



STANFORD

Lecture 7

Capacity Region: Multiple Access Channel

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Announcements & Agenda

- Announcements
 - Problem Set #4 is due Tuesday April 30 (no late, so solutions can distribute).
 - Midterm is 5/2 in class.
- Agenda
 - General Capacity region
 - MAC $\mathcal{C}(\mathbf{b})$ via partial rate sums
 - Scalar Gaussian MAC
 - Vector Gaussian MAC
 - mu_mac.m software
 - Back-up
 - Capacity Region for frequency-indexed channels



General MU Capacity Region and related optima

Section 2.6.4

3 General Search Steps

- Search 1: Find \mathcal{I}_{min} for given $\mathbf{\Pi}$ and p_{xy}
- Search 2: Generate these \mathcal{I}_{min} 's convex hull over all orders $\mathbf{\Pi}$ for the achievable region $\mathcal{A}(\mathbf{b}, p_{xy})$
- Search 3: Generate a 2nd Convex hull over all probability distributions p_x for $\mathcal{C}(\mathbf{b})$
- These searches can be complex for general case, but do simplify for Gaussian MAC, BC, and IC.



Order-and-Distribution-Dependent Region

- **Order Step** forms a first convex hull of all \mathcal{I}_{min} vectors FOR EACH GIVEN ORDER and input distribution.

$$\mathcal{A}(\mathbf{b}, p_{xy}) = \bigcup_{\Pi}^{conv} \mathcal{I}_{min}(\Pi, p_{xy})$$

**Achievable
Region**

- Any point outside $\mathcal{A}(\mathbf{b}, p_x)$ will in the AEP sense have large error probability for at least one receiver.
 - The orders are “dimension shared” across different designs (the convex hull / union) operation ... sub users.
 - Every order and all convex combinations thereof have been considered, so each point can be decoded if inside $\mathcal{A}(\mathbf{b}, p_x)$.
- **Distribution Step** forms hull over the allowed input distributions (a 2nd convex hull operation).

$$\mathcal{C}(\mathbf{b}) = \bigcup_{p_x}^{conv} \mathcal{A}(\mathbf{b}, p_{xy})$$

**MU Capacity
Region**

- The order search is “NP-hard.”
- The subusers need to collapse back to U -dimensional region through sum.
- **Admissibility:** Is $\mathbf{b} \in \mathcal{C}(\mathbf{b})$? (often easier fortunately)

many cases
simplify



Collapse Region Dimensionality

- Collapse Dimensionality: add the subuser rates for each user at each point in the region.

$$\mathcal{C}_{U \times 1}(\mathbf{b}_{U \times 1}) = \left\{ \mathbf{b}_{U \times 1} \mid b_u = \sum_{(u, u') \in \mathcal{S}_u} b'_{u, u'}, \forall \mathbf{b}' \in \mathcal{C}_{U' \times 1}(\mathbf{b}_{U' \times 1}) \text{ and } \forall u \in [1 : U] \right\}$$

- Alternative Capacity Region Expression notes:

- the imposition of the $\mathcal{I}_{\min}(\Pi, p_{xy})$ is an intersection (which is a convex operation also) of MACs.

$$\mathcal{C}(\mathbf{b}) = \sum_{u'=1}^U \left[\bigcup_{u \in \mathcal{E}[1:U^2]}^{\text{hull}} \bigcup_{\Pi}^{\text{hull}} \bigcap_{r \in [1:U]} \text{MAC}(r, \Pi, \mathcal{E}) \right]_{u, u'}$$

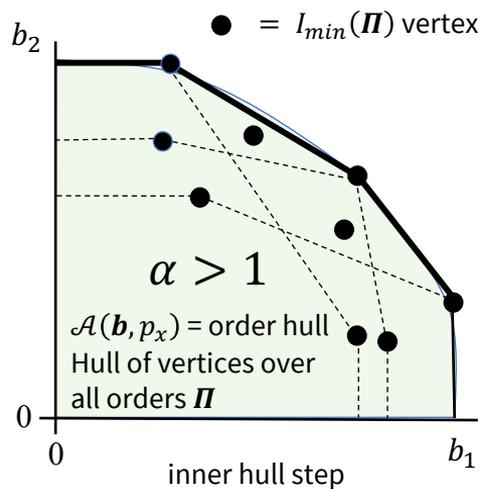
Shortcut the notation on summing 😊

- Union distributes over intersection (and vice versa), so many interpretations

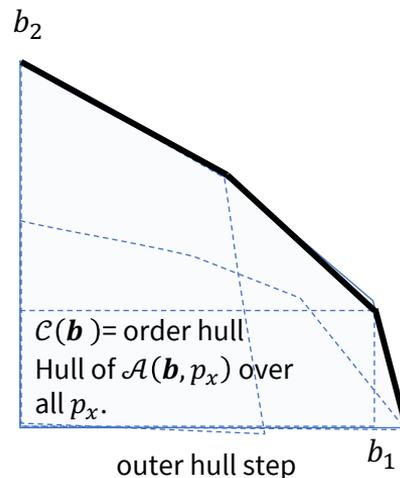


The two convex-hull steps

- The **order-vertices'** hull

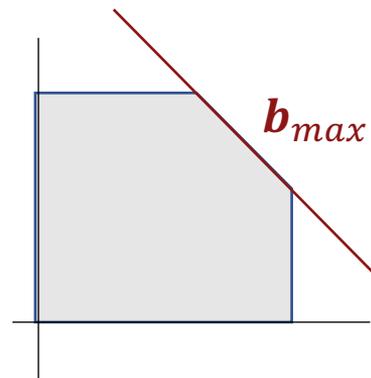
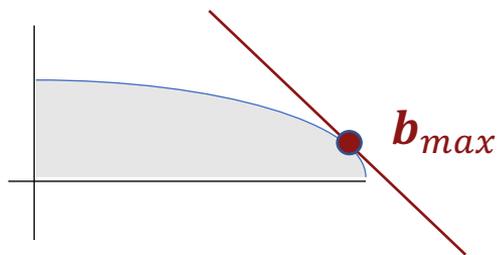


- The **input-distributions'** hull



Maximum Rate Sum

- The **rate sum** is $\mathbf{1}^* \mathbf{b}$, or simply the sum of the user bits/symbol.
- This is a hyperplane in U -space.
- This plane with normal vector $\mathbf{1}$ will be tangent to $\mathcal{C}(\mathbf{b})$ at \mathbf{b}_{max} , where $\mathbf{1}^* \cdot \mathbf{b}_{max} = b_{max}$, the maximum sum rate.



Degraded-Matrix AWGN

Definition 2.6.7 [(Subsymbol) Degraded multiuser Gaussian Channel] A (subsymbol)-degraded AWGN multiuser channel has matrix ranks for H and/or $R_{\mathbf{x}\mathbf{x}}$ that are ρ_H and $\rho_{R_{\mathbf{x}\mathbf{x}}}$ respectively, such that

$$\min \left\{ \rho_{R_{\mathbf{x}\mathbf{x}}}, \rho_H \right\} < U \quad . \quad (2.284)$$

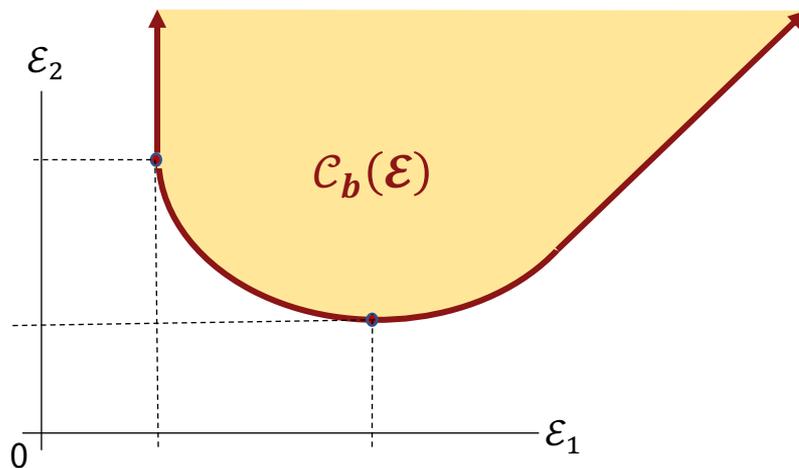
Otherwise, the channel is **non-degraded**. The literature often omits the word “subsymbol,” but it is tacit in degraded-channel definitions.

