STANFORD

Lecture 6

Multiuser Channels

and the Chain Rule

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Announcements & Agenda

- Announcements
  - Mid term is April 29
  - If you want early start on PS4, here it is.

- Problem Set 4 = PS4 (due April 29)
  1. 2.21 Multiuser Channel Types
  2. 2.22 Multiuser Detector Margin
  3. 2.23 Mutual-Information Vector
  4. 2.24 Time-Division Multiplexing region
  5. 2.25 MAC regions

- Agenda
  - Multi-User (MU) Introduction
    - Where used?; What is a multi-user data rate?; order & decodability
  - The 3 basic MU types and the matrix AWGN
  - Rate Bounds and Detection
  - General MU Capacity Region and other optima

Multiuser (MU) Introduction

(definitions and fundamentals)

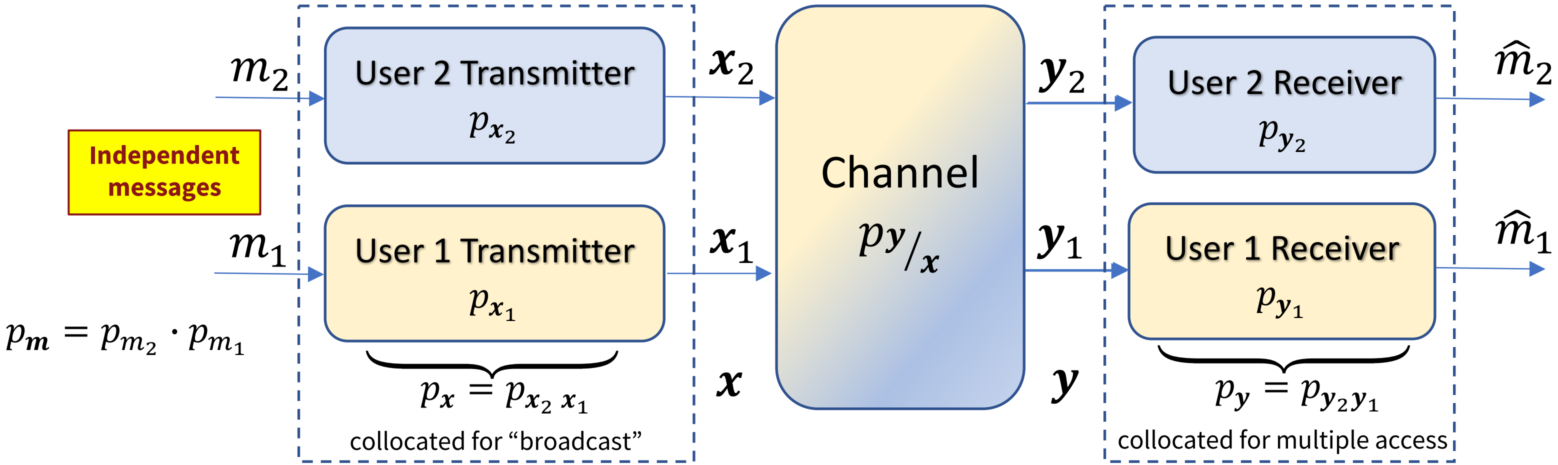
Section 2.6 intro

U>1 users

- Downlink/stream – one to many (“broadcast”)
- Uplink/stream – many to one (“multiple access)
- Relay signals (“mesh”)
- Overlapping combinations (Wi-Fi, or cell, or really all) – “interference”

MU Mathematical Model (Section 2.6)

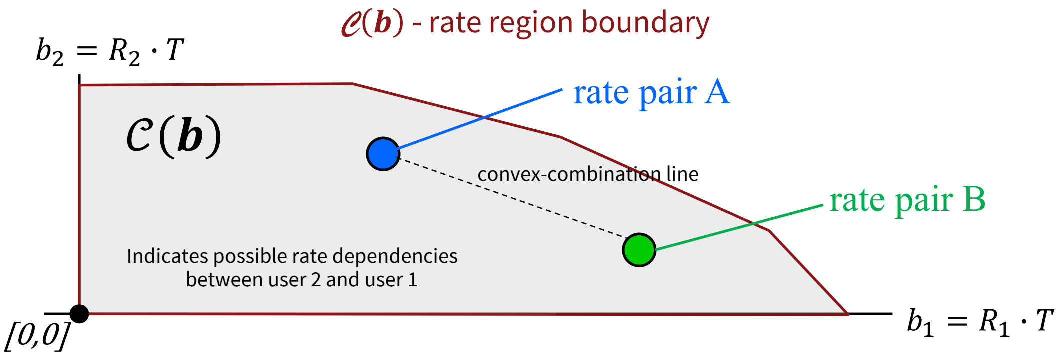
- There is a joint probability distribution $p_{\boldsymbol{xy}}$ that determines all marginals (e.g., input) and conditionals (channel), $p_{\boldsymbol{y}/\boldsymbol{x}} = \frac{p_{\boldsymbol{xy}}}{p_{\boldsymbol{x}}}$.



- User 1 & 2's data rates are mutually dependent (otherwise just two single-user channels).
- $b \rightarrow \boldsymbol{b} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} = \boldsymbol{R} \cdot T = \begin{bmatrix} R_2 \cdot T \\ R_1 \cdot T \end{bmatrix}$; the bits/sub-symbol becomes a U −dimensional vector, $u = 1, \ldots, U$.
- Single-user is a (degenerate) subset of multiuser.

The Rate Region

- "Reliably decodable" set of users' bits/subsymbol vectors that can be achieved $P_e \rightarrow 0\ (AEP)$.



- All "convex combinations" (on the line connecting points) must trivially be achievable too.

- What is $\mathcal{C}(\boldsymbol{b})$ if independent single-user channels?
  "crosstalk free"

  rectangle (2), prism (3), Orthotope (U)

- The region is "convex hull" (union) of achievable points over all "allowed" $p_{\boldsymbol{xy}}$, or really over $p_{\boldsymbol{x}}$,
  - because $p_{\boldsymbol{y}/\boldsymbol{x}}$ (the general MU channel description) is given (fixed).

