Lecture 7

Multiple Access Channels

April 20, 2026

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Announcements & Agenda

- Announcements
  - Problem Set #4 is due **TUESDAY** April 28
    - no late, so solutions can distribute.
  - Midterm is 4/29 in class.

- Agenda
  - General Capacity region
  - MAC $\mathcal{C}(\boldsymbol{b})$ via partial rate sums
  - Scalar Gaussian MAC
  - Vector Gaussian MAC
    - mu_mac.m software
  - Back-up
    - Capacity Region for frequency-indexed channels

- Problem Set 4 = PS4, due April 28, 6pm
  1. 2.21 multiuser channel types
  2. 2.22 multiuser margins
  3. 2.23 minimum information vector
  4. 2.24 TDM capacity region
  5. 2.25 Gaussian Vector MAC

General MU Capacity Region and related optima

Section 2.6.4

3 General Search Steps

- Search 1: Find $\mathcal{I}_{\boldsymbol{min}}$ for given $\boldsymbol{\Pi}$ and $p_{\boldsymbol{xy}}$
- Search 2: Generate these $\mathcal{I}_{\boldsymbol{min}}$'s convex hull over all orders $\boldsymbol{\Pi}$ for the achievable region $\mathcal{A}\left(\boldsymbol{b}\,, p_{\boldsymbol{xy}}\right)$
- Search 3: Generate a 2nd Convex hull over all probability distributions $p_{\boldsymbol{x}}$ for $\mathcal{C}(\boldsymbol{b}\ )$
- These searches can be complex for general case, but do simplify for Gaussian MAC, BC, and IC.

Order-and-Distribution-Dependent Region

- **Order Step** forms a first convex hull of all $\mathcal{I}_{\boldsymbol{min}}$ vectors FOR EACH GIVEN ORDER and input distribution.

$$\mathcal{A}(\boldsymbol{b}\ ,p_{\boldsymbol{xy}})=\bigcup_{\boldsymbol{\Pi}}^{conv}\mathcal{I}_{\boldsymbol{min}}(\boldsymbol{\Pi},p_{\boldsymbol{xy}})$$

Achievable Region

- Any point outside $\mathcal{A}(\boldsymbol{b},p_{\boldsymbol{x}})$ will in the AEP sense have large error probability for at least one receiver.
  - The orders are "dimension shared" across different designs (the convex hull / union) operation …. sub users.
  - Every order and all convex combinations thereof have been considered, so each point can be decoded if inside $\mathcal{A}(\boldsymbol{b},p_{\boldsymbol{x}})$.

- **Distribution Step** forms hull over the allowed input distributions (a 2nd convex hull operation).

$$\mathcal{C}(\boldsymbol{b}\ )=\cup_{p_{\boldsymbol{x}}}^{conv}\mathcal{A}(\boldsymbol{b}\ ,p_{\boldsymbol{xy}})$$

MU Capacity Region

- The order search is "NP-hard."
- The subusers rates sum to a U-dimensional region.
- **Admissibility:** Is $\boldsymbol{b}\in\mathcal{C}(\boldsymbol{b})$? (often easier fortunately)

many cases simplify

Collapse Region Dimensionality

- Collapse Dimensionality: add the subuser rates for each user at each point in the region.

$$\mathcal{C}_{U\times 1}(\boldsymbol{b}_{U\times 1}) = \left\{ \boldsymbol{b}_{U\times 1} \mid b_u = \sum_{(u,u')\in\mathcal{S}_u} b'_{u,u'} \ , \ \forall\, \boldsymbol{b}' \in \mathcal{C}_{U''\times 1}(\boldsymbol{b}_{U''\times 1}) \text{ and } \forall\, u \in \ [1:U] \right\}$$

- Alternative Capacity Region Expression notes:
  - the imposition of the $\mathcal{I}_{\boldsymbol{min}}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}})$ is an intersection (which is a convex operation also) of MACs.

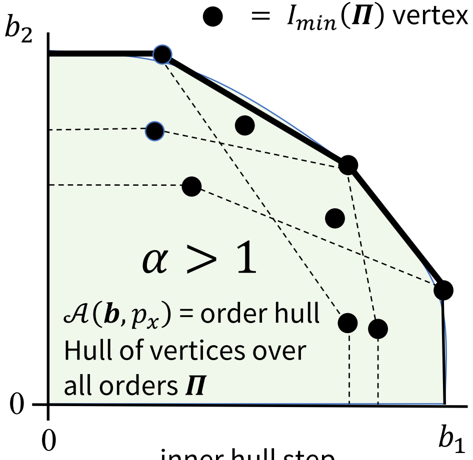
$$C(\boldsymbol{b}) = \sum_{u=1}^{U} \left[\bigcup_{u'\in\mathcal{E}[1:U'']}^{hull} \ \bigcup_{\boldsymbol{\Pi}}^{hull} \ \bigcap_{r\in[1:U]} \boldsymbol{MAC}\,(r, \boldsymbol{\Pi}, \mathcal{E}) \right]_{u,u'}$$

Shortcut the notation on summing 😇

- Union distributes over intersection (and vice versa), so many interpretations

The two convex-hull steps

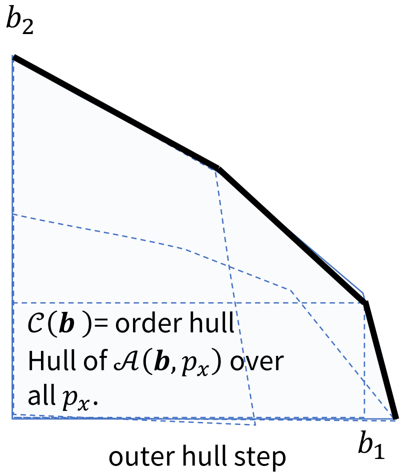
- The **order-vertices’** hull



$\mathcal{A}(\boldsymbol{b}, p_x)$ = order hull
Hull of vertices over
all orders $\boldsymbol{\Pi}$

inner hull step

- The **input-distributions’** hull

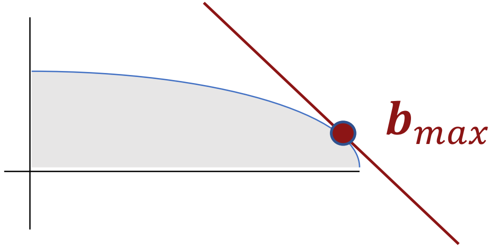


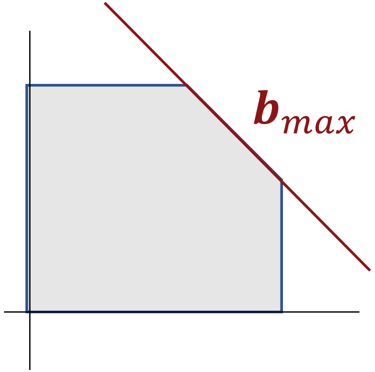
$\mathcal{C}(\boldsymbol{b}\ )$= order hull
Hull of $\mathcal{A}(\boldsymbol{b}, p_x)$ over
all p_x.

outer hull step

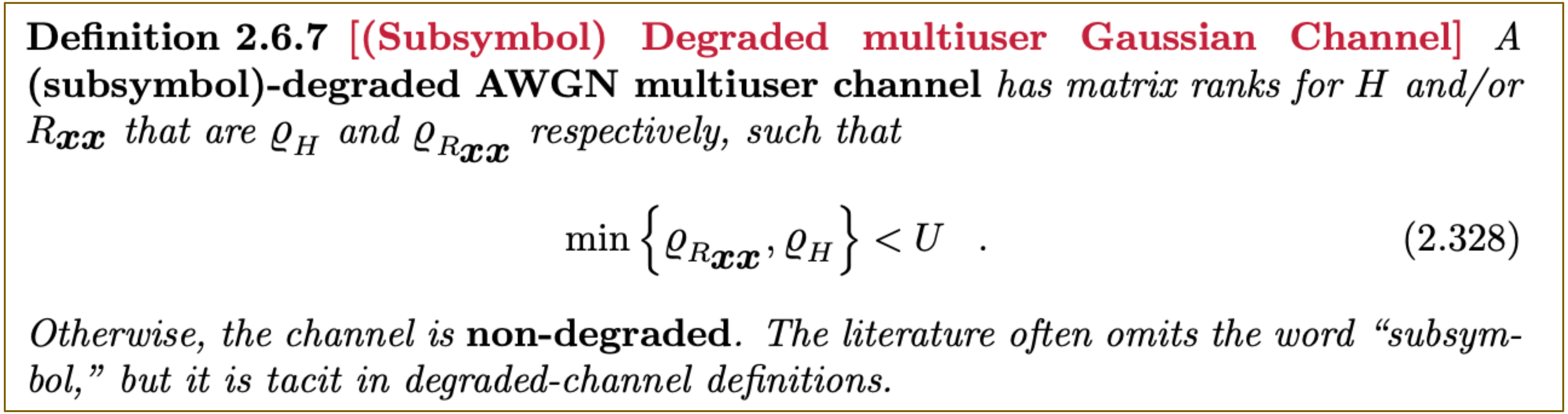
Maximum Rate Sum

- The **rate sum** is $\mathbf{1}^* \boldsymbol{b}$, or simply the sum of the user bits/symbol.
- This is a hyperplane in U-space.
- This plane with normal vector $\mathbf{1}$ will be tangent to $\mathcal{C}(\boldsymbol{b})$ at $\boldsymbol{b}_{max}$, where $\mathbf{1}^* \cdot \boldsymbol{b}_{max}$= b_{max} , the maximum sum rate.