This degraded definition depends on channel AND input.

- What “degraded” means physically is that there are not enough dimensions to carry all users independently.
 - There are other chain-rule conditional-probability definitions, but they appear equivalent.
- If all users energize, some must co-exist on the available (subsymbol) dimensions.
 - A name is NOMA (new name for old subject) – Non-Orthogonal Multiple Access (associated with IoT where U can be very large).
- Non-degraded channels (Massive MIMO is an example) have a surplus of dimensions (less likely to be degraded).
- $R_{\mathbf{m}\mathbf{m}}$ is never singular on real channels, so noise whitening should not reduce the rank.
 - however, we will see a special case where design will assume a fictitious singular noise, so we’ll need care on this when used.



Capacity-Energy Region (AWGN only)



- Essentially redraws the capacity regions for different energy vectors with fixed \mathbf{b} .
 - Trivially, any point within is reliably achievable, while points outside have insufficient energy.
- If a given $\mathcal{E}_x \in \mathcal{C}_b(\mathcal{E})$, then \mathbf{b} is **admissible** when also $\mathbf{b}_{\mathcal{E}_x} \in \mathcal{C}(\mathbf{b})$.



Ergodic Capacity Region

- Design averages the capacity region over the variable-channel's parameter (joint if multiparameters) distribution.
 - This assumes messages are independent of parameters.
- Example: The **ergodic capacity region** is $\langle \mathcal{C}(\mathbf{b}) \rangle = \mathbb{E}_H[\mathcal{C}(\mathbf{b})]$ for the matrix AWGN:
 - *interesting result* – The distribution p_x that maximizes the ergodic capacity when H is **Raleigh (any user) fading** is a discrete distribution (so then not Gaussian); extends well-known result for single user.
 - The AEP results don't hold because AEP assumes the INPUT distribution is ergodic – and that is not necessarily true because the optimizing input distribution varies with the channel.
 - AEP's tacit interchange of input/channel limits for large blocklength may not hold and Rayleigh is example.
 - This presumably extends to multiuser case; however most channel variation for wideband (e.g. modern wireless) have codeword lengths/delays for good codes that are less than the coherence time, so Gaussian good codes remain in wide use. Thus, might as well go with Gaussian/known-good-codes for “quasi-stationary” assumption.
- **Outage Capacity Region?**
 - There is some work on “zero-outage” capacity regions (depending on definition may not be same as $\langle \mathcal{C}(\mathbf{b}) \rangle$).
 - Not necessarily just $(1 - P_{out}) \cdot \langle \mathcal{C}(\mathbf{b}) \rangle$, like single-user case because of “which user outage?” question, although it probably is a decent measure anyway.
 - More prudent analysis might instead focus on user input-rate variation (and contention for which point in $\mathcal{C}(\mathbf{b})$) and layer 2/3 buffer overflow outages, etc. as per L6's QPS, or see EE384S from N. Bambos for more complete address of queuing/network theory.

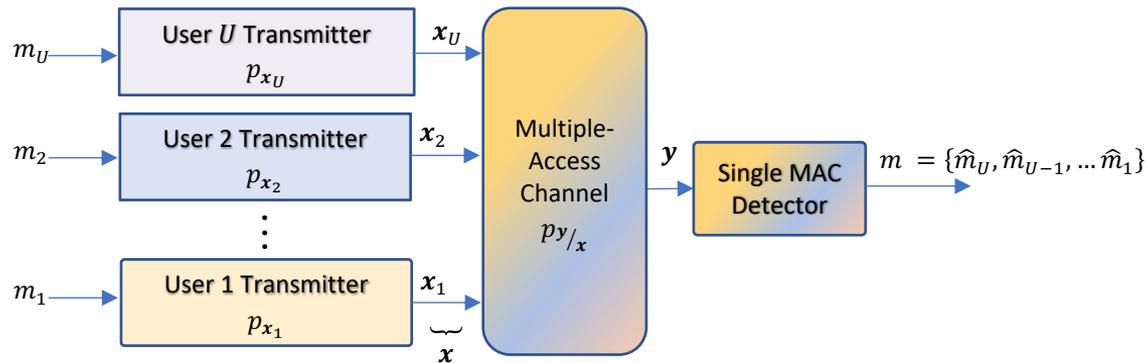


MAC $\mathcal{C}(b)$ via partial rate sums

PS4.3 - 2.23

The MAC's partial rate sums

$p_x = \prod_{u=1}^U p_{x_u}$ independent user inputs



- User u has maximum bit rate, when all other users are given (cancelled):

$$b_u \leq \mathcal{I}(x_u; y / x_{U \setminus u})$$

- The single receiver can process any user subset $\mathbf{u} \subseteq \mathbf{U}$.
 - This has a single-macro-user interpretation with summed bits/subsymbol:
 - $b_{\mathbf{u}} = \sum_{u \in \mathbf{u}} b_u \leq \mathcal{I}(x_{\mathbf{u}}; y / x_{U \setminus \mathbf{u}})$.
 - This defines a hyperplane with $|\mathbf{u}| - 1$ dimensions ($\in \mathbb{R}^{|\mathbf{u}|}$).
- MAC order simplifies (receiver) to $\mathbf{\Pi} = \mathbf{\pi}_1$.
 - The user order within \mathbf{u} does not change the sum $\mathcal{I}(x_{\mathbf{u}}; y / x_{U \setminus \mathbf{u}})$, nor does the order within $\mathbf{U} \setminus \mathbf{u}$.
 - The number of planes (lines ... hyperplanes) to search decreases substantially to $2^U - 1$ (null set excluded) $\ll (U!)^U$ (large U).



Chain-Rule Reminder Lemma 2.3.4

$$\mathcal{I}(\mathbf{x}; \mathbf{y}) = \mathcal{I}(\mathbf{x}_u; \mathbf{y} / \mathbf{x}_{U \setminus u}) + \mathcal{I}(\mathbf{x}_{U \setminus u}; \mathbf{y})$$

2^U possible choices of \mathbf{u}

User (set) \mathbf{u} is detected with all other users $\mathbf{x}_{U \setminus u}$ given (cancelled).
Other-user (set) $U \setminus \mathbf{u}$ is detected with users \mathbf{x}_u as noise.

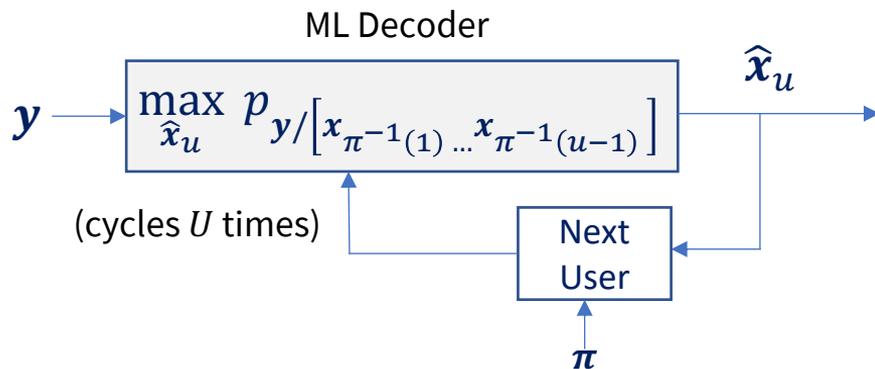
- $b \leq \mathcal{I}(\mathbf{x}; \mathbf{y})$ - This rate sum corresponds to the choice $\mathbf{u} = U$.
- A (hyperplane) **face**: $b_1 + b_2 + \dots + b_{|u|} \leq \mathcal{I}(\mathbf{x}_u; \mathbf{y} / \mathbf{x}_{U \setminus u})$ - defines $(2^{|u|} - 1)$ partial rate sums.
 - There are also U trivial faces for positive bits/subsymbol $b_u \geq 0$, so really $2^U - 1 + U$ faces that bound $\mathcal{A}(\mathbf{b}, p_x)$.
- A **vertex** corresponds to a specific $\mathbf{b} = \mathcal{I}$ for a specific order $\boldsymbol{\pi}$; examples include for $U = 2$:

$$\begin{bmatrix} \mathcal{I}(\mathbf{x}_2; \mathbf{y} / \mathbf{x}_1) \\ \mathcal{I}(\mathbf{x}_1; \mathbf{y}) \end{bmatrix} \quad \begin{bmatrix} \mathcal{I}(\mathbf{x}_1; \mathbf{y} / \mathbf{x}_2) \\ \mathcal{I}(\mathbf{x}_2; \mathbf{y}) \end{bmatrix}.$$

In general, $\exists U!$ vertices for a specific p_x .