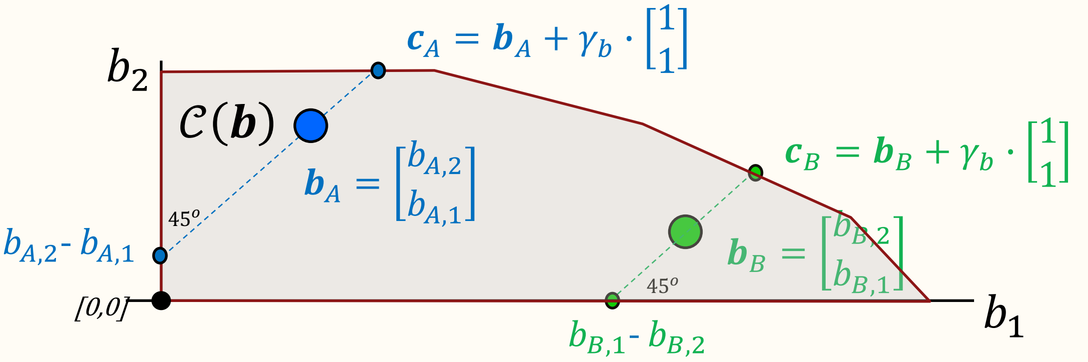
Multiuser Margin

- Single-user (energy) margin $\bar{b} = \frac{1}{2} \cdot \log_2\left(1 + \frac{SNR}{\Gamma \cdot \gamma_m}\right)$ measures safety for $\bar{b}$ if SNR changes.

$$\gamma_m = \frac{1}{\Gamma} \cdot \frac{SNR}{2^{2\bar{b}} - 1}$$

- The **bit gap** is $\gamma_b = \mathcal{C} - b$ where $\bar{\mathcal{C}} = \frac{1}{2} \cdot \log_2(1 + SNR) = \bar{b} + \bar{\gamma}_b$ - so measures rate gap to $\bar{\mathcal{C}}$.
  - $\Gamma \cdot \gamma_m \cong 6 \cdot \bar{\gamma}_b$ dB , $\bar{\gamma}_b = 0$ if the code achieves capacity (6 dB/bit-dimension).

- **Multiuser bit gap** measures to $\boldsymbol{c}_{\boldsymbol{b}'} \in \mathcal{C}(\boldsymbol{b})$, the rate region boundary, so $\gamma_{\boldsymbol{b}'} \cdot \mathbf{1} = \boldsymbol{c}_{\boldsymbol{b}'} - \boldsymbol{b}'$



$$\begin{aligned} b_2 &= b_1 + (b_{A,2} - b_{A,1}) \\ b_1 &= b_2 + (b_{B,1} - b_{B,2}) \end{aligned}$$

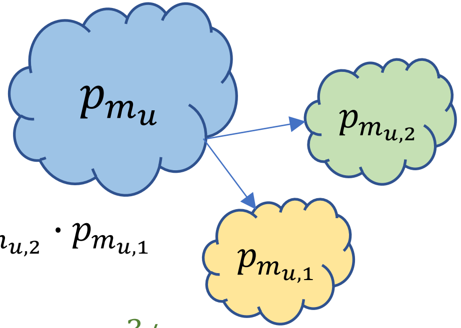
$$umax \triangleq \arg\left(\max_u b'_u\right)$$

$$b_{umax} = \left(\sum_{i \neq umax} b_i\right) + b'_{umax} - \left(\sum_{i \neq umax} b'_i\right)$$

- Multiuser (energy) margin still is then same as single-user margin. $\Gamma \cdot \gamma_m \cong 6 \cdot \bar{\gamma}_b$ dB

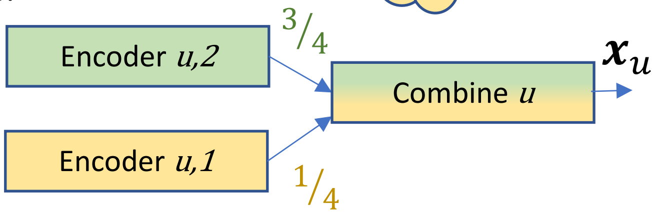
User Components (a.k.a. “time/dimension-sharing”)

- Two independent **user components or subusers** have the
  - same transmitter and same receiver (“different components of same user”).
- These two subusers (codes used) can be separately encoded and decoded.

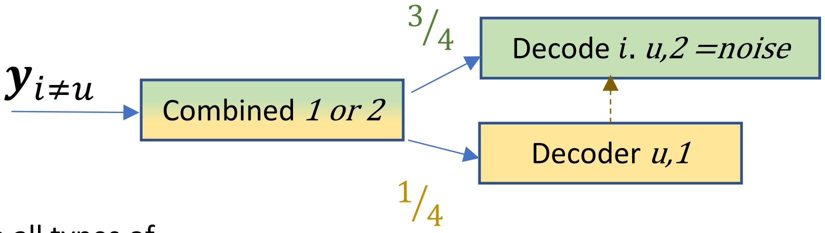


$p_{m_u} = p_{m_{u,2}} \cdot p_{m_{u,1}}$

- Bits per symbol is $b_u = b_{u,1} + b_{u,2}$.
- Other users’ receivers $i \neq u$ may decode all, none, or some of these components:
  - which they should do and remove if possible, or
  - otherwise they are averaged in marginal (remains as noise when Gaussian).



- The two subusers may simultaneously share dimensions, apportioning fractional information (or energy when Gaussian) to each.
- MIMO U can increase to $U' = \sum_{u=1}^{U} L_{x,u}$, or more generally $2^{U'} - 1$ components.
  - $L_{x,u}$ is user u’s number of spatial dimensions (antennas).



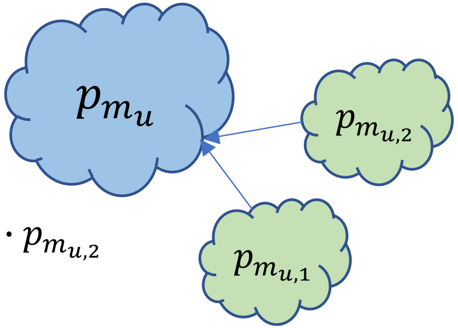
- $\mathcal{C}(\boldsymbol{b})$, and b, can also expand to $2^{U'} - 1$ dimensions:
  - Original $\mathcal{C}(\boldsymbol{b})$ adds together the sub-users’ dimensional rates,
  - and thus decreases its dimensionality.
- Some information theorists call this “**time-sharing**,”
  - but user components is more accurate and general and extrapolates to all types of dimensions and combinations, space, time, frequency, and energy.