Degraded-Matrix AWGN



Definition 2.6.7 [(Subsymbol) Degraded multiuser Gaussian Channel] *A* **(subsymbol)-degraded AWGN multiuser channel** *has matrix ranks for H and/or $R_{\boldsymbol{xx}}$ that are ϱ_H and $\varrho_{R_{\boldsymbol{xx}}}$ respectively, such that*

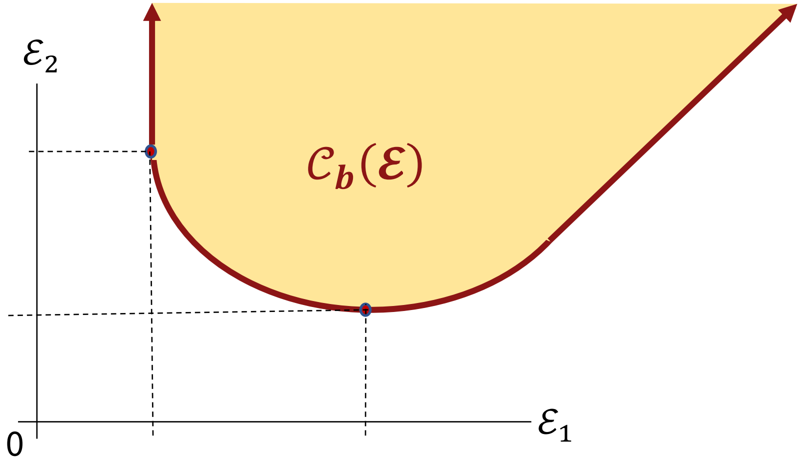
$$\min\left\{\varrho_{R_{\boldsymbol{xx}}}, \varrho_H\right\} < U \quad . \tag{2.328}$$

Otherwise, the channel is **non-degraded**. *The literature often omits the word "subsymbol," but it is tacit in degraded-channel definitions.*

This degraded definition depends on channel AND input.

- What "degraded" means physically is that there are not enough dimensions to carry all users independently.
  - There are other chain-rule conditional-probability definitions, but they appear equivalent.
- If all users energize, some must co-exist on the available (subsymbol) dimensions.
  - A name is NOMA (new name for old subject) – Non-Orthogonal Multiple Access (associated with IoT where U can be very large).
- Non-degraded channels (Massive MIMO is an example) have a surplus of dimensions (less likely to be degraded).
- $R_{\boldsymbol{nn}}$ is never singular on real channels, so noise whitening should not reduce the rank.
  - however, we will see a special case where design will assume a fictitious singular noise, so we'll need care on this when used.

Capacity-Energy Region (AWGN only)



- Essentially redraws the capacity regions for different energy vectors with fixed $\boldsymbol{b}$.
  - Trivially, any point within is reliably achievable, while points outside have insufficient energy.

- If a given $\boldsymbol{\mathcal{E}}_x \in \mathcal{C}_{\boldsymbol{b}}(\boldsymbol{\mathcal{E}})$, then $\boldsymbol{b}$ is **admissible** when also $\boldsymbol{b}_{\boldsymbol{\mathcal{E}}_x} \in \mathcal{C}(\boldsymbol{b})$.

Ergodic Capacity Region

- Design averages the capacity region over the variable-channel's parameter (joint if multiparametric) distribution.
  - This assumes messages are independent of parameters.

- Example: The **ergodic capacity region** is $\langle \mathcal{C}(\boldsymbol{b}) \rangle = \mathbb{E}_{\boldsymbol{H}}[\mathcal{C}(\boldsymbol{b})]$ for the matrix AWGN:

  - *interesting result* – The distribution $p_{\boldsymbol{x}}$ that maximizes the ergodic capacity when H is Raleigh (any user) fading is a discrete distribution (so then not Gaussian); extends well-known result for single user.

  - The AEP results don't hold because AEP assumes the INPUT distribution is ergodic – and that is not necessarily true because the optimizing input distribution varies with the channel.
    - AEP's tacit interchange of input/channel limits for large blocklength may not hold and Rayleigh is example.

  - This presumably extends to multiuser case; however, most channel variation for wideband (e.g. modern wireless) have codeword lengths/delays for good codes that are less than the coherence time, so Gaussian good codes remain in wide use. Thus, design might as well go with Gaussian/known-good-codes for "quasi-stationary" assumption.

- **Outage Capacity Region?**
  - There is some work on "zero-outage" capacity regions (depending on definition may not be same as $\langle \mathcal{C}(\boldsymbol{b}) \rangle$).

  - Not necessarily just $(1 - P_{out}) \cdot \langle \mathcal{C}(\boldsymbol{b}) \rangle$), like single-user case because of "which user outage?" question, although it probably is a decent measure anyway.

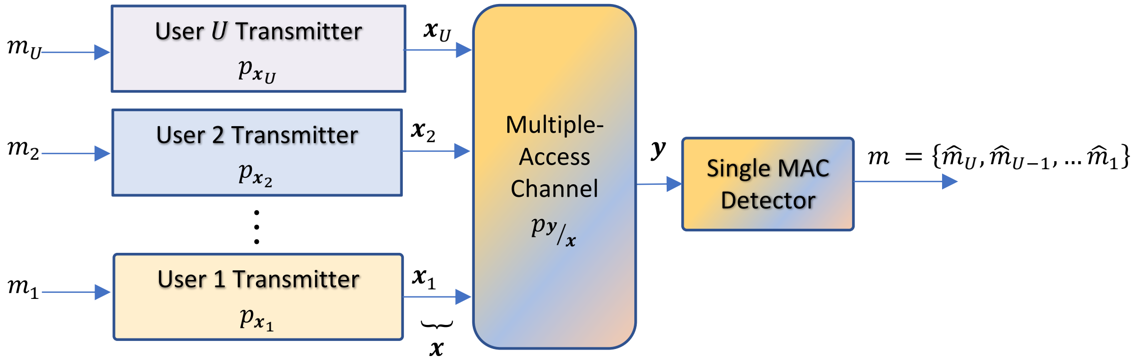
  - More prudent analysis might instead focus on user input-rate variation (and contention for which point in $\mathcal{C}(\boldsymbol{b})$) and layer 2/3 buffer overflow outages, etc. as per L6's QPS, or see EE384S from N. Bambos for more complete address of queuing/network theory.

MAC $\mathcal{C}(\boldsymbol{b})$ via partial rate sums

PS4.3 - 2.23

The MAC’s partial rate sums

$p_{\boldsymbol{x}} = \prod_{u=1}^{U} p_{\boldsymbol{x}_u}$ independent user inputs



- User u has maximum bit rate, when all other users are given (cancelled): $\boxed{b_u \le \mathbb{I}\left(\boldsymbol{x}_u; \boldsymbol{y} / \boldsymbol{x}_{U\setminus u}\right)}$
- The single receiver can process any user subset $\boldsymbol{u} \subseteq \boldsymbol{U}$.
  - This has a single-macro-user interpretation with summed bits/subsymbol:
    - $b_{\boldsymbol{u}} = \sum_{u \in \boldsymbol{u}} b_u \le \mathbb{I}\left(\boldsymbol{x}_{\boldsymbol{u}}; \boldsymbol{y} / \boldsymbol{x}_{U\setminus \boldsymbol{u}}\right)$.
  - This defines a hyperplane with $|\boldsymbol{u}| - 1$ dimensions $\left(\in \mathbb{R}^{|\boldsymbol{u}|}\right)$.
- MAC order simplifies (receiver) to $\boldsymbol{\Pi} = \boldsymbol{\pi}_1$.
  - The user order within $\boldsymbol{u}$ does not change the sum $\mathbb{I}\left(\boldsymbol{x}_{\boldsymbol{u}}; \boldsymbol{y} / \boldsymbol{x}_{U\setminus \boldsymbol{u}}\right)$, nor does the order within $\boldsymbol{U} \setminus \boldsymbol{u}$.
  - The number of planes (lines … hyperplanes) to search decreases substantially to $2^U - 1$ (null set excluded) << $(U \cdot U''!)^U$ (large U).

Chain-Rule Reminder Lemma 2.3.4

$$\boxed{\mathbb{I}(\boldsymbol{x};\boldsymbol{y}) = \mathbb{I}\left(\boldsymbol{x}_{\boldsymbol{u}};\boldsymbol{y}\,/\boldsymbol{x}_{\boldsymbol{U}\backslash\boldsymbol{u}}\right)+\mathbb{I}\left(\boldsymbol{x}_{\boldsymbol{U}\backslash\boldsymbol{u}}\,;\boldsymbol{y}\,\right)}$$

2^U possible choices of $\boldsymbol{u}$

User (set) $\boldsymbol{u}$ is detected with all other users $\boldsymbol{x}_{\boldsymbol{U}\backslash\boldsymbol{u}}$ given (cancelled).