Chain-Rule Decoder

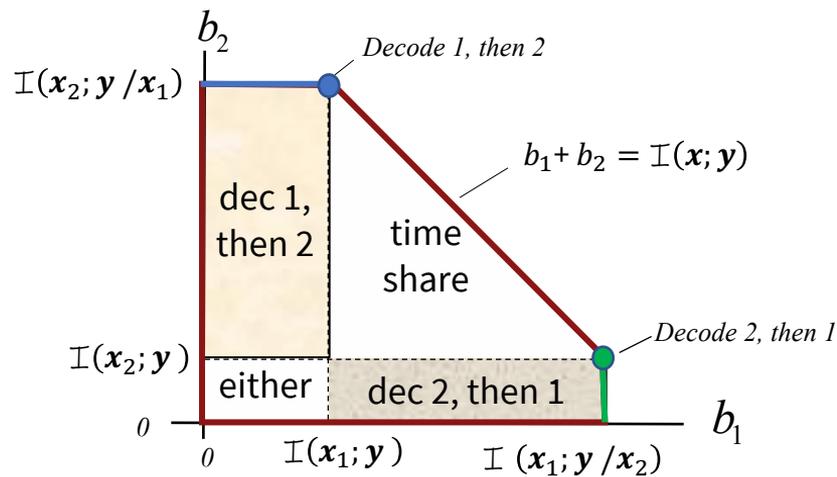


**Successive Decoding or ...
Generalized Decision Feedback Eq
(or “NOMA”)**

- For the given order, the receiver first decodes all lower-indexed users and then decodes the current user.
- Since there is only one order, analysis can relabel users and avoid all the $\pi^{-1}(\cdot)$ notation.
- There is no loss of generality.



A 2-user MAC rate region

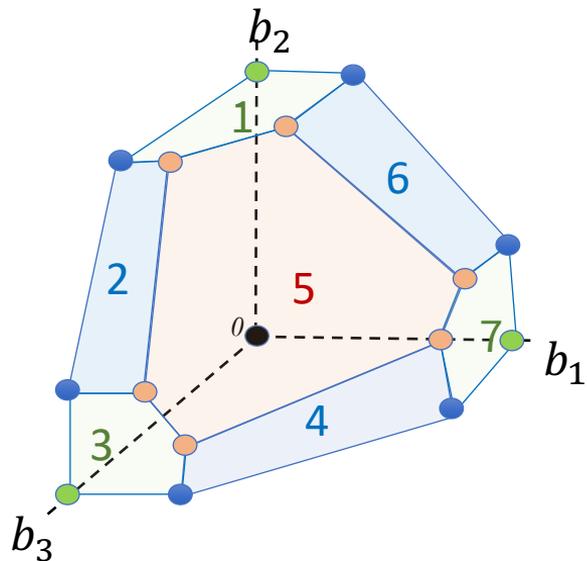


- Pentagon – 5 vertices and 5 faces
 - $2^U - 1 + U$ Faces are the $I(x_u; y/x_{U \setminus u})$ & $b_u \geq 0$
 - $U! = 2$ vertices are the both-user order points π
 - 2 more are single-user points, one for each user
 - 1 more is the origin
 - 5 total

- b_2 vertex (short blue line) decodes 1 first (given), then 2 as if 1 is “cancelled.”
 - Similar statement holds for b_1 vertex (and short green) line.
- Line with slope -1 is **time-share or really vertex-share**; it also is constant maximum rate sum (for this p_{xy}).
 - There are two codes for each user (4 codes); This is example of user components (or subusers, sometimes called “rate splitting”)



A 3-user rate region



- Decahedron – 10 faces
 - $2^U - 1 + U$ Faces are the $\mathcal{I}(x_u; y / x_{U \setminus u})$
 - $U! = 6$ vertices (rose) are the 3-user order points π

- = vertex for 3 users $U!$ (= 6)
- = vertex for 2 users $U!$ (= 6)
- = vertex for 1 user U (= 3)
- = vertex for 0 users (= 1)

$$\sum_{k=0}^U \binom{U}{k} \cdot k! = 16 \text{ vertices}$$

- b_2 horizontal plane (pentagon) decodes 1 and 3 first (given), then 2 as if 1 and/or 3 are “cancelled.”
 - 1 and 3 form a two-user horizontal pentagon region.
 - Similar statements hold for b_1 vertical-plane pentagon and b_3 facial-plane pentagon.
- Rose plane normal to $\mathbf{1} = [1 \ 1 \ 1]^*$ dimension-shares the rose vertices; it has constant maximum rate sum (for this p_{xy}).
 - There could be as many as 3 codes/components for each user on a time-share of vertices.
- The blue and green planes may also dimension-share vertices.
- $\mathcal{A}(\mathbf{b}, p_x)$ is the entire interior plus faces and vertices. Any point outside violates at least one single-user mutual-information bound.



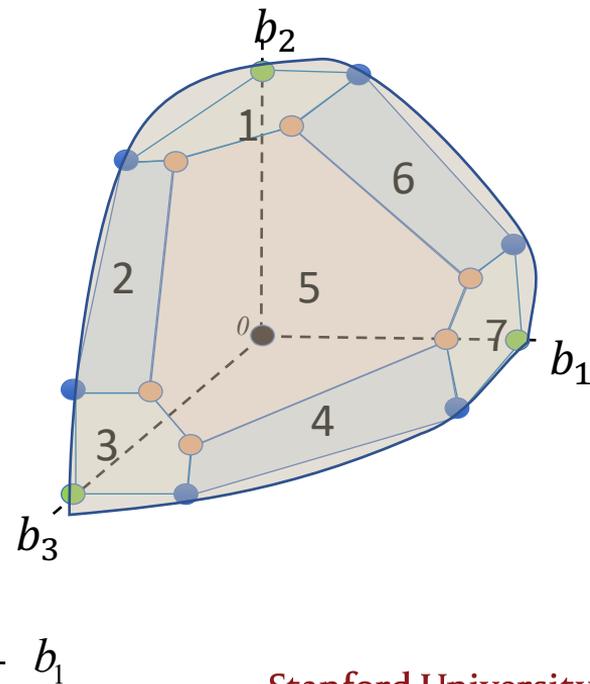
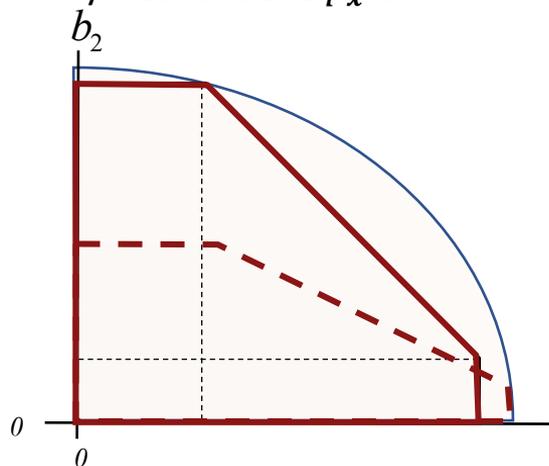
MAC Capacity Region

- More formally, the MAC's achievable region is bounded by hyperplanar regions

$$\mathcal{A}(\mathbf{b}, p_x) = \bigcap_{u \subseteq U} \left\{ \mathbf{b} \mid 0 \leq \sum_{i \in \{u\}} b_i \leq \mathcal{I}(x_i; \mathbf{y} / x_{u \setminus i}) \right\}.$$