Macro Users

- Two users (or user components) that have identical impact/influence create **a macro user.**
  - $p_{\ldots x_u \ldots . x_i \; y} = p_{\ldots x_i \ldots . x_u \; y}$ - interchange of the users does not change the joint probability distribution.
  - These two could be considered one macro user, where any partition of this macro user's rate to the two original users is feasible.

- Simple example is $y = x_1 + x_2 + n$, where both users 1 and 2 share the same energy.
  - This is ~ single-user channel with macro user $x = x_1 + x_2$, for which any division of $b = b_1 + b_2$ is possible.

- Macro/Micro users can simplify some capacity-region construction.



$$p_{m_u} = p_{m_{u,1}} \cdot p_{m_{u,2}}$$

Brief Scheduling

(Section 2.6.7, see also Appendix A, and also EE384S)

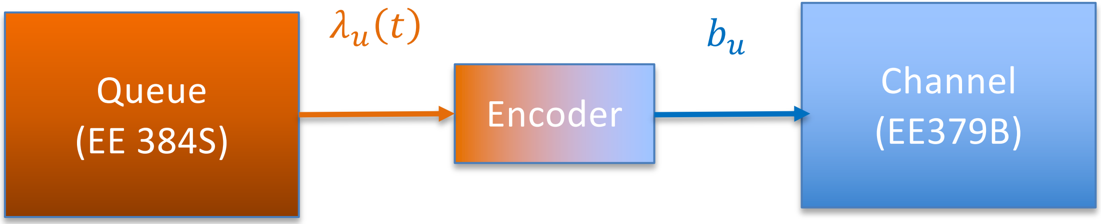
A variation "culprit" – the users' rates

Network Designer

says the "channel has a continuous rate that apportions dynamically as needed to any user, $\sum_u \lambda_u(t) = constant$."

Modem Designer

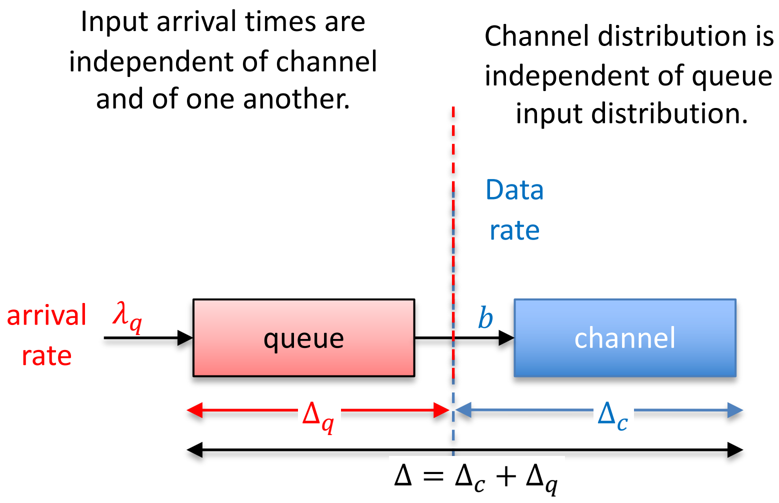
says the "source has a continuous rate that is always on, each b_u is constant."



- Neither of these two design perspectives is (always) correct.
  - See also Appendix A's queuing theory basics.

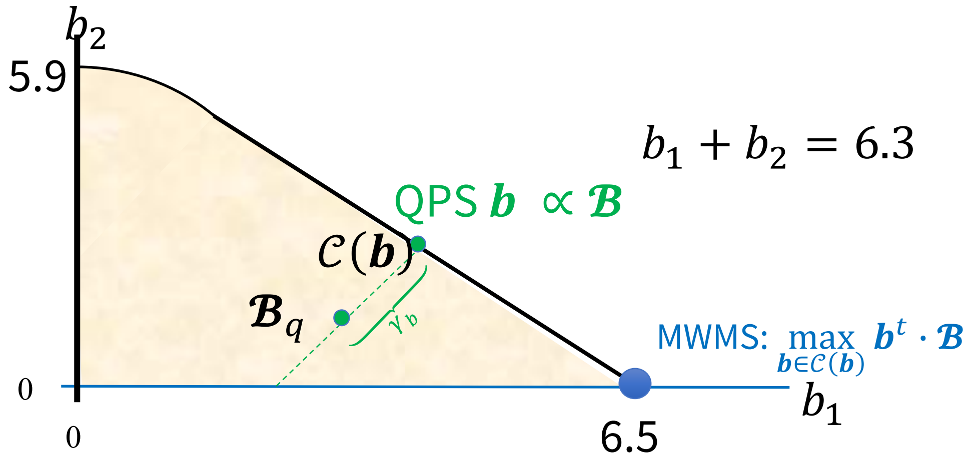
Queuing Basics

- Arrivals are independent of channel variation.
- $\mathcal{B} = \lambda_q \cdot \Delta \longrightarrow \mathbb{E}[\mathcal{B}] = \mathbb{E}[\lambda_q] \cdot \mathbb{E}[\Delta]$ = number of bits in system (Little's Theorem).
- $\mathbb{E}[\lambda_q] \leq \mathbb{E}[b]$ for stable operation.
- Multiuser Form
  - $\mathbb{E}[\boldsymbol{\mathcal{B}}] = \mathbb{E}[\boldsymbol{\lambda}_q] \odot \mathbb{E}[\boldsymbol{\Delta}]$.



A Solution: Queue Proportional Scheduling

- Send data rate in capacity region that has user rate vector as scaled version of user queue depths.



We'll learn later how to find if a point is admissible (the green QPS point on the boundary).

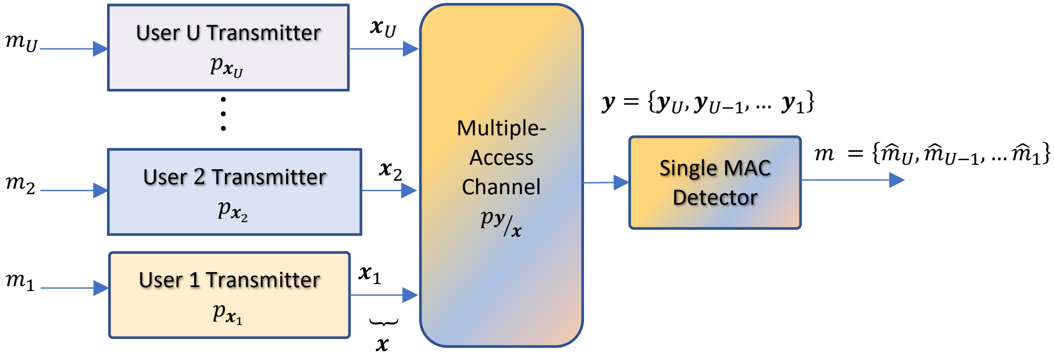
- The QPS design $\boldsymbol{b}$ is proportional to users' relative queue depths $\boldsymbol{\mathcal{B}}$, and has margin γ_b.
- QPS (Queue Proportional Scheduling) has lowest average delay of all scheduling methods.
- Less jitter than MWMS, which tends to empty the largest user into channel (QPS empties the overall queues faster).