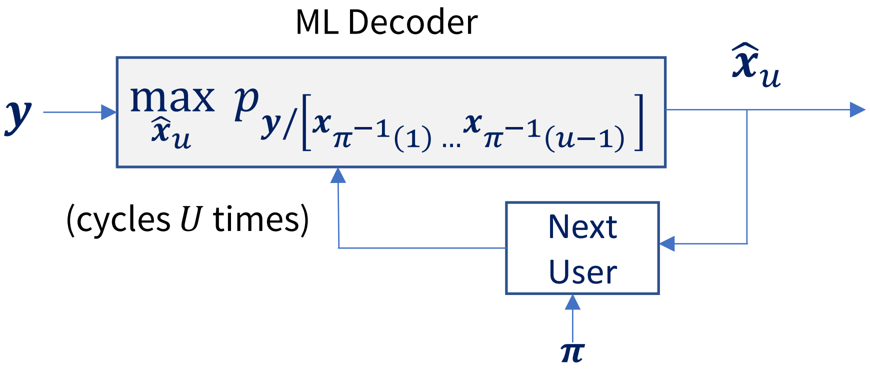
Other-user (set) $\boldsymbol{U}\backslash\boldsymbol{u}$ is detected with users $\boldsymbol{x}_{\boldsymbol{u}}$ as noise.

- $b \leq \mathbb{I}(\boldsymbol{x};\boldsymbol{y})$ - This rate sum corresponds to the choice $\boldsymbol{u}=\boldsymbol{U}$.
- A (hyperplane) **face**: $b_1 + b_2 + \cdots + b_{|\boldsymbol{u}|} \leq \mathbb{I}\left(\boldsymbol{x}_{\boldsymbol{u}};\boldsymbol{y}\,/\boldsymbol{x}_{\boldsymbol{U}\backslash\boldsymbol{u}}\right)$ - defines $(2^{|\boldsymbol{u}|}-1)$ partial rate sums.
    - There are also U trivial faces for positive bits/subsymbol $b_u \geq 0$, so really 2^u -1+U faces that bound $\mathcal{A}(\boldsymbol{b},p_{\boldsymbol{x}})$.
- A **vertex** corresponds to a specific $\boldsymbol{b} = \mathbb{I}$ for a specific order $\boldsymbol{\pi}$; *examples include for* $U = 2$ *:*

$$\begin{bmatrix}\mathbb{I}(\boldsymbol{x}_2;\boldsymbol{y}\,/\boldsymbol{x}_1)\\ \mathbb{I}(\boldsymbol{x}_1;\boldsymbol{y}\,)\end{bmatrix}\quad\begin{bmatrix}\mathbb{I}(\boldsymbol{x}_1;\boldsymbol{y}\,/\boldsymbol{x}_2)\\ \mathbb{I}(\boldsymbol{x}_2;\boldsymbol{y})\end{bmatrix}.$$

In general, $\exists\ U!$ MAC vertices for a specific $p_{\boldsymbol{x}}$.

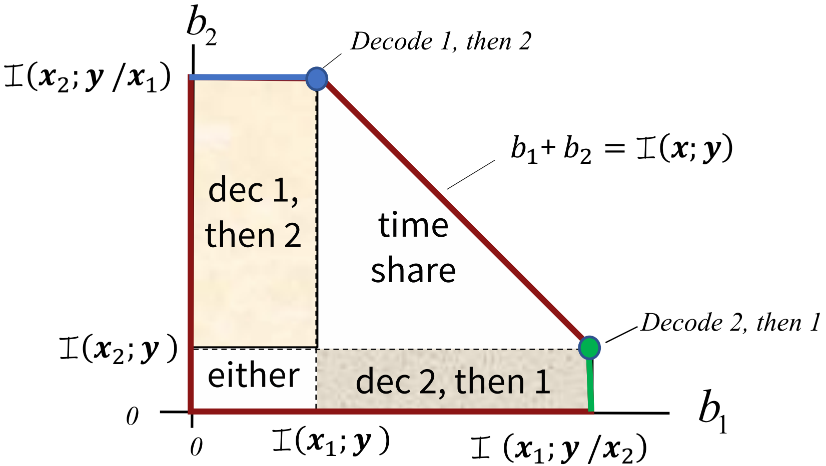
Chain-Rule Decoder



Successive Decoding or …
Generalized Decision Feedback Eq
(or "NOMA")

- For the given order, the receiver first decodes all lower-indexed users and then decodes the current user.
- Since there is only one order, analysis can relabel users and avoid all the $\pi^{-1}(\cdot)$ notation.
- There is no loss of generality.

A 2-user MAC rate region

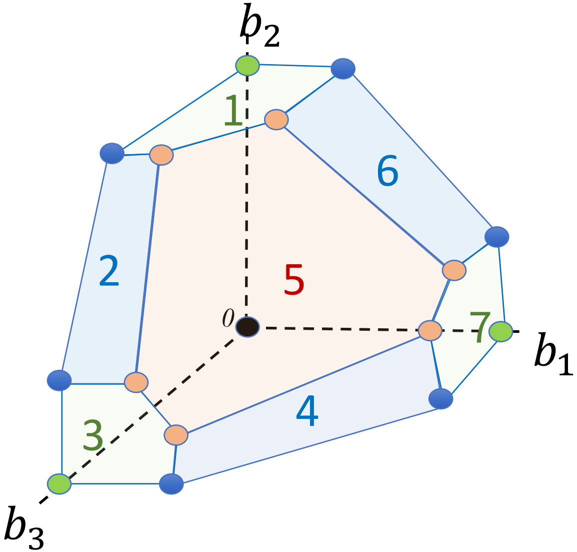


Specific to a $p_{\boldsymbol{xy}}$

- Pentagon – 5 vertices and 5 faces
  - $2^U - 1 + U$ Faces are the $\mathbb{I}\big(\boldsymbol{x_u}\,;\boldsymbol{y}\,/\boldsymbol{x}_{\boldsymbol{U\backslash u}}\big)$ & $b_u \geq 0$
  - $U! = 2$ vertices are the both-user order points $\boldsymbol{\pi}$
    - 2 more are single-user points, one for each user
    - 1 more is the origin
    - 5 total

- b_2 vertex (short blue line) decodes 1 first (given), then 2 as if 1 is "cancelled."
  - Similar statement holds for b_1 vertex (and short green) line.
- Line with slope -1 is **time-share or really vertex-share**; it also is constant maximum rate sum (for this $p_{\boldsymbol{xy}}$).
  - There are two codes for each user (4 codes); This is example of user components (or subusers, sometimes called "**rate splitting**")

A 3-user rate region



- Decahedron – 10 faces
  - $2^U - 1 + U$ Faces are the $\mathcal{I}(\boldsymbol{x}_{\boldsymbol{u}}\,; \boldsymbol{y} \,/ \boldsymbol{x}_{U\setminus\boldsymbol{u}})$
  - $U! = 6$ vertices (rose) are the 3-user order points $\boldsymbol{\pi}$

🟠 = vertex for 3 users $U!\ (= 6)$
🔵 = vertex for 2 users $U!\ (= 6)$
🟢 = vertex for 1 user $U\ \ (= 3)$
⚫ = vertex for 0 users $\ (= 1)$

$\sum_{k=0}^{U} \binom{U}{k} \cdot k!$ =16 vertices

- b_2 horizontal plane (pentagon 1) decodes 1 and 3 first (given), then 2 as if 1 and/or 3 are "cancelled."
  - 1 and 3 form a two-user horizontal pentagon region.
  - Similar statements hold for b_1 vertical-plane pentagon and b_3 facial-plane pentagon.
- Rose plane normal to $\mathbf{1} = [1 \quad 1 \quad 1]^*$ dimension-shares the rose vertices; it has constant maximum rate sum (for this $p_{\boldsymbol{xy}}$).
  - There could be as many as 3 codes/components for each user on a time-share of vertices.
- The blue and green planes may also dimension-share vertices.
- $\mathcal{A}(\boldsymbol{b}, p_{\boldsymbol{x}})$ is the entire interior plus faces and vertices. Any point outside violates at least one single-user mutual-information bound.