- The vertices are where hyperplanes intersect at a point.
 - Or, lines (smaller dimensional hyperplanes) may also bound.
- Convex hull over all multi-user input probability distributions p_x is

$$\mathcal{C}_{MAC}(\mathbf{b}) = \bigcup_{u \subseteq p_x}^{conv} \mathcal{A}(\mathbf{b}, p_x).$$

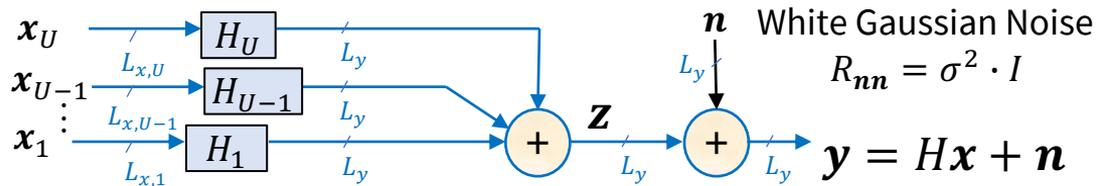


Scalar Gaussian MAC

PS4.3 - 2.25 Time-Division Multiplexing region

Section 2.7.2

General Gaussian MAC



$$\mathbf{y} = \underbrace{[H_U \quad H_{U-1} \quad \cdots \quad H_1]}_{L_y \times \mathcal{L}_x} \cdot \underbrace{\begin{bmatrix} \mathbf{x}_U \\ \mathbf{x}_{U-1} \\ \vdots \\ \mathbf{x}_1 \end{bmatrix}}_{\mathcal{L}_x \times 1} + \mathbf{n}$$

$L_y \times 1$ $L_y \times \mathcal{L}_x$ $\mathcal{L}_x \times 1$

More generally, variable-dim inputs have

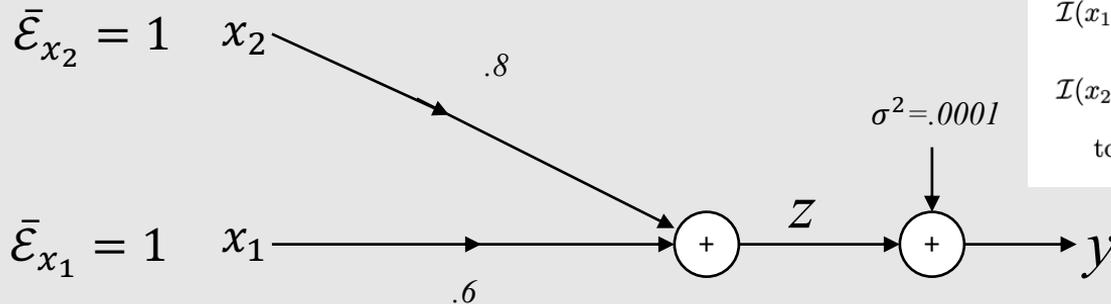
$$\mathcal{L}_x = \sum_{u=1}^U L_{x,u} \sim U \cdot L_x$$

- Inputs are independent.
 - R_{xx} is block diagonal.
 - Only 1 output and 1 noise.
- One Receiver will estimate all inputs.
 - It can do so in any order.
 - “Given an input” \mathbf{x}_u means cancel it from \mathbf{y} .
 - This does not necessary mean subtract $H_u \cdot \mathbf{x}_u$ from \mathbf{y}
 - Unless $L_y = L_{x,u} = 1$; or H_u is diagonal and noise is white.

\mathcal{P}_H is the matrix H 's **rank**:
 = number of linearly independent rows (or columns)
 = # of non-zero singular values.



Example

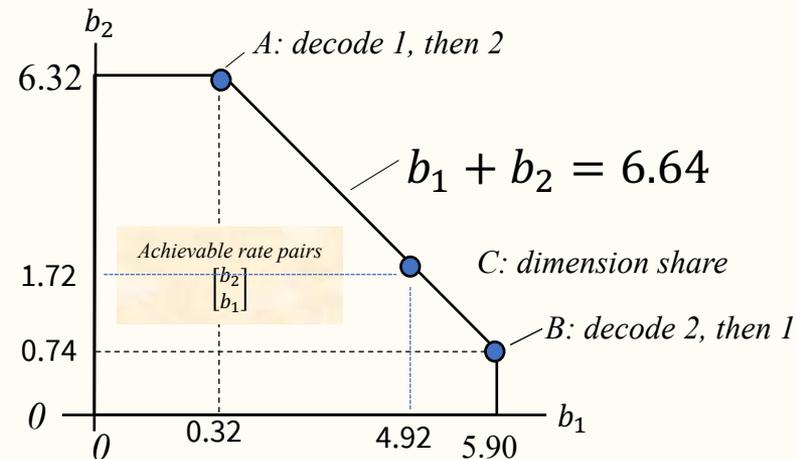


$$\begin{aligned} \mathcal{I}(x_1; y) &= \frac{1}{2} \log_2 \left(1 + \frac{.36 \cdot 1}{.0001 + .64} \right) = .32 \text{ bits/dimension} \\ \mathcal{I}(x_2; y) &= \frac{1}{2} \log_2 \left(1 + \frac{.64 \cdot 1}{.0001 + .36} \right) = .74 \text{ bits/dimension} \\ \text{total} &= 1.06 \text{ bits/dimension} \end{aligned}$$

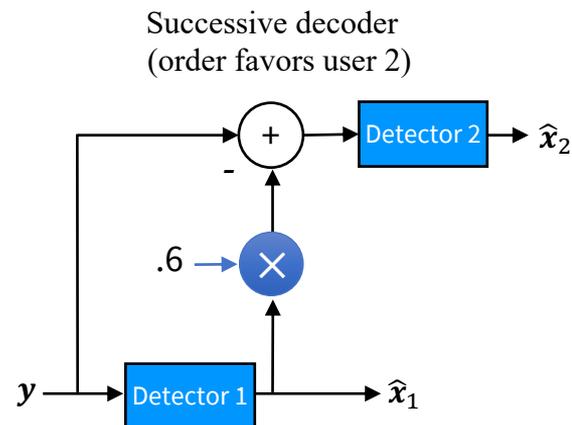
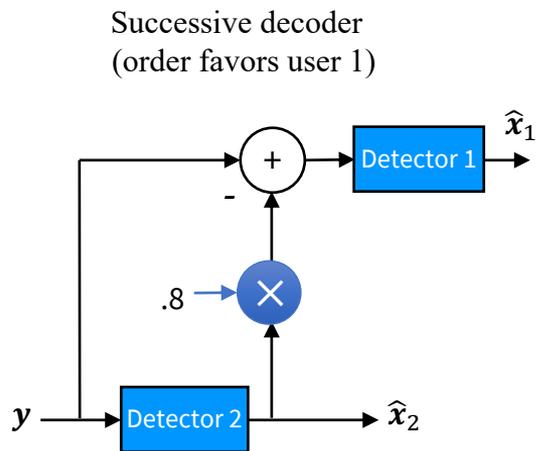
$$\mathcal{I}(x_2; y/x_1) = \frac{1}{2} \log_2 (1 + \text{SNR}_2) = \frac{1}{2} \log_2 \left(1 + \frac{.64 \cdot 1}{.0001} \right) = 6.32 \text{ bits/dimension}$$

$$\mathcal{I}(x_1; y/x_2) = \frac{1}{2} \log_2 (1 + \text{SNR}_1) = \frac{1}{2} \log_2 \left(1 + \frac{.36 \cdot 1}{.0001} \right) = 5.90 \text{ bits/dimension}$$

- Point C is $\frac{1}{4}$ share B and $\frac{3}{4}$ share A.



Successive decoding for scalar example



- Only 2 orders are possible for 2 users.
- $\exists U!$ in general (corresponding to each possible order).
- The last user is “favored” in decoding (first accepts other as noise).