The 3 Basic MUs & Matrix AWGN

PS4.1 - 2.21 Multiuser Channel Types

Section 2.6.1

Multiple Access Channel (MAC)



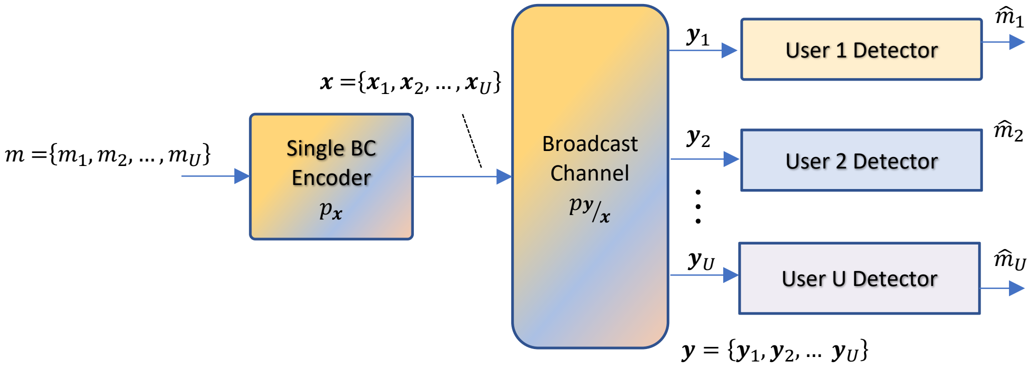
Sec 2.6.1 and 2.7

- User transmitters are in different locations (cannot coordinate to encode/modulate $\boldsymbol{x}$).
  - All use a good single-user code (see 379A, Chapter 8).
- Single receiver detects all users and:
  - separates the users,
  - reliably decodes, $P_e \to 0$, by decoding and removing some (none or all) other users first, which
  - suggests "user **order**" $\boldsymbol{\pi}$ (vector "priority") is important (decode $\boldsymbol{\pi}$'s 1st/bottom element first, … U … last at top).
  - There are $U!$ possible MAC orders.
- **Order is fundamental** to best MU design, each order has a $\boldsymbol{b}$.
  - and all potential convex combinations thereof.
  - There is also an input $p_{\boldsymbol{x}}$ choice (or code choice), and all potential convex combinations thereof.
  - The MAC simplifies so that $U!$ is sufficient (<< $2^{U'} - 1$).

Design

1. $\boldsymbol{\pi}$
2. $p_{\boldsymbol{x}}$

Broadcast Channel (BC)



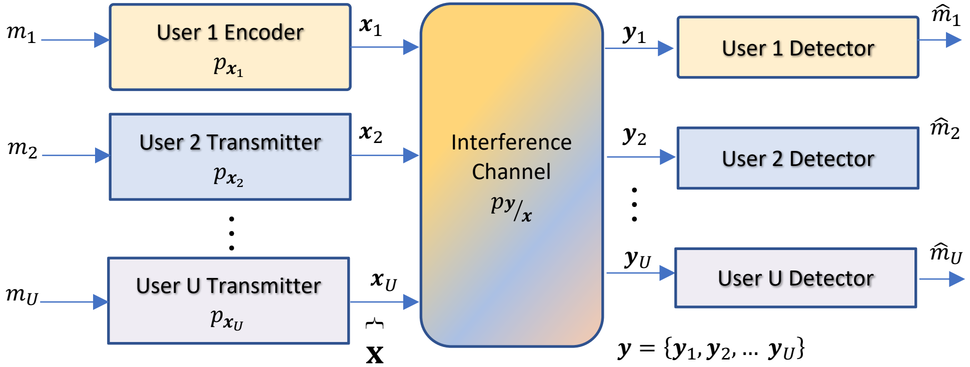
Sec 2.6.1 and 2.8

- The BC is the **"dual"** of a special type of MAC.
  - This eventually allows a common design method, at least for Matrix AWGN case.
- Receivers are in different places, so they cannot "co-process" $\boldsymbol{y}$'s user outputs.
- Transmitter can co-encode/generate $\boldsymbol{x}$, although input messages remain independent.
  - Who encodes first? (may be at disadvantage)
  - Who encodes last? (knowing other users' signals is an advantage)
  - What then is the **order**?
  - Can reduce to just U'! Orders, although $U' > U$.

Design

1. $\boldsymbol{\pi}$
2. $p_{\boldsymbol{x}}$

Interference Channel (IC)



Sec 2.6.1 and 2.9

- Users' transmitters, and also receivers, are in different locations.
  - No co-encoding of user messages nor coordinated reception is possible.
  - Views: A set of MACs with same inputs or a set of BCs with same outputs.

- Each receiver can use a decoding **order** to detect others first, if decoding is possible.
  - The rcvr treats other users as noise if not possible to decode/remove first.
  - Each receiver's order is column of **matrix order $\boldsymbol{\Pi}$**.