MAC Capacity Region

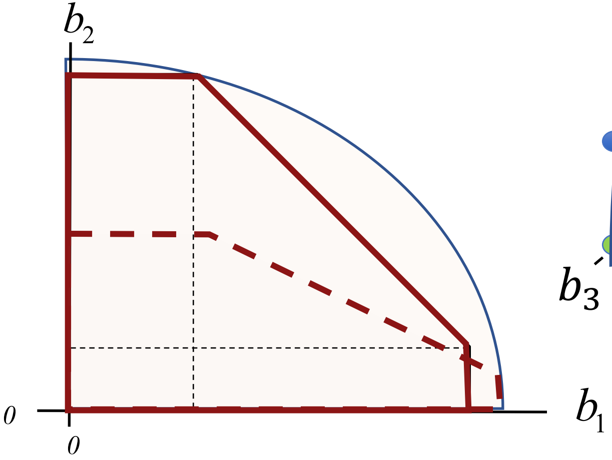
- More formally, the MAC's achievable region is bounded by hyperplanar regions

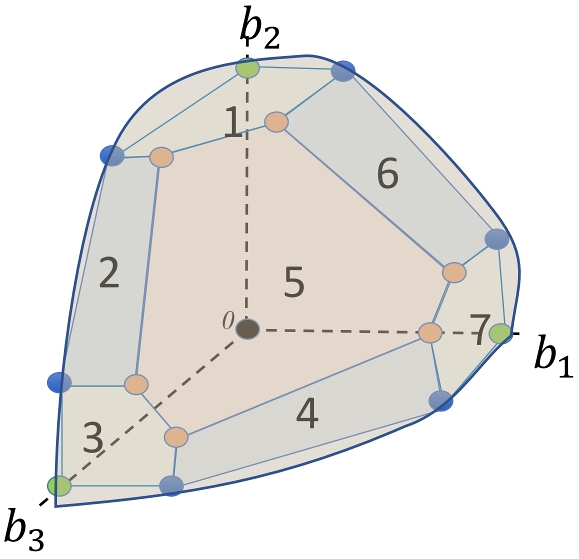
$$\mathcal{A}(\boldsymbol{b}, p_{\boldsymbol{x}}) = \bigcap_{\boldsymbol{u}\subseteq \boldsymbol{U}} \left\{ \boldsymbol{b} \,\middle|\, 0 \le \sum_{i\in\{\boldsymbol{u}\}} b_i \le \mathcal{I}\left(\boldsymbol{x}_i ; \boldsymbol{y} / \boldsymbol{x}_{\boldsymbol{u}\setminus i}\right) \right\} .$$

- The vertices are where hyperplanes intersect at a point.
  - Or, lines (smaller dimensional hyperplanes) may also bound.

- Convex hull over all multi-user input probability distributions $p_{\boldsymbol{x}}$ is

$$\mathcal{C}_{MAC}(\boldsymbol{b}) = \bigcup_{\boldsymbol{u}\subseteq p_{\boldsymbol{x}}}^{conv} \mathcal{A}(\boldsymbol{b}, p_{\boldsymbol{x}}).$$



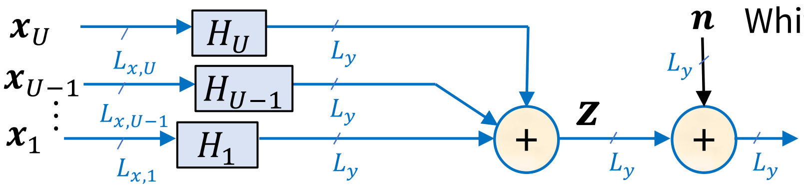


Scalar Gaussian MAC

PS4.3 - 2.25 Time-Division Multiplexing region

Section 2.7.2

General Gaussian MAC



White Gaussian Noise

$R_{\boldsymbol{nn}} = \sigma^2 \cdot I$

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{n}$$

$$\underset{L_y\times 1}{\boldsymbol{y}} = \underbrace{[H_U \quad H_{U-1} \quad \cdots \quad H_1]}_{\substack{H \\ L_y \times \mathfrak{L}_x}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{x}_U \\ \boldsymbol{x}_{U-1} \\ \vdots \\ \boldsymbol{x}_1 \end{bmatrix}}_{\substack{\boldsymbol{x} \\ \mathfrak{L}_x \times 1}} + \boldsymbol{n}$$

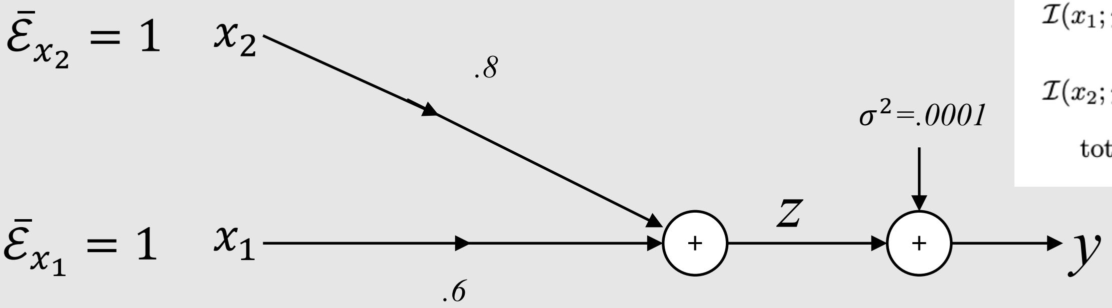
More generally, variable-dim inputs have

$$\mathfrak{L}_x = \sum_{u=1}^{U} L_{x,u} \sim U \cdot L_x$$

- Inputs are independent.
  - $R_{\boldsymbol{xx}}$ is block diagonal.
  - Only 1 output and 1 noise.
- One Receiver will estimate all inputs.
  - It can do so in any order.
  - "Given an input" $\boldsymbol{x}_u$ means cancel it from $\boldsymbol{y}$.
  - This does not necessary mean subtract $H_u \cdot \boldsymbol{x}_u$ from $\boldsymbol{y}$
    - Unless $L_y = L_{x,u} = 1$; or H_u is diagonal and noise is white.

$\wp_H$ is the matrix H's **rank:**
= number of linearly independent rows (or columns)
= # of non-zero singular values.

Example

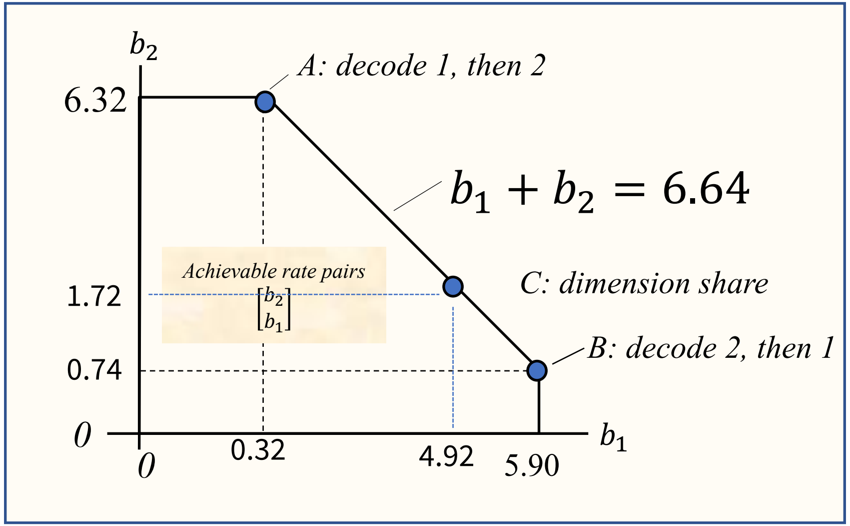


$$\begin{aligned}
\mathcal{I}(x_1;y) &= \frac{1}{2}\log_2\left(1+\frac{.36\cdot 1}{.0001+.64}\right) = .32 \text{ bits/dimension} \\
\mathcal{I}(x_2;y) &= \frac{1}{2}\log_2\left(1+\frac{.64\cdot 1}{.0001+.36}\right) = .74 \text{ bits/dimension} \\
\text{total} &= 1.06 \text{ bits/dimension} \quad ,
\end{aligned}$$

$$\mathcal{I}(x_2;y/x_1) = \frac{1}{2}\log_2(1+\text{SNR}_2) = \frac{1}{2}\log_2\left(1+\frac{.64\cdot 1}{.0001}\right) = 6.32 \text{ bits/dimensio}$$

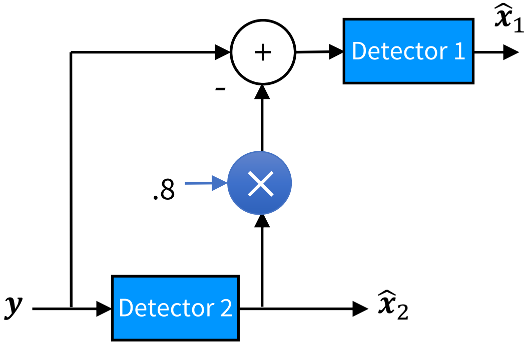
$$\mathcal{I}(x_1;y/x_2) = \frac{1}{2}\log_2(1+\text{SNR}_1) = \frac{1}{2}\log_2\left(1+\frac{.36\cdot 1}{.0001}\right) = 5.90 \text{ bits/dimension}$$

- Point C is ¼ share B and ¾ share A.