2 – User Scalar $L_x = L_y = 1$

General formula
Scalar MAC

$$\bar{I}(x_{\mathbf{u}}; y/x_{\mathbf{U} \setminus \mathbf{u}}) = \frac{1}{2} \log_2 \left(1 + \frac{\sum_{i \in \mathbf{u}} \bar{\mathcal{E}}_i \cdot |H_i|^2}{\sigma^2} \right)$$

2 users

$$\text{SNR}_1 = \frac{\mathcal{E}_1 \cdot |h_1|^2}{\sigma^2}$$

$$\text{SNR}_2 = \frac{\mathcal{E}_2 \cdot |h_2|^2}{\sigma^2}$$

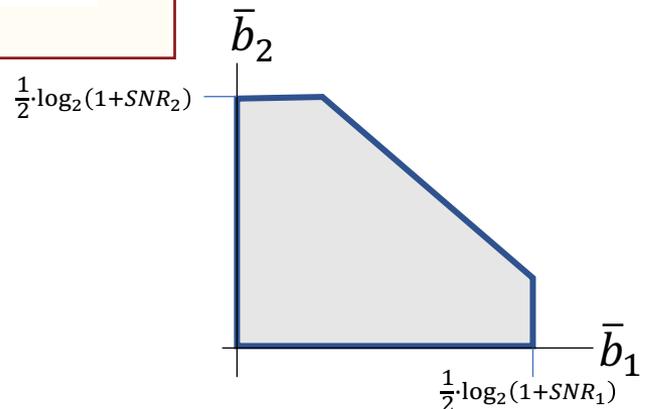
$$\text{SNR} = \frac{\mathcal{E}_1 \cdot |h_1|^2 + \mathcal{E}_2 \cdot |h_2|^2}{\sigma^2}$$

$$\bar{b}_1 \leq \frac{1}{2} \log_2 (1 + \text{SNR}_1)$$

$$\bar{b}_2 \leq \frac{1}{2} \log_2 (1 + \text{SNR}_2)$$

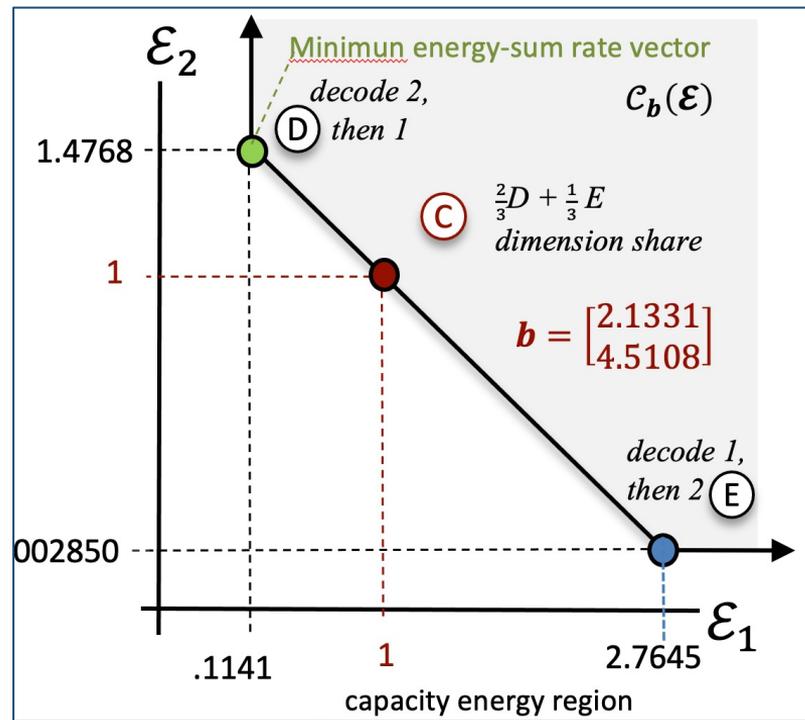
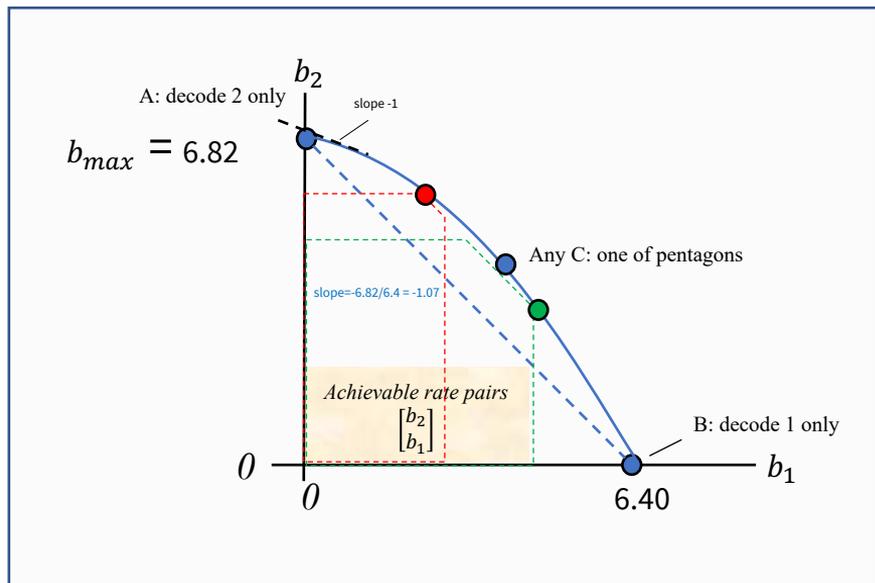
$$\bar{b}_1 + \bar{b}_2 \leq \frac{1}{2} \log_2 (1 + \text{SNR})$$

- 3 planes (lines) ~ 3 SNR's
- 1 sum rate
- Nonzero individual rates



Energy-Sum MAC

- \exists a single energy constraint $\mathcal{E}_1 + \mathcal{E}_2 \leq \mathcal{E}_x$ (instead of 2 constraints)
- Capacity region is a union of pentagons (and 1 triangle),
 - one for each combination of energies that add to total.
- Or \exists an Energy-Capacity Region,
 - one for each bit vector \mathbf{b} .



Time-Sharing Conundrum (Coding Theorist's Fallacy in disguise)

- What is meaning of time-sharing? (“convex hull”)
 - The different codes correspond to user components, each used for its respective fraction of “time” (dimensions).
- With time-sharing, what does \mathcal{E}_u mean?
 - Energy constant at \mathcal{E}_u : Is this then for every symbol/subsymbol in the sharing?
 - Or the average over the “time-shared” subsymbols?
- The second instance of averaging often enlarges the capacity region.
- So, “time-sharing” is somewhat ill-defined.
 - Despite most info/com texts on MAC using it.
- Lecture 4’s Separation Theorem actually allows different mutual information \mathcal{I}_A and \mathcal{I}_B to be represented by their average information – *for the same user*.
 - $\mathcal{I} = \alpha \cdot \mathcal{I}_A + (1 - \alpha) \cdot \mathcal{I}_B$.
 - ST uses same constellation with average \mathbf{b} for each symbol, possible very large $|C|$.
 - If the shared same-user codes correspond to vertices with different orders, this creates issues for Separation Thm application.
 - But it is still possible, although the successive decoding needs to become “iterative-user” successive decoding (“turbo xtalk cancelling”).
 - Of course, each user can use subusers; each user has subcode for A and for B, but then constellation varies.



Primary and Secondary Components (E-sum MAC)

Primary-user component: has nonzero energy for E-sum MAC's maximum rate.

Secondary-user component: has zero energy for E-sum MAC's maximum rate.

- Primary components dominate with largest pass-space gains (dimensions used for component).
- Secondary users “free load” on these primary-component dimensions.

Previous example (.8 and .6):

The pass-space is just one dimension ($L_y = 1$).
user 2 is all primary (.8) ; user 1 is all secondary (.6).
max sum is 6.82 (all energy on user 2).

Rate-sum decreases if secondary user components energize (see slide L7:24).



How Many Primary Components (E-sum MAC)?

- The MAC has no more than $U^o \leq p_H$ primary components, to find them first do U SVD's:

$$\tilde{H}_u = R_{noise}^{-1/2}(u) \cdot H_u = F_u \cdot \Lambda_u \cdot M_u^* \quad \text{with} \quad |\tilde{H}_u| \triangleq \prod_{l=1}^{p_{H_u}} \lambda_{u,l} > 0.$$

- Each user can excite up to p_{H_u} possible independent dimensions per subsymbol.
 - The $R_{noise}(u)$ includes all other user components' crosstalk for whatever energies they use (knows all $R_{xx}(u)$'s).
 - Each user can have vector-coding modulator without loss, or some linear combination of the pass-space dimensions.