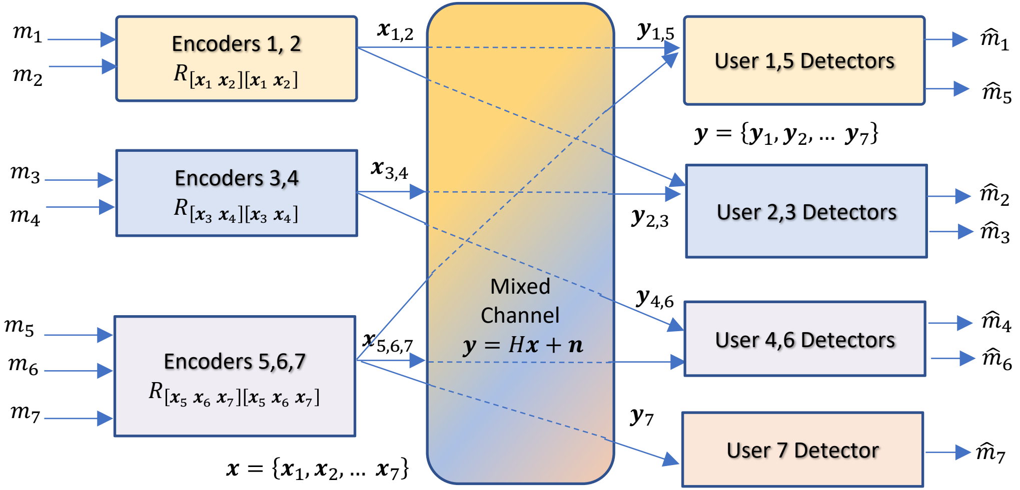
- There are $(U^2!)^U$ possible IC **orders:** … $U^2!$ at each receiver (still less than $\left[U \cdot (2^{U'} - 1)!\right]^U$.
  - Each user may have a subuser component for every user's receiver to detect, large **FINITE.**

Design

1. $\boldsymbol{\Pi}$
2. p_x

Other MU Types / Combinations

- Mixed



Different Macro views:

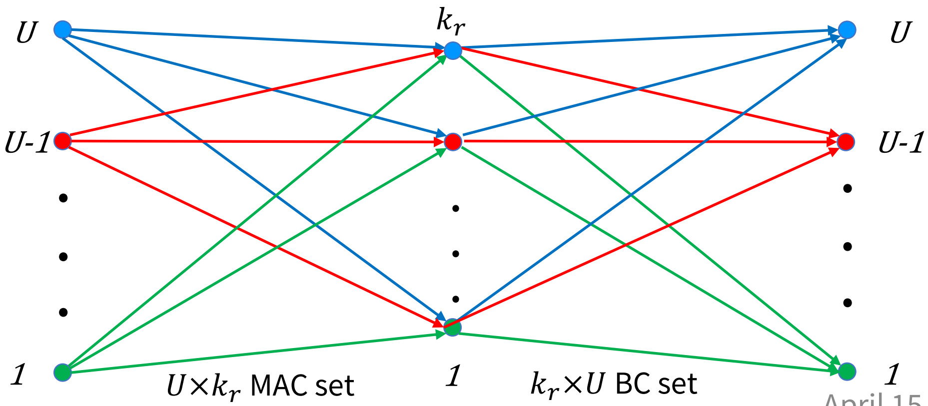
IC of 4 MAC macros:
{1,5}, {2,3}, {4,5}
{7} is single user.

IC of 3 BC macros:
{1,2}, {3,4}, {5,6,7}.

Macro design can shrink $\mathcal{C}(\boldsymbol{b})$ dimensionality.

- Mesh/Relay

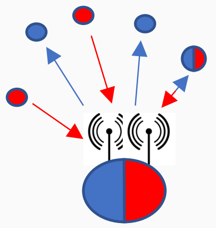
- Single User



k_r **relays**

Nested MU Channel Examples

- Each **nested MU** channel → 1 macro user.
- These macro users crosstalk into each other.
  - Some users with macro group may decode
  - with any given order in that group.
  - Those not decoded are undecodable "noise."
- Design treats as IC of macro users:
  - Orders users within the macro's subusers.
  - Rcvrs decode all subusers within the local macro group
  - or use "multi-level" waterfilling (end of 379B).



a) radio node edge (**cellular base station** or **Wi-Fi Access point**)

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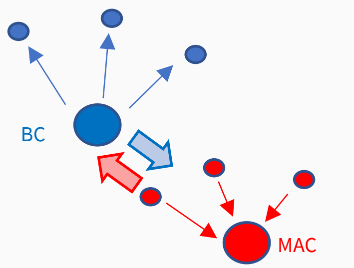
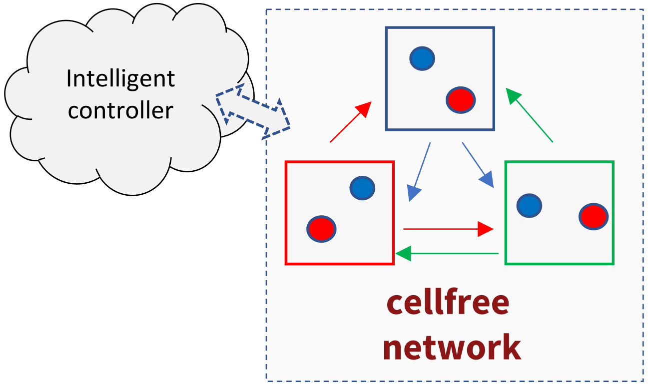


b) two nested MUs: **down** (BC) and **up** (MAC) with IC network between them



c) three nested IC: (BC, MAC) like those in b), nested into 3x3 IC network

Rate Bounds & Detection

PS4.3 - 2.23 Mutual-Information Vector

Chain-Rule Reminder/Review

$$\mathbb{I}(\boldsymbol{x};\boldsymbol{y}) = \sum_{n=1}^{N} \mathbb{I}(\boldsymbol{x}_n;\boldsymbol{y}\,/[\boldsymbol{x}_{n-1}\ \ \cdots\ \ \boldsymbol{x}_1])$$

Lemma 2.3.4

- Think of the input components $\boldsymbol{x}_n$ as users, so $U \to N$ and $u \to n$ (may replace U with U' or $2^{U'} - 1$ in general).
- Any receiver output (or combination of them), $\boldsymbol{y}$, has chain-rule decomposition(s); for the given $p_{\boldsymbol{xy}}$, this $\mathbb{I}(\boldsymbol{x};\boldsymbol{y})$ represents a maximum (sum-user) data rate by AEP.
- Each sum term has similar interpretation, given the "previously decoded" (given) other users.
- The capacity region points correspond to chain-rule $\mathbb{I}$ terms in $\mathcal{C}(\boldsymbol{b})$ for each user receiver in that point's construction.
- User **decoding order** characterizes the different "chain-rule" compositions.

Some data rate bounds

- **Sum-Rate bound:** $b = \sum_{u=1}^{U} b_u \leq \mathbb{I}(\boldsymbol{x};\boldsymbol{y})$ - full transmit/receiver coordination is vector coding.