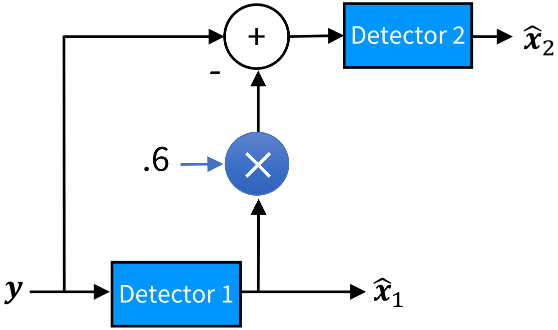


Successive decoding for scalar example

Successive decoder
(order favors user 1)



Successive decoder
(order favors user 2)



- Only 2 orders are possible for 2 users.
- $\exists\ U!$ in general (corresponding to each possible order).
- The last user is "favored" in decoding (first accepts other as noise).

2 – User Scalar $L_x = L_y = 1$

General formula
Scalar MAC

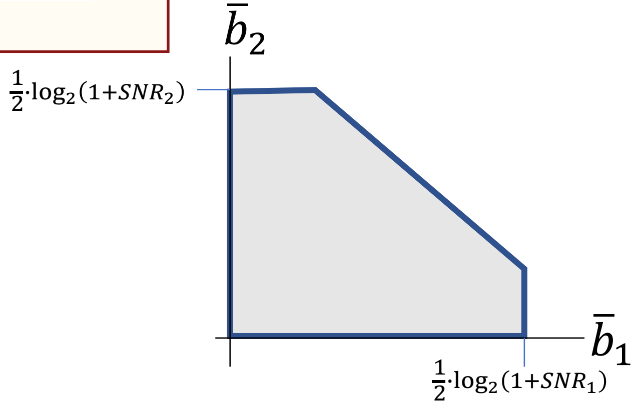
$$\bar{I}(x_{\boldsymbol{u}}; y/x_{\boldsymbol{U}\setminus\boldsymbol{u}}) = \frac{1}{2}\log_2\left(1+\frac{\sum_{i\in\boldsymbol{u}}\bar{\mathcal{E}}_i\cdot|H_i|^2}{\sigma^2}\right)$$

2 users

$$\begin{aligned}\mathrm{SNR}_1 &= \frac{\mathcal{E}_1\cdot|h_1|^2}{\sigma^2}\\ \mathrm{SNR}_2 &= \frac{\mathcal{E}_2\cdot|h_2|^2}{\sigma^2}\\ \mathrm{SNR} &= \frac{\mathcal{E}_1\cdot|h_1|^2+\mathcal{E}_2\cdot|h_2|^2}{\sigma^2}\end{aligned}$$

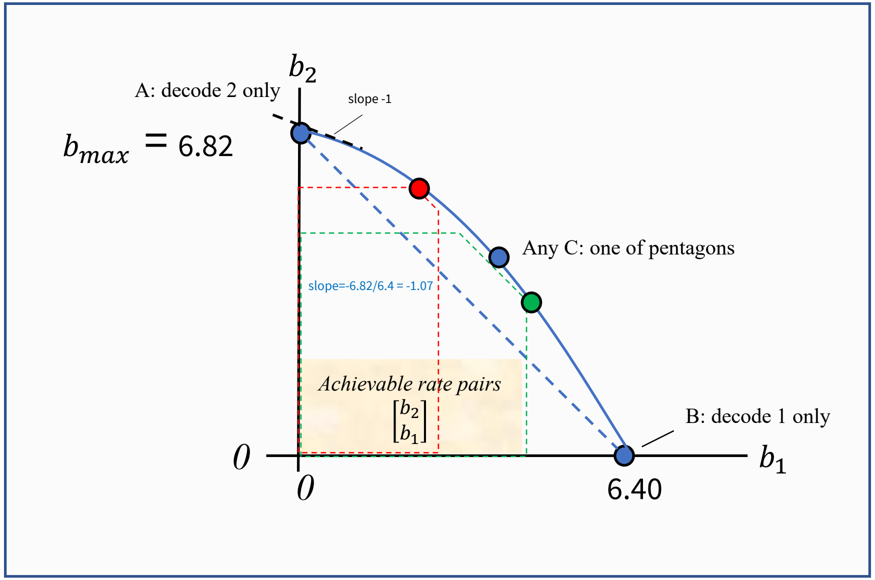
$$\begin{aligned}\bar{b}_1 &\le \frac{1}{2}\log_2\left(1+\mathrm{SNR}_1\right)\\ \bar{b}_2 &\le \frac{1}{2}\log_2\left(1+\mathrm{SNR}_2\right)\\ \bar{b}_1+\bar{b}_2 &\le \frac{1}{2}\log_2\left(1+\mathrm{SNR}\right)\end{aligned}$$

- 3 planes (lines) ~ 3 SNR's
- 1 sum rate
- Nonzero individual rates

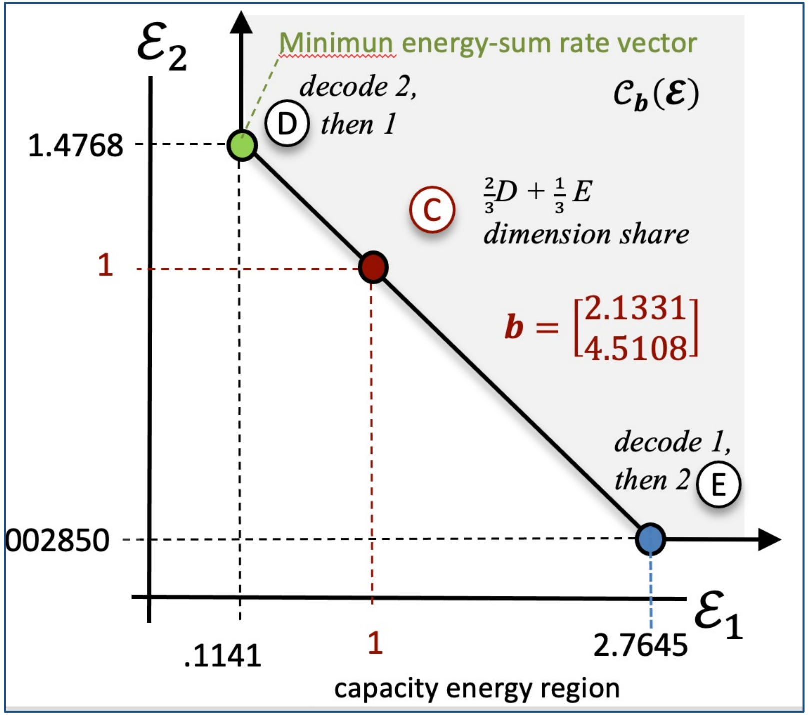


Energy-Sum MAC

- ∃ a single energy constraint $\mathcal{E}_1 + \mathcal{E}_2 \leq \mathcal{E}_x$ (instead of 2 constraints)
- Capacity region is a union of pentagons (and 1 triangle),
  - one for each combination of energies that add to total.



- Or ∃ an Energy-Capacity Region,
  - one for each bit vector ***b*****.**

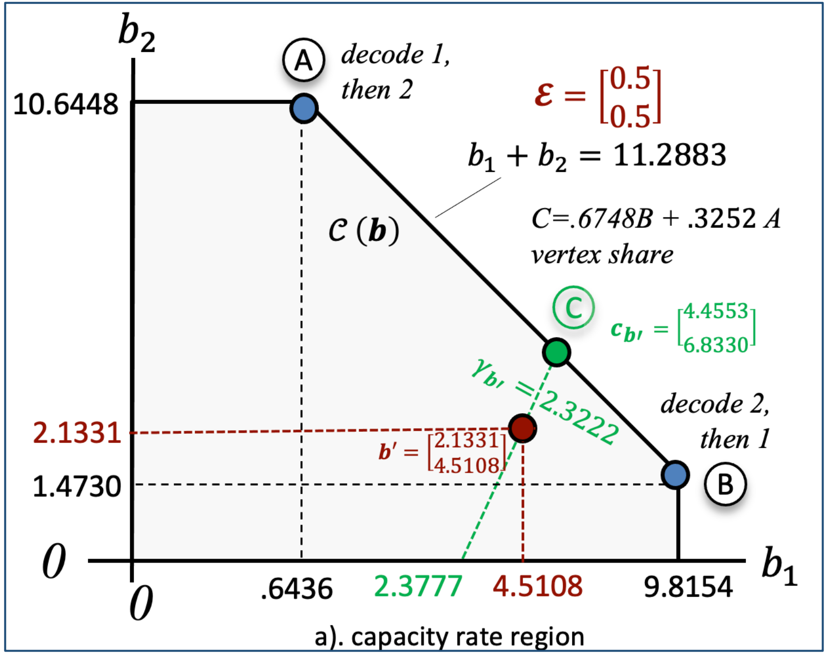


Time-Sharing Conundrum (Coding Theorist's Fallacy in disguise)