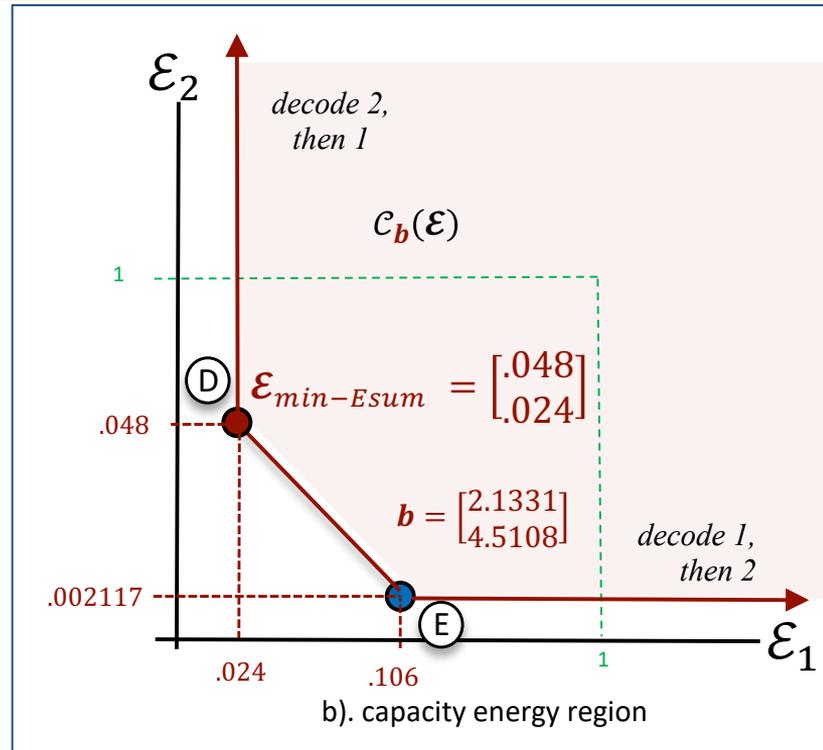
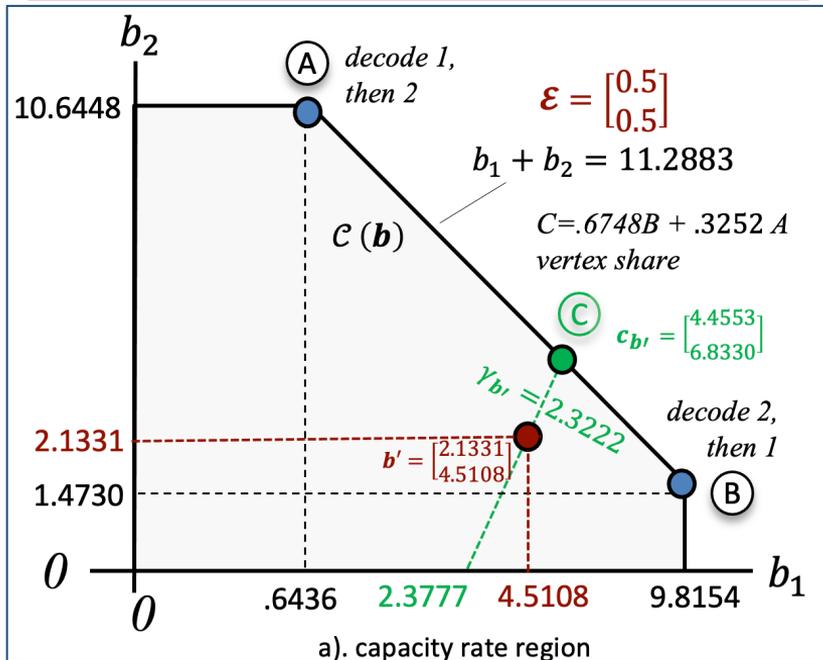
- For the channel gains in the VC,
$$g_u = |\tilde{H}_u|^2 = \prod_{l=1}^{p_{H_u}} \lambda_{u,l} \quad .$$

- The **primary-user components** correspond to those energized in achieving max rate sum on the E-sum MAC. All others are **secondary-user components**.
 - With MIMO MAC, components can be viewed also as “subuser components.”
- The “components” idea is helpful when individual users' transmitters have >1 dimension (MIMO), via
 - time-sharing, DMT, and/or multiple antennas.



Conundrum: double-sampling-rate Example

[80 60] channel again at twice sampling rate



- The vector \mathbf{b} is now in the interior of the region, although is it the same channel?
 - The time-sharing needs to occur at the same sampling rate, meaning the symbol period increases, for the original $\mathcal{C}(\mathbf{b})$ to apply.

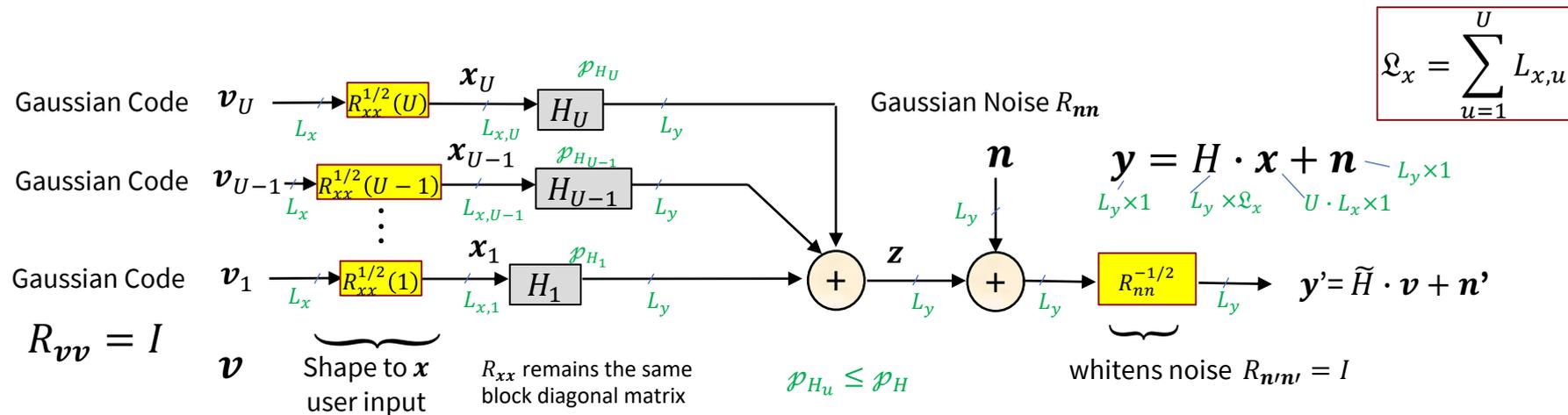


Vector Gaussian MAC

PS4.4 - 2.24 MAC regions

Section 2.7.2.2

MAC ~ single channel with white input



- This normalizes (redefines, not $R_{noise}(u)$ here) individual user MAC channels to $\tilde{H}_u \triangleq R_{nn}^{-1/2} \cdot H_u \cdot R_{xx}^{1/2}(u)$.
- **Normalized MAC** is now $\mathbf{y}' = \tilde{H} \cdot \mathbf{v} + \mathbf{n}'$, where:
 - New input(s) is (are) “white”, $R_{vv} = I$.
 - New noise is “white”, $R_{nn'} = I$.
 - We drop the primes going forward; $\mathbf{y} = \tilde{H} \cdot \mathbf{v} + \mathbf{n} \rightarrow \tilde{H}'$ ’s dimensions carry the information (secondary may freeload).