- **Average User u bound:** $\mathbb{I}_u(x_u;\boldsymbol{y}) \leq \mathbb{I}(\boldsymbol{x};\boldsymbol{y})$- this does NOT bound b_u (could remove other user(s) first)
  - $\mathbb{I}_u(x_u;\boldsymbol{y})$ treats **all** other users as ``noise.''

- The **conditional mutual information** ~first decodes the conditioning users' messages correctly (reliably) and then removes them from the detection process.
  - For AWGN, this operation corresponds to remodulating, filtering by MMSE-based xtalk estimate, and subtracting this estimate from receiver u 's signal.

- All chain-rule bounds apply also if $U \rightarrow U' \; or \; U^2$ or $2^{U'} - 1$.

Fundamental: User Priority → "order"

- Receiver u decodes who first?, last? $\boldsymbol{\pi}_u = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ or …. $(2^{U'} - 1)!$ Choices)
- Why is this important?
  - The as-yet un-decoded users are "noise" (averaged to compute marginal dist'n $p_{\boldsymbol{y}_u/\boldsymbol{x}_i}$, upon which ML detector is based).
  - Are the other users reliably decodable? (They must be treated as "noise" if not.)

If others decoded first, then it is successive decoding or "generalized decision feedback," for some order π_u.

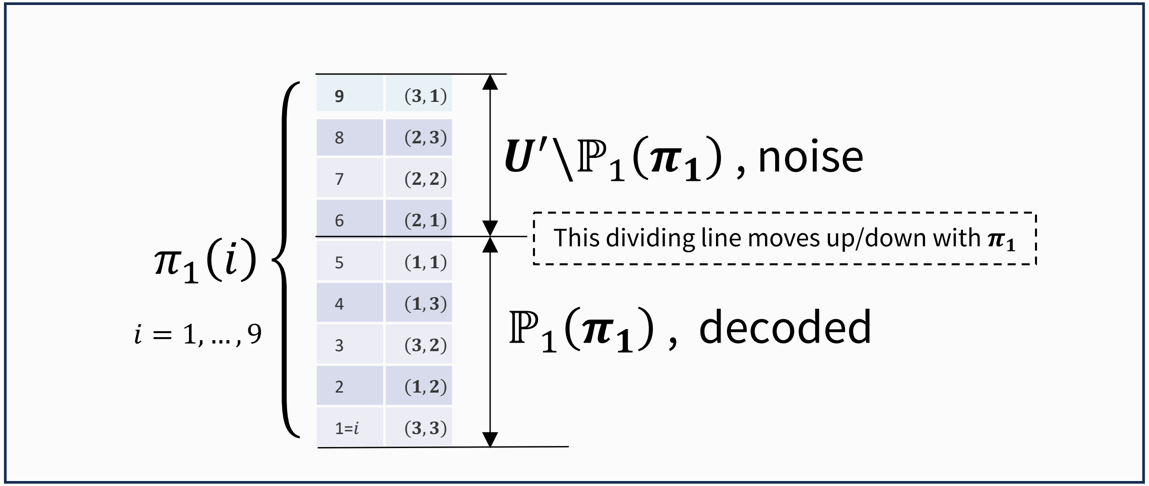
- Same for all users – so there is a "global" order possibility: $\boldsymbol{\Pi} = [\boldsymbol{\pi}_U \quad \cdots \quad \boldsymbol{\pi}_1]$ with $\left(U \cdot (2^{U'} - 1)!\right)^U$ choices
- The designer might check all $\boldsymbol{\Pi}$, and then take convex combinations for each and every allowed $p_{\boldsymbol{x}}$.
  - It simplifies in many situations (including MAC, BC, and IC).
- **Order vector** and **inverse:**
  - Any permutation vector has inverse.
  - Same as 379A interleave, different use.

$$\boldsymbol{\pi}_u = \begin{bmatrix} \pi(U') \\ \vdots \\ \pi(1) \end{bmatrix} \quad \boldsymbol{\pi}_u^{-1} = \begin{bmatrix} U' \\ \vdots \\ 1 \end{bmatrix} \qquad j = \pi(i) \ \rightarrow i = \pi^{-1}(j)$$

Prior-User Set for SCALAR ($U = U'$) IC Example

- U^2 subusers subdivide into ordered pairs (u, u') where $u = 1, \ldots, U$ and $u' = 1, \ldots, U$.
  - Receiver u observes components from users $u' = 1, \ldots, U$.

- In order $\pi(u, u')$, when receiver u has decoded all $u' = 1, \ldots, U$ (all its subuser components), it is done.
  - Any higher-in-order user components are noise (marginals).
  - Can reindex order so that $\pi(u, u') = \pi(i), i = 1, \ldots, U^2$ where i counts from bottom up.
  - The prior-user set for this order $\mathbb{P}_u(\boldsymbol{\pi})$ occurs for smallest i that contains $\{(u,1), (u,2), \ldots, (u,U)\} \subseteq \mathbb{P}_u(\boldsymbol{\pi})$.
- Example for $U = 3$:



More Formal Prior-User Set

- Define the receiver-indexed sets
  - $S_1 \triangleq \{(1,1), \ldots, (1,U)\}$
  - $S_2 \triangleq \{(2,1), \ldots, (2,U)\}$
  - $\vdots$
  - $S_U \triangleq \{(U,1), \ldots, (U,U)\}$
  - $S = S_1 \cup S_2 \cup \cdots \cup S_U$.

$$i_u^*(\boldsymbol{\pi}_u) \triangleq \arg\min_j [\pi_u(S_u) \subseteq \{1{:}j\}]$$

- Prior-User Set is $P_u\left(\pi_{i_u^*}\right) \triangleq \{(u,i)|\pi(u,i) \le i_u^*(\boldsymbol{\pi}_u)\}$.
- For given order and input $p_{\boldsymbol{x}}$, find each user's worst data rate.
- Example – let's say this particular order $\boldsymbol{\Pi}$ just happens to have the 4 subusers/user aligned successively at each receiver,
  - which simplifies illustration from 16 to 4.

rcvr/ User i	$\pi_4(i)$	$\pi_3(i)$	$\pi_2(i)$	$\pi_1(i)$
$i = 4$	3	3	4	3
$i = 3$	4	2	3	2
$i = 2$	1	4	2	1
$i = 1$	2	1	1	4
$\mathbb{P}_u(\boldsymbol{\pi}_u)$	$\{1,2\}$	$\{2,4,1\}$	$\{1\}$	$\{4\}$

$$\boldsymbol{\Pi} = \begin{bmatrix} 3 & 3 & 4 & 3 \\ 4 & 2 & 3 & 2 \\ 1 & 4 & 2 & 1 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

- Designer lists data-rate entries (mutual information bounds really) for those users who are decoded, and ∞ for those treated as noise.