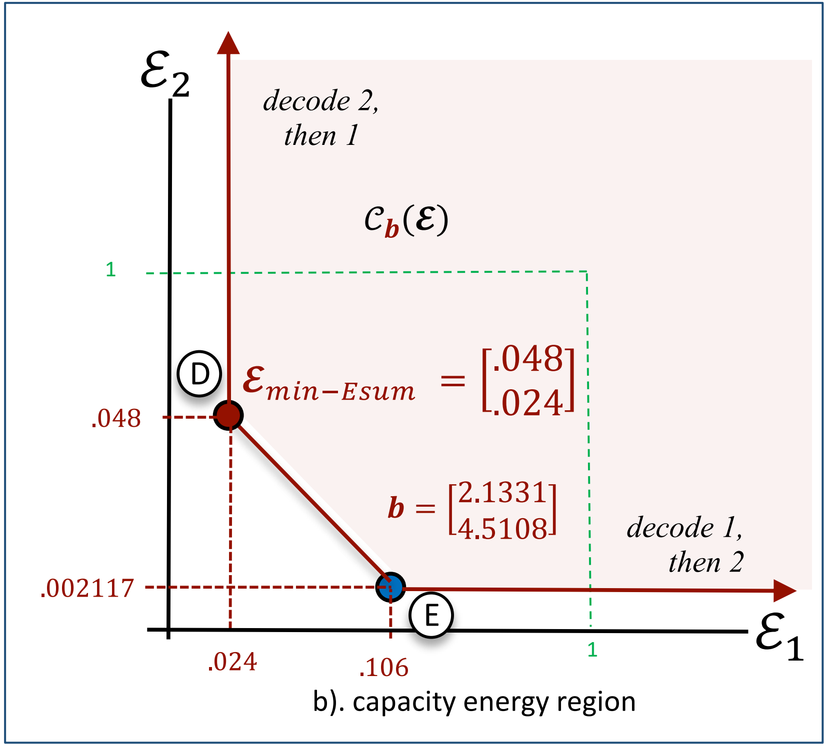
- What is meaning of time-sharing? (“convex hull”)
  - The different codes correspond to user components, each used for its respective fraction of “time” (dimensions).
- With time-sharing, what does $\mathcal{E}_u$ mean?
  - Energy constant at $\mathcal{E}_u$: Is this then for every symbol/subsymbol in the sharing?
  - Or the average over the “time-shared” subsymbols?
- The second instance of averaging often enlarges the capacity region.
- So, “time-sharing” is somewhat ill-defined.
  - Despite most info/com texts on MAC using it.
- Lecture 4's Separation Theorem actually allows different mutual information $\mathcal{I}_A$ and $\mathcal{I}_B$ to be represented by their average information – *for the same user.*
  - $\mathcal{I} = \alpha \cdot \mathcal{I}_A + (1-\alpha) \cdot \mathcal{I}_B$.
  - ST uses same constellation with average $\boldsymbol{b}$ for each symbol, possible very large$|C|$).
  - If the shared same-user codes correspond to vertices with different orders, this creates issues for Separation Thm application.
    - But it is still possible, although the successive decoding needs to become “iterative-user” successive decoding (“turbo xtalk cancelling”).
    - Of course, each user can use subusers; each user has subcode for A and for B, but then constellation varies.

Conundrum: double-sampling-rate Example

[80 60] channel again at twice sampling rate



a). capacity rate region



b). capacity energy region

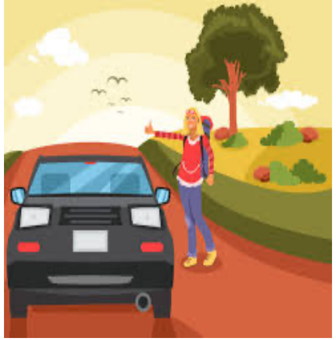
- The vector $\boldsymbol{b}$ is now in the interior of the region, although is it the same channel?
  - The time-sharing needs to occur at the same sampling rate, meaning the symbol period increases, for the original $\mathcal{C}(\boldsymbol{b})$ to apply.

Primary and Secondary Components (E-sum MAC)

Primary-user (subuser): has nonzero energy for E-sum MAC's maximum rate.

Secondary-user (subuser): has zero energy for E-sum MAC's maximum rate.

- Primary subusers dominate with largest pass-space gains (dimensions used for component).
- Secondary subusers "free load" on these primary-component dimensions.



Previous example (.8 and .6):

The pass-space is just one dimension ($L_y = 1$).

user 2 is all primary (.8) ; user 1 is all secondary (.6).

max sum is 6.82 (all energy on user 2).

Rate-sum decreases if secondary user components energize (see slide L7:24).

How Many Primary Subusers (E-sum MAC)?

- The MAC has no more than $U^o \leq \wp_H$ primary subusers, to find them first do U SVD's:

$$\widetilde{H}_u = R_{noise}^{-1/2}(u) \cdot H_u = F_u \cdot \Lambda_u \cdot M_u^* \quad \text{with} \quad \left|\widetilde{H}_u\right| \triangleq \prod_{l=1}^{\wp_{H_u}} \lambda_{u,l} > 0 \,.$$

- Each user can excite up to $\wp_{H_u}$ possible independent dimensions per subsymbol.
  - The $R_{noise}(u)$ includes all other user components' crosstalk for whatever energies they use (knows all $R_{\boldsymbol{xx}}(u)$'s).
  - Each user can have vector-coding modulator without loss, or some linear combination of the pass-space dimensions.

- For the channel gains in the VC,

$$g_u = \left|\widetilde{H}_u\right|^2 = \prod_{l=1}^{\wp_{H_u}} \lambda_{u,l} \qquad .$$

- The **primary-user components** correspond to those energized in achieving max rate sum on the E-sum MAC. All others are **secondary-user components**.
  - With MIMO MAC, components can be viewed also as "subuser components."

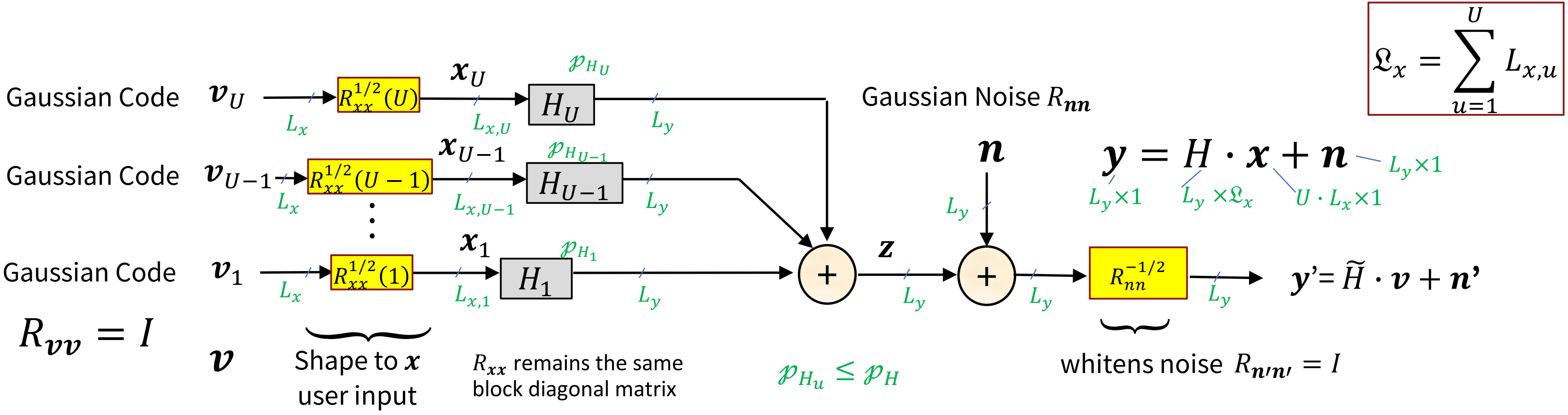
- The "components" idea is helpful when individual users' transmitters have >1 dimension (MIMO), via
  - time-sharing, frequency-sharing, and/or space sharing.

Vector Gaussian MAC

PS4.4 - 2.24 MAC regions

Section 2.7.2.2

MAC ~ single channel with white input



$$\mathfrak{L}_x = \sum_{u=1}^{U} L_{x,u}$$

$$y = H \cdot x + n$$

$$y' = \tilde{H} \cdot v + n'$$

$R_{vv} = I$

Shape to x user input

R_{xx} remains the same block diagonal matrix

$\wp_{H_u} \leq \wp_H$

whitens noise $R_{n'n'} = I$

- This normalizes (redefines, not $R_{noise}(u)$ here) individual user MAC channels to $\tilde{H}_u \triangleq R_{nn}^{-1/2} \cdot H_u \cdot R_{xx}^{1/2}(u)$.
- **Normalized MAC** is now $y' = \tilde{H} \cdot v + n'$, where:
  - New input(s) is (are) "white" , $R_{vv} = I$.
  - New noise is "white" , $R_{n'n'} = I$.
  - We drop the primes going forward; $y = \tilde{H} \cdot v + n$ → $\tilde{H}$ ′ s dimensions carry the information (secondary may freeload).

Cholesky Factorization

- This is related to MMSE linear-prediction (see Appendix D).