Cholesky Factorization

- This is related to MMSE linear-prediction (see Appendix D).
- Positive definite Hermitian symmetric matrix factors as $R = G^* \cdot S \cdot G$, where
 - G is upper triangular monic (1's on diagonal), &
 - S is positive real diagonal matrix (even if R is complex).
- Matlab command is “**chol**” for *lower* \times *upper* (lower is upper*) – **produces upper**.
 - $G_{temp} = \text{chol}(R)$
 - $G = \text{inv}(\text{diag}(\text{diag}(G_{temp}))) * G_{temp}$
 - $S = \text{diag}(\text{diag}(G_{temp})) * \text{diag}(\text{diag}(G_{temp}))$
- See website's lohcm program for *lower* \times *upper*.

```
>> R=[2 1
1 2];
>> Gtemp=chol(R) %=
    1.4142    0.7071
     0    1.2247
>> G= inv(diag(diag(Gtemp)))*Gtemp %=
    1.0000    0.5000
     0    1.0000
S= diag(diag(Gtemp))*diag(diag(Gtemp)) %=
    2.0000     0
     0    1.5000
>> G'*S*G %=
    2.0000    1.0000
    1.0000    2.0000
```



Forward and Backward Canonical Channels

- **Forward Canonical** Channel is $\mathbf{y}' = \underbrace{\tilde{H}^* \cdot \tilde{H}}_{R_f} \cdot \boldsymbol{\nu} + \underbrace{\tilde{H}^* \cdot \mathbf{n}}_{\mathbf{n}'},$
 - the output of matched-filter matrix.

- **MMSE Estimator** for backward channel

$$R_{\boldsymbol{\nu}\mathbf{y}'} \cdot R_{\mathbf{y}'\mathbf{y}'}^{-1} = R_f \cdot [R_f \cdot R_f + R_f]^{-1} = [R_f + I]^{-1} = R_b$$

- **Backward Canonical** Channel $\boldsymbol{\nu} = R_b \cdot \mathbf{y}' + \mathbf{e} \quad R_{\mathbf{e}\mathbf{e}} = R_b$

- Use **Cholesky** on backward-channel inverse $R_b^{-1} = R_f + I = G^* \cdot S_0 \cdot G$

$$\mathbf{y}'' = S_0^{-1} \cdot G^{-*} \mathbf{y}' \quad (\text{algebra})$$

$$\mathbf{y}'' = G \cdot \boldsymbol{\nu} - \mathbf{e}'$$

$$\text{where } R_{\mathbf{e}'\mathbf{e}'} = S_0^{-1}.$$



Back Substitution

- Not quite ML/MAP, but **successive decoding**,
 - but **canonical** – achieves \mathcal{I} reliably, each user,
 - if decisions are correct** (asymptotic MMSE = MAP again).
 - If $\Gamma > 0$ dB, then iterative decoding that \rightarrow ML may be needed.
- Each of these is MMSE based,
 - which is related to conditional \mathcal{I} .
- The decoder is much simpler decoder (“GDFE”).
- SNR (biased) for each decision/dimension is $S_{0,u,l}$.
- But also

$$G = \begin{bmatrix} 1 & g_{U,U-1} & \dots & g_{U,2} & g_{U,1} \\ 0 & 1 & \dots & g_{U-1,2} & g_{U-1,1} \\ \vdots & \ddots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & g_{2,1} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

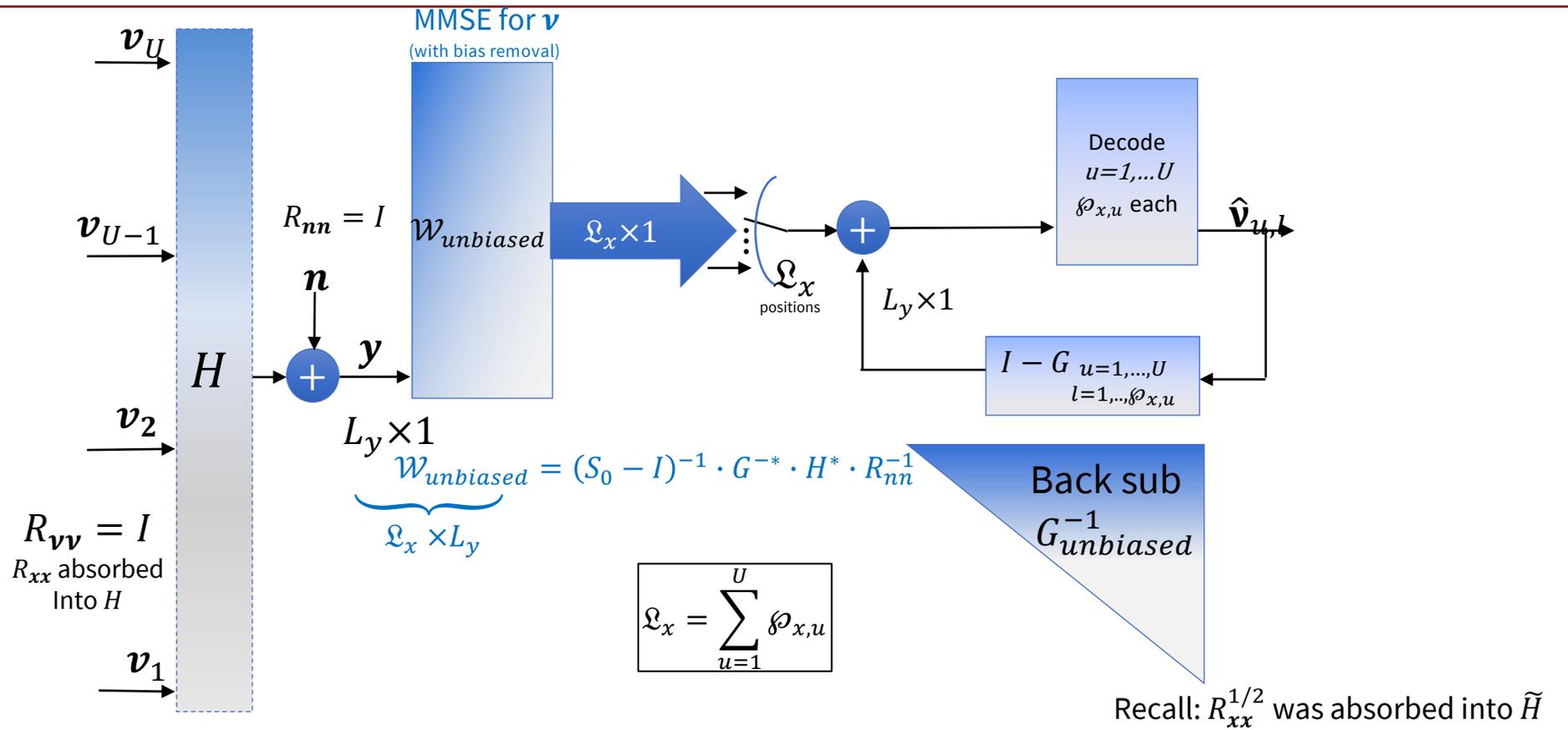
$$\begin{aligned} \hat{\mathbf{v}}_1 &= \text{decision}(\mathbf{y}_1'') \\ \hat{\mathbf{v}}_2 &= \text{decision}(\mathbf{y}_2'' - g_{2,1} \cdot \hat{\mathbf{v}}_1) \\ &\vdots \\ \hat{\mathbf{v}}_{U'} &= \text{decision}\left(\mathbf{y}_U'' - \sum_{i=1}^{U-1} g_{U,i} \cdot \hat{\mathbf{v}}_i\right) \end{aligned}$$

$$\mathcal{I}(\mathbf{x}; \mathbf{y}) = \log_2(|\underbrace{\tilde{H}^* \tilde{H} + I}_{R_b^{-1}}|) = \log_2 |S_0| = \log_2 \left\{ \prod_{u=1}^{U'} \prod_{\ell=1}^{L_{x,u}} SNR_{mmse,u,\ell} \right\} \text{ bits / complex symbol .}$$

**New parallel
“independent”
subchannels**

CANONICAL RECEIVER (any R_{xx})

Vector MAC Receiver



- Each user/decoder achieves $\mathcal{I}(\mathbf{v}_u; \mathbf{y} / [\mathbf{v}_{u-1} \ \dots \ \mathbf{v}_1])$