$\mathfrak{I}$	$\mathfrak{I}_4$	$\mathfrak{I}_3$	$\mathfrak{I}_2$	$\mathfrak{I}_1$
top	∞	$\mathcal{I}_3(3/1,2,4)$ 20	∞	∞
	$\mathcal{I}_4(4/1,2)$ 10	$\mathcal{I}_3(2/1,4)$ 9	∞	∞
	$\mathcal{I}_4(1/2)$ 5	$\mathcal{I}_3(4/1)$ 4	$\mathcal{I}_2(2/1)$ 4	$\mathcal{I}_1(1/4)$ 2
bottom	$\mathcal{I}_4(2)$ 1	$\mathcal{I}_3(1)$ 2	$\mathcal{I}_2(1)$ 2	$\mathcal{I}_1(4)$ 5

$$\mathcal{I}_{min}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) = \begin{bmatrix} 4 \\ 20 \\ 1 \\ 2 \end{bmatrix}$$

The minimum Mutual – Info Vector $\mathcal{I}_{min}$

- **Mutual-information-like** quantity
  - follows the prior-user set.

$$\mathscr{I}_u\left(\boldsymbol{x}_{\pi_u(i)}; \boldsymbol{y}_u / \mathbb{P}_{\pi_u(i)}(\boldsymbol{\pi}_u)\right) \triangleq \begin{cases} \infty & i > \pi_u^{-1}(u) \\ \mathcal{I}_u\left(\boldsymbol{x}_{\pi_u(i)}; \boldsymbol{y}_u / \mathbb{P}_{\pi_u(i)}(\boldsymbol{\pi}_u)\right) & i \le \pi_u^{-1}(u) \end{cases}$$

- A worst rate for each and every user $\mathcal{I}_{min,u}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}})$ to compare to the user's implemented rate b_u.

$$\mathcal{I}_{min,u}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) = \min_{i \in \{1,\ldots,U\}} \left\{ \mathscr{I}_i\left(\boldsymbol{x}_u; \boldsymbol{y}_i / \mathbb{P}_u(\boldsymbol{\pi}_i)\right) \right\}$$

The $\mathcal{I}_{min}$ vector

$$\mathcal{I}_{min}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) = \begin{bmatrix} \mathcal{I}_{min,U}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) \\ \vdots \\ \mathcal{I}_{min,u}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) \\ \vdots \\ \mathcal{I}_{min,1}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) \end{bmatrix}$$

$$\boldsymbol{b} \preceq \mathcal{I}_{min}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}})$$

- This tacitly implies sum over each user's subuser components.

Lemma 2.6.1 [Best Decodable Set] *When good codes (with $\Gamma = 0$ dB), given $\boldsymbol{\Pi}$ and $p_{\boldsymbol{xy}}$, and with*

$$\boldsymbol{b} \preceq \mathcal{I}_{min}(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}) \quad , \qquad (2.234)$$

then

$$\mathbb{P}_u(\boldsymbol{\pi}_u) \subseteq \mathscr{D}_u(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}, \boldsymbol{b}) \qquad (2.235)$$

and receiver u reliably achieves the data rate $b = \mathcal{I}_u(\boldsymbol{x}_u; \boldsymbol{y}_u / \mathbb{P}_u(\boldsymbol{\pi}_u))$ with order $\boldsymbol{\pi}_u$.

Example: sum of 3 users (MAC)

$$y = x_1 + x_2 + x_3 + n$$

real subsymbols

Order $\mathbf{\Pi}$	b_1	b_2	b_3
$[1\ 2\ 3]^*$	$\frac{\log_2\left(1+\frac{\mathcal{E}_1}{\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_2}{\mathcal{E}_1+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_3}{\mathcal{E}_1+\mathcal{E}_2+\sigma^2}\right)}{2}$
$[1\ 3\ 2]^*$	$\frac{\log_2\left(1+\frac{\mathcal{E}_1}{\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_2}{\mathcal{E}_1+\mathcal{E}_3+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_3}{\mathcal{E}_1+\sigma^2}\right)}{2}$
$[3\ 1\ 2]^*$	$\frac{\log_2\left(1+\frac{\mathcal{E}_1}{\mathcal{E}_3+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_2}{\mathcal{E}_1+\mathcal{E}_3+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_3}{\sigma^2}\right)}{2}$
$[2\ 3\ 1]^*$	$\frac{\log_2\left(1+\frac{\mathcal{E}_1}{\mathcal{E}_2+\mathcal{E}_3+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_2}{\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_3}{\mathcal{E}_2+\sigma^2}\right)}{2}$
$[2\ 1\ 3]^*$	$\frac{\log_2\left(1+\frac{\mathcal{E}_1}{\mathcal{E}_2+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_2}{\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_3}{\mathcal{E}_1+\mathcal{E}_2+\sigma^2}\right)}{2}$
$[3\ 2\ 1]^*$	$\frac{\log_2\left(1+\frac{\mathcal{E}_1}{\mathcal{E}_2+\mathcal{E}_3+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_2}{\mathcal{E}_3+\sigma^2}\right)}{2}$	$\frac{\log_2\left(1+\frac{\mathcal{E}_3}{\mathcal{E}_2+\mathcal{E}_3+\sigma^2}\right)}{2}$

Position in order determines whether other signals are noise or pre-decoded and then pre-subtracted.

- With the MAC's one receiver:
  - the $\boldsymbol{\pi}_u \equiv \boldsymbol{\pi}$ vectors,
  - $U' = U$,
  - $U! = 6$ for this example, so there are only 6 orders to consider.
- There are many other situations that simplify also.

If energies are $\sigma_n^2 = .001$, $\mathcal{E}_1 = 3.072$, $\mathcal{E}_2 = 1.008$, and $\mathcal{E}_3 = .015$, then with Gaussian codes ($p_{\boldsymbol{x}}$ Gaussian) the order $[123]^*$ corresponds to $b_1 = 1$, $b_2 = 3$, and $b_3 = 2$.

