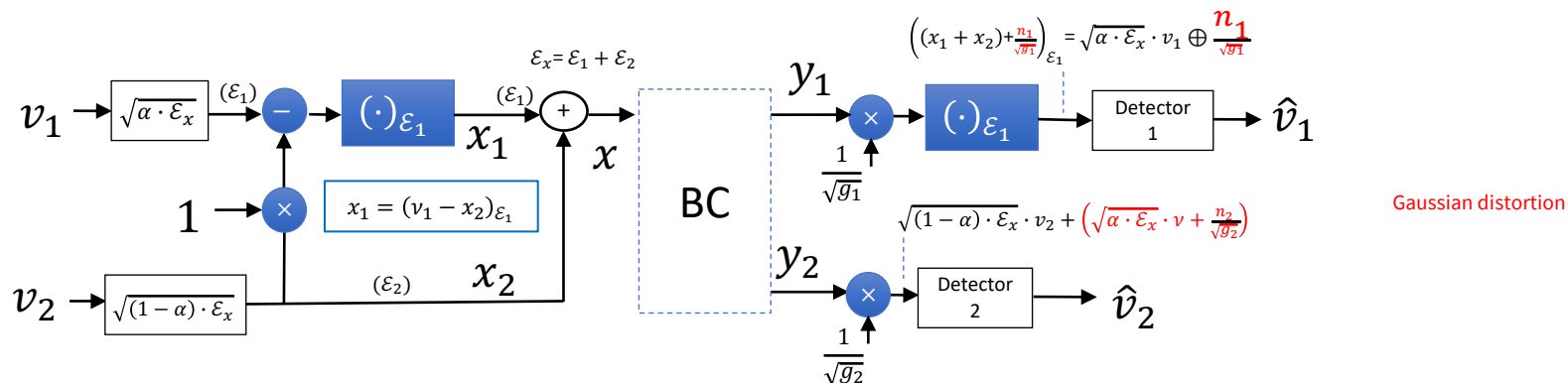
BC Successive Decoders



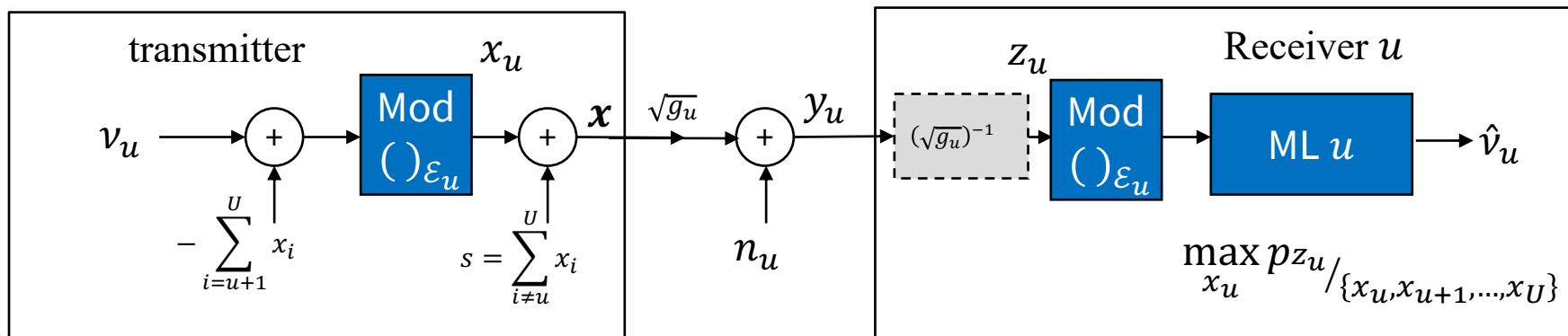
- U ML- U detectors; or really $\sum_{u=1}^U u = \frac{U}{2} \cdot (U + 1)$ total detectors.
- A precoder simplifies to U uses of the same modulo at transmitter (+ 1 modulo at each receiver).



Scalar Precoder



- The side information becomes x_2 and $\mathcal{E}_x = \mathcal{E}_1 + \mathcal{E}_2$; receiver 1's modulo removes x_2 .
- Precoder applies inductively (recursively) applied from $U \dots 1$.

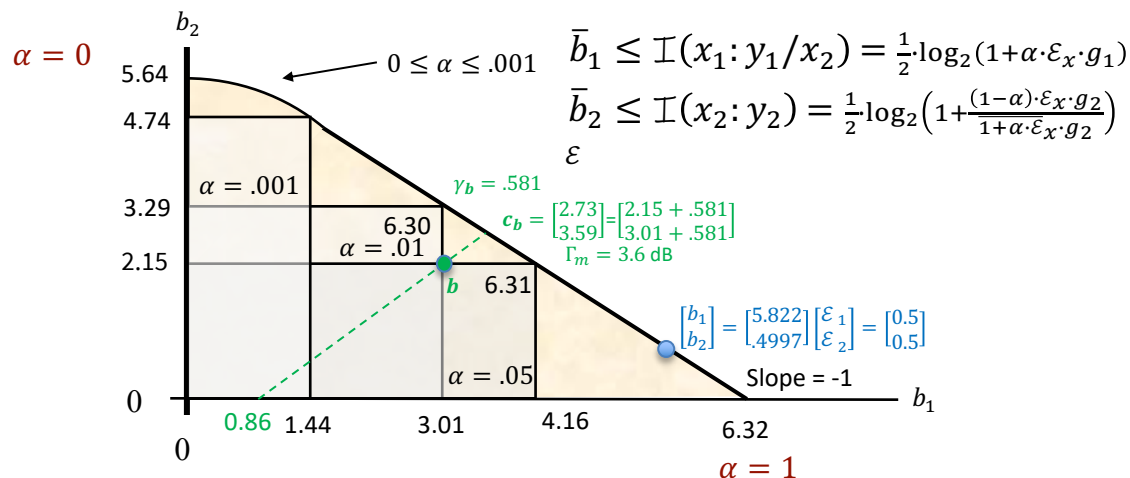


Example

- $h_1 = 0.8 ; h_2 = 0.5 ; \sigma_1^2 = \sigma_2^2 = .0001$

$$\mathcal{I}(\mathbf{x} : \mathbf{y}) = \frac{1}{2} \cdot \log_2 \left(\frac{|R_{yy}|}{|R_{mm}|} \right) = \frac{1}{2} \cdot \log_2 \left(\frac{(.6401) \cdot (.2501) - .4^2}{.01^2} \right) = 6.56$$

α	\bar{b}_1	\bar{b}_2	$\bar{b} = \bar{b}_1 + \bar{b}_2$
1.0	6.32	0	6.32
.75	6.12	.20	6.32
.50	5.82	.50	6.32
.25	5.32	1.0	6.32
.10	4.66	1.66	6.32
.05	4.16	2.15	6.31
.01	3.01	3.29	6.30
.001	1.44	4.74	6.18
0	0	5.64	5.64



- User 1 has highest sum rate when User 2 has zero energy.
 - User 1 is a primary user/component.
 - User 2 is a secondary user/component.





End Lecture 8