STANFORD

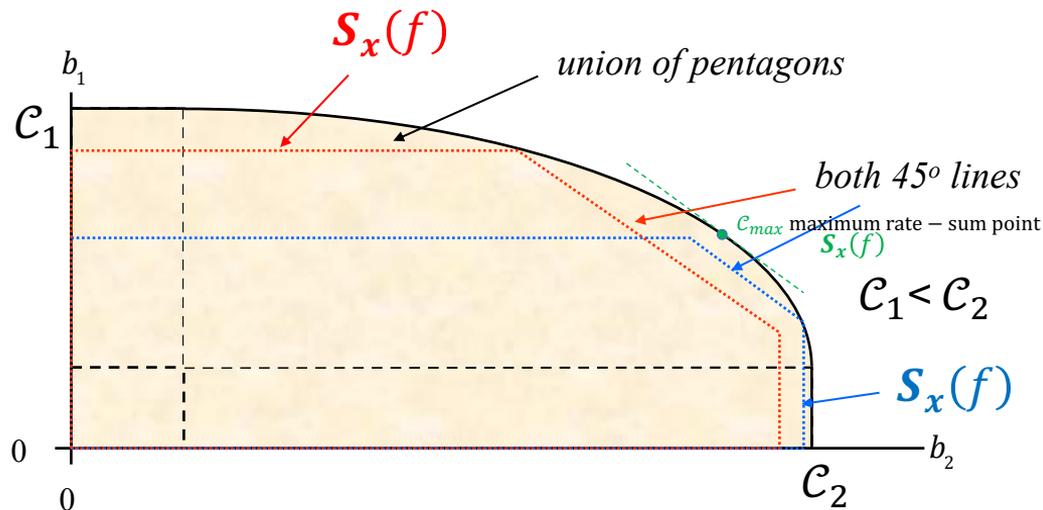
End Lecture 7

(back-up material FYI)

Capacity Region for continuous-frequency-indexed channels

Sections 2.7.4.1-2

$\mathcal{C}(b)$ is Union of $S_x(f)$ -indexed Pentagons



$$\bar{b} = \sum_{u=1}^U \bar{b}_u \leq \bar{\mathcal{I}}(\mathbf{x}; \mathbf{y}) = \int_{-\infty}^{\infty} \frac{1}{2} \cdot \log_2 \left[1 + \frac{\sum_{u=1}^U S_{x,u}(f) \cdot |H_u(f)|^2}{S_n(f)} \right] df$$

Calculus of variations again, decomposes into U water-fills, each with other users as noise – more details in L9.

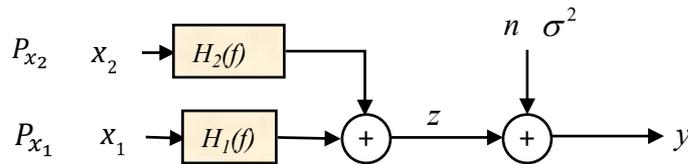
$$S_{x,u}(f) + \frac{\sigma^2 + \sum_{i \neq u} S_{x,i}(f) + |H_i(f)|^2}{|H_u(f)|^2} = K_u$$

Simultaneous water-filling
→ Maximum rate sum

- Each pentagon corresponds to an $S_x(f)$ choice.
 - The pentagons become triangles for the sum-energy MAC.
- The union (convex hull is union when inputs are Gaussian) can dimension-share in frequency as $N \rightarrow \infty$.



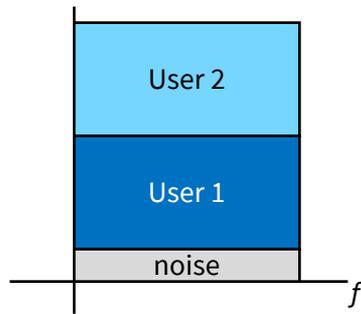
MT MAC



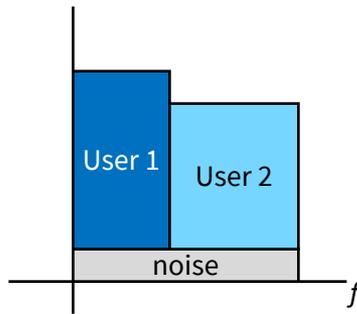
- The users have continuous-time/frequency channels \rightarrow use MT on each, theoretically.
- This really means dimensionality is infinite (or very large) so “dimension-sharing” may be inherent.
- SWF applies, but with some interpretation (like power instead of energy, etc and power per dimension instead of power-spectral density, etc.)



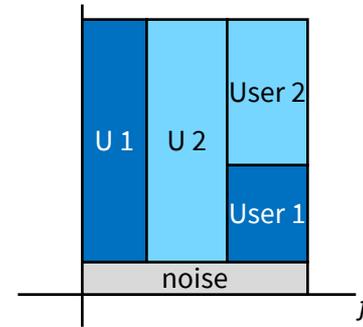
Decoders and SWF



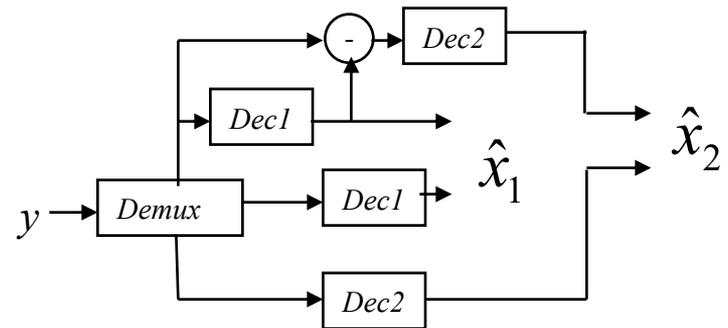
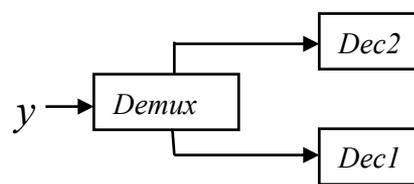
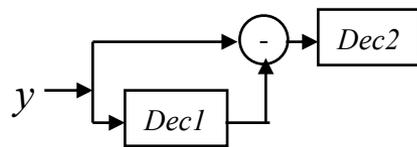
a). both flat



b). FDM



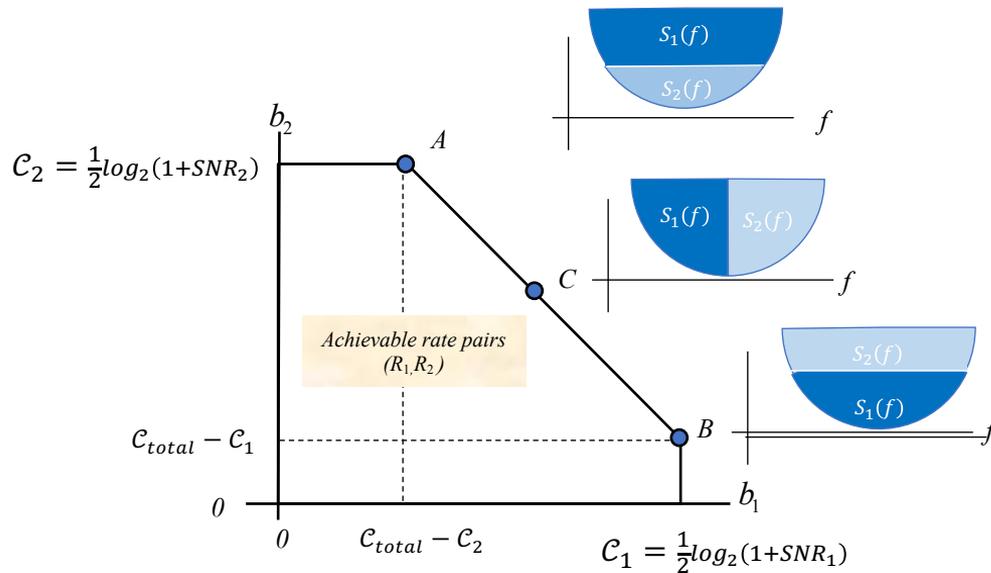
c). mixed



- FDM is clearly simplest decoder for max rate sum case.



Symmetric 2-user channel and SWF



- Symmetric means $H_1(f) = H_2(f)$ (noise is one-dimensional and added to sum)
- Each of points A, B, and C have different SWF spectra – all have same (max) rate sum