- Positive definite Hermitian symmetric matrix factors as $R = G^* \cdot S \cdot G$, where
  - G is upper triangular monic (1’s on diagonal), &
  - S is positive real diagonal matrix (even if R is complex).

- Matlab command is “**chol**” for *lower* × *upper* (lower is upper*) – **produces upper.**
  - Gtemp=chol (R)
  - G= inv(diag(diag(Gtemp)))*Gtemp
  - S= diag(diag(Gtemp))*diag(diag(Gtemp))

- See website’s lohc.m program for *lower* × *upper* .

```
>> R=[2 1
1 2];
>> Gtemp=chol(R) % =
   1.4142    0.7071
        0    1.2247
>> G= inv(diag(diag(Gtemp)))*Gtemp %=
   1.0000    0.5000
        0         1.0000
 S= diag(diag(Gtemp))*diag(diag(Gtemp)) % =
   2.0000         0
        0    1.5000
>> G'*S*G % =
   2.0000    1.0000
   1.0000    2.0000
```

Forward and Backward Canonical Channels

- **Forward Canonical** Channel is
  - the output of matched-filter matrix.

$$\boldsymbol{y}' = \underbrace{\widetilde{H}^* \cdot \widetilde{H}}_{R_f} \cdot \boldsymbol{\nu} + \underbrace{\widetilde{H}^* \cdot \boldsymbol{n}}_{\boldsymbol{n}'} ,$$

- **MMSE Estimator** for backward channel

$$R_{\boldsymbol{\nu}\boldsymbol{y}'} \cdot R^{-1}_{\boldsymbol{y}'\boldsymbol{y}'} = R_f \cdot [R_f \cdot R_f + R_f]^{-1} = [R_f + I]^{-1} = R_b$$

- **Backward Canonical** Channel

$$\boldsymbol{\nu} = R_b \cdot \boldsymbol{y}' + \boldsymbol{e} \qquad R_{\boldsymbol{ee}} = R_b$$

- Use **Cholesky** on backward-channel inverse $R_b^{-1} = R_f + I = G^* \cdot S_0 \cdot G$

$$\boldsymbol{y}'' = S_0^{-1} \cdot G^{-*} \boldsymbol{y}'$$ (algebra)

$$\boxed{\boldsymbol{y}'' = G \cdot \boldsymbol{\nu} - \boldsymbol{e}'}$$

where $R_{\boldsymbol{e}'\boldsymbol{e}'} = S_0^{-1}$.

Back Substitution

- Not quite ML/MAP, but **successive decoding,**
  - but **canonical** **–** achieves $\mathcal{I}$ reliably, each user**,**
  - **if decisions are correct** (asymptotic MMSE = MAP again).
  - If $\Gamma > 0$ dB, then iterative decoding that →ML may be needed.
- Each of these is MMSE based,
  - which is related to conditional $\mathcal{I}$.
- The decoder is much simpler (“GDFE”).
- SNR (biased) for each decision/dimension is $S_{0,u,l}$.
- But also

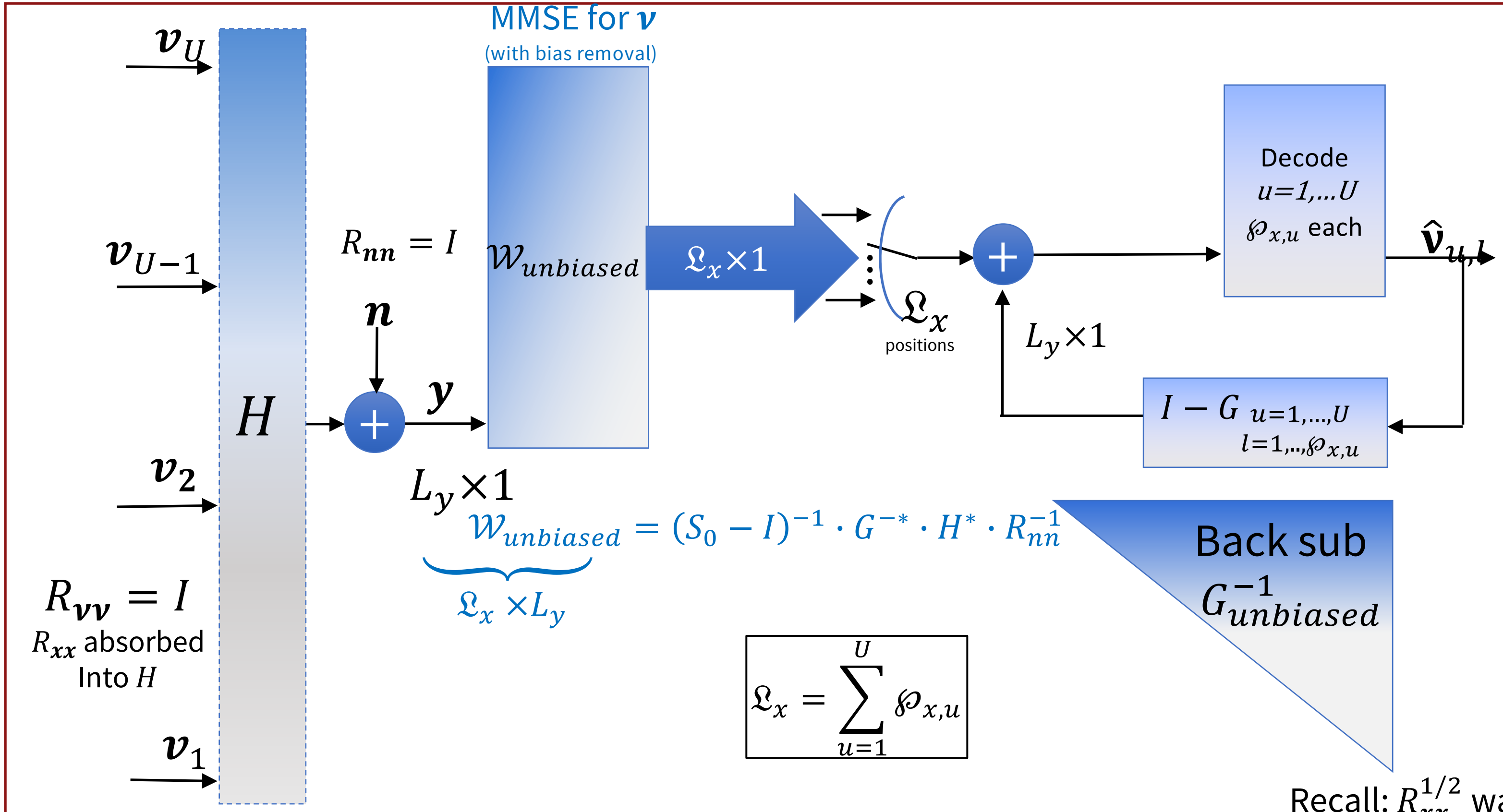
$$G = \begin{bmatrix} 1 & g_{U,U-1} & \cdots & g_{U,2} & g_{U,1} \\ 0 & 1 & \cdots & g_{U-1,2} & g_{U-1,1} \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & g_{2,1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \hat{\boldsymbol{\nu}}_1 &= \text{decision}\left(\boldsymbol{y}''_1\right) \\ \hat{\boldsymbol{\nu}}_2 &= \text{decision}\left(\boldsymbol{y}''_2 - g_{2,1}\cdot\hat{\boldsymbol{\nu}}_1\right) \\ \vdots \quad & \vdots \quad \vdots \\ \hat{\boldsymbol{\nu}}_{U'} &= \text{decision}\left(\boldsymbol{y}''_U - \sum_{i=1}^{U-1} g_{U,i}\cdot\hat{\boldsymbol{\nu}}_i\right) \end{aligned}$$

$$\mathcal{I}(\boldsymbol{x};\boldsymbol{y}) = \log_2(|\underbrace{\tilde{H}^*\tilde{H}+I}_{R_b^{-1}}|) = \log_2|S_0| = \log_2\left\{\prod_{u=1}^{U'}\prod_{\ell=1}^{L_{x,u}} SNR_{mmse,u,\ell}\right\} \text{bits / complex symbol} \ .$$

New parallel “independent” subchannels

Vector MAC Receiver



MMSE for $\boldsymbol{v}$

(with bias removal)

$\boldsymbol{v}_U$

$\boldsymbol{v}_{U-1}$

$\boldsymbol{v}_2$

$\boldsymbol{v}_1$

H

$R_{nn} = I$

$\boldsymbol{n}$

$\boldsymbol{y}$

$L_y \times 1$

$\mathcal{W}_{unbiased}$

$\mathfrak{L}_x \times 1$

$\mathfrak{L}_x$ positions

Decode $u=1,\ldots U$ $\wp_{x,u}$ each

$\hat{\boldsymbol{v}}_{u,l}$

$L_y \times 1$

$I - G_{\substack{u=1,\ldots,U \\ l=1,\ldots,\wp_{x,u}}}$

$$\underbrace{\mathcal{W}_{unbiased}}_{\mathfrak{L}_x \times L_y} = (S_0 - I)^{-1} \cdot G^{-*} \cdot H^{*} \cdot R_{nn}^{-1}$$

Back sub $G_{unbiased}^{-1}$

$R_{\boldsymbol{vv}} = I$

$R_{\boldsymbol{xx}}$ absorbed Into H

$$\boxed{\mathfrak{L}_x = \sum_{u=1}^{U} \wp_{x,u}}$$

Recall: $R_{\boldsymbol{xx}}^{1/2}$ was absorbed into $\tilde{H}$

- Each user/decoder achieves $\mathcal{I}(\boldsymbol{\nu}_u; \boldsymbol{y} / [\boldsymbol{\nu}_{u-1} \quad \cdots \quad \boldsymbol{\nu}_1])$

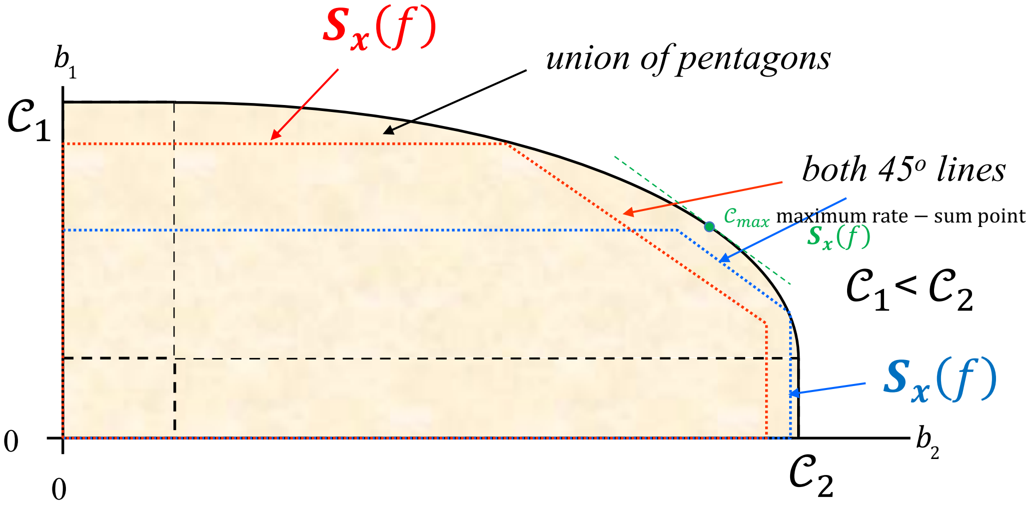
End Lecture 7

(back-up material FYI)

Capacity Region for continuous-frequency-indexed channels

Sections 2.7.4.1-2

$\mathcal{C}(b)$ is Union of $S_x(f)$-indexed Pentagons



$$\bar{b} = \sum_{u=1}^{U} \bar{b}_u \le \bar{\mathcal{I}}(\boldsymbol{x};\boldsymbol{y}) = \int_{-\infty}^{\infty} \frac{1}{2} \cdot \log_2 \left[1 + \frac{\sum_{u=1}^{U} S_{x,u}(f) \cdot |H_u(f)|^2}{S_n(f)} \right] df$$

Calculus of variations again, decomposes into U water-fills, each with other users as noise – more details in L9.

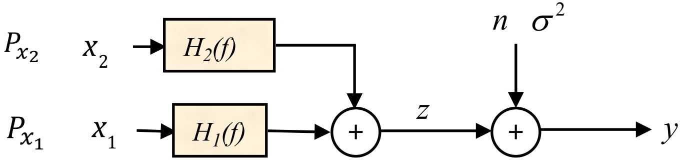
$$S_{x,u}(f) + \frac{\sigma^2 + \sum_{i \neq u} S_{x,i}(f) + |H_i(f)|^2}{|H_u(f)|^2} = K_u$$

Simultaneous water-filling
→ Maximum rate sum

- Each pentagon corresponds to an $\boldsymbol{S}_{\boldsymbol{x}}(f)$ choice.
  - The pentagons become triangles for the sum-energy MAC.

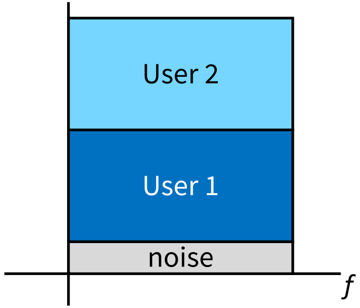
- The union (convex hull is union when inputs are Gaussian) can dimension-share in frequency as $N \to \infty$.

MT MAC

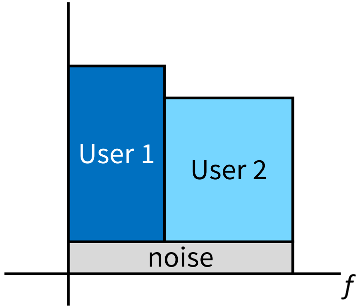


- The users have continuous-time/frequency channels → use MT on each, theoretically.
- This really means dimensionality is infinite (or very large) so “dimension-sharing” may be inherent.
- SWF applies, but with some interpretation (like power instead of energy, etc and power per dimension instead of power-spectral density, etc.)

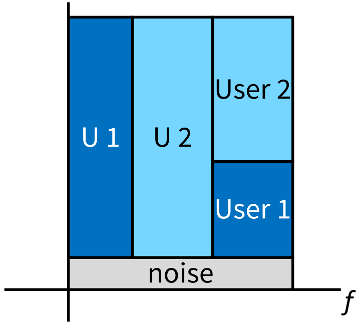
Decoders and SWF



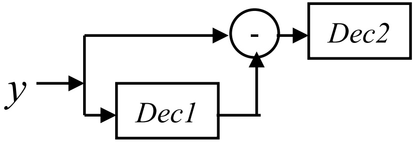
a). both flat

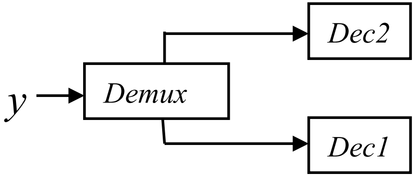


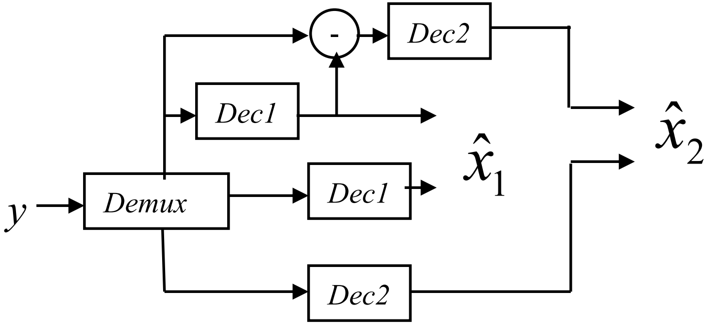
b). FDM



c). mixed

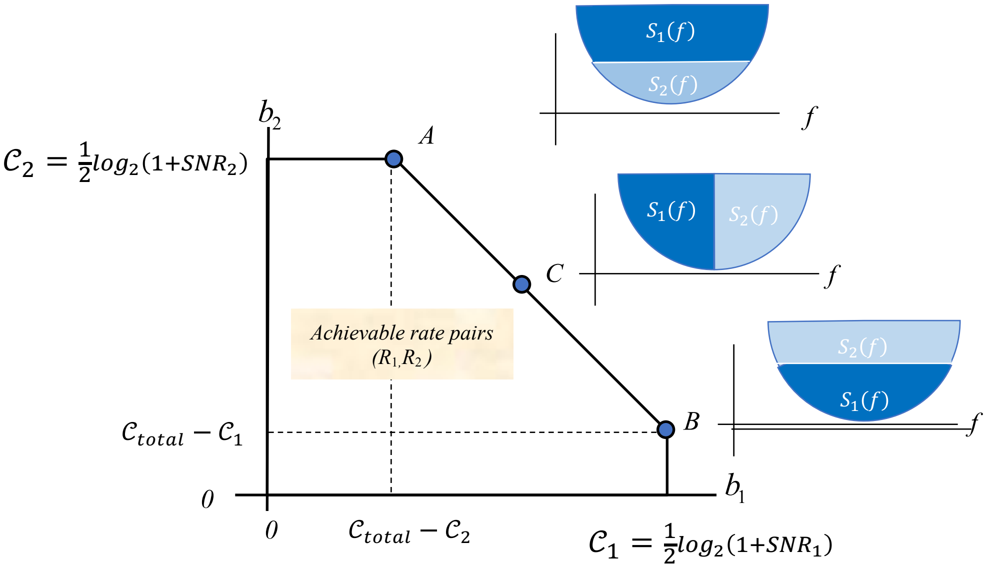






- FDM is clearly simplest decoder for max rate sum case.

Symmetric 2-user channel and SWF



- Symmetric means $H_1(f) = H_2(f)$ (noise is one-dimensional and added to sum)
- Each of points A, B, and C have different SWF spectra – all have same (max) rate sum