Optimum Detectors (2.6.3)

- Section 2.6.3 formalizes (general, including non-Gaussian, case) optimum detection.
- There are various integrals/sums and definitions.
- More simply, each user's optimum detector -- for any given order $\boldsymbol{\pi}$ – must:
  - first (asymptotically/reliably) detect all other "earlier" users (MMSE→MAP, chain rule) reliably ("no errors"), and
  - then consider all "later" users as noise (this generalizes to integration over margin distribution on non AWGNs).
- The error-probability calculation follows single-user, simply with any "pre-users" no longer present and any "post-users" averaged (treated like noise).
- Thus, these are the same 379A decoders – just more complex notation for the multiuser case,
  - with some modulation-level preprocessing.

Multi-User Detection (MUD) – 2.6.2

- Optimum remains $\max_{\hat{x}_u} \left\{ p_{x_u/y} \right\}$ where the y is the receiver input for detection (MAP detection).

$$P_e(u) = 1 - P_c(u) = 1 - \sum_{i=1}^{M_u} P_{c/i}(u) \cdot p_i(u)$$

$P_{c/i}$ is the probability for message i averaged over all the possible y's for which i is selected (Decision Region).

- But the receiver now might estimate another user earlier (order), so P_e becomes order dependent.

- More generally for any given $p_{\boldsymbol{xy}}$:
  - All **decodable users**, $\mathfrak{D}_u(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}, \boldsymbol{b})$, are first detected and then "cancelled" – they contribute no "noise" (earlier in order). Abbreviate here $\mathfrak{D}_u(\boldsymbol{\Pi})$ inputs, but remains a function of all 3 $(\boldsymbol{\Pi}, p_{\boldsymbol{xy}}, \boldsymbol{b})$.
  - Other users, $\overline{\mathfrak{D}}_{u'}(\boldsymbol{\Pi})\backslash u$ are not first detected and are "averaged" (treated as noise).

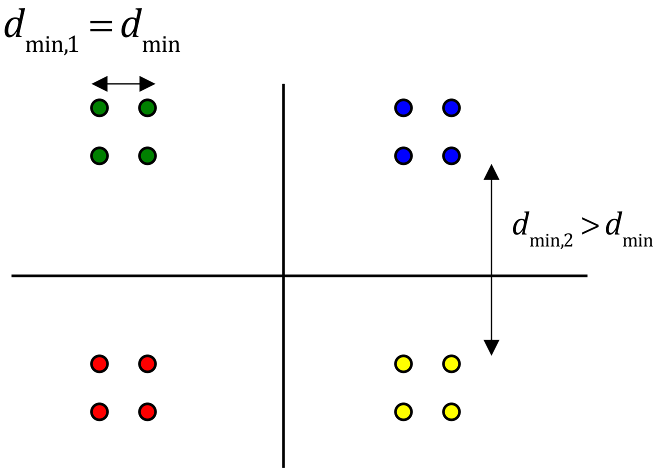
$$p_{\boldsymbol{x}_u/[\boldsymbol{y}\ \boldsymbol{x}_{i\in\mathscr{D}_u(\boldsymbol{\Pi})}]}(\boldsymbol{\chi}_u, \boldsymbol{x}_{i\in\mathscr{D}_u(\boldsymbol{\Pi})}, \boldsymbol{y}) = \int_{\boldsymbol{\chi}\in\boldsymbol{x}_{\{\overline{\mathscr{D}}_u(\boldsymbol{\Pi})\backslash u\}}} p_{\boldsymbol{x}/[\boldsymbol{y}\ \boldsymbol{x}_{\{i\in\mathscr{D}_u(\boldsymbol{\Pi})\}}]}(\boldsymbol{\chi}, \boldsymbol{x}_{i\in\mathscr{D}_u(\boldsymbol{\Pi})}, \boldsymbol{y}) \cdot d\boldsymbol{\chi}$$

Integration/sum is over the noise ave

$$p_{\boldsymbol{x}/[\boldsymbol{y}\ \boldsymbol{x}_{i\in\{\mathscr{D}_u(\boldsymbol{\Pi})\}}]}(\boldsymbol{\chi}_u, \boldsymbol{\chi}_{i\in\mathscr{D}_u(\boldsymbol{\Pi})}, \boldsymbol{y}) = \frac{p_{\boldsymbol{y}/\boldsymbol{x}}(\boldsymbol{\chi}_{i\in\overline{\mathscr{D}}_u(\boldsymbol{\Pi})}, \boldsymbol{x}_{i\in\mathscr{D}_u(\boldsymbol{\Pi})}, \boldsymbol{y}) \cdot p_{\boldsymbol{x}}(\boldsymbol{\chi}_{i\in\overline{\mathscr{D}}_u(\boldsymbol{\Pi})})}{p_{\boldsymbol{y}/\boldsymbol{x}_{\{i\in\mathscr{D}_u(\boldsymbol{\Pi})\}}}(\boldsymbol{x}_{i\in\mathscr{D}_u(\boldsymbol{\Pi})}, \boldsymbol{y})}$$

Term inside integral Derives from $p_{\boldsymbol{xy}}$.

Simple Example



Design can (optimally) reuse all the single-user good codes !!!

- The decoder should decode first red, green, blue, yellow; this treats the variation within each color as "noise."
- Then the decoder re-centers the constellation and decides further which of the 4 same-color points.
  - This effectively cancels the noise from the first step.
- Yes, an overall decoder performs the same if earlier decisions are correct, but the basic concept expands.
  - Again, MMSE (which is chain rule) is optimum detector if previous users (asymptotically reliable – no errors) are correct.

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End Lecture 6

MU Matrix AWGN Channels

- $\mathcal{C}(\boldsymbol{b})$ for a multi-user AWGN channel $\boldsymbol{y} = H \cdot \boldsymbol{x} + \boldsymbol{n}$ will have all users input distributions as Gaussian at the region's (non-zero) boundary, $\mathscr{C}(\boldsymbol{b})$.
  - Each of these points is a mutual information that for each receiver/user $b_u = \mathcal{I}$ has a chain-rule decomposition.
  - For any subset of output dimensions $\boldsymbol{y}$ and any subset of inputs $\boldsymbol{x_u}$, $\mathcal{I}(\boldsymbol{x};\boldsymbol{y}) = \mathcal{I}\left(\boldsymbol{x_u};\boldsymbol{y}\;/\boldsymbol{x}_{\boldsymbol{U}\backslash\boldsymbol{u}}\right) + \mathcal{I}\left(\boldsymbol{x}_{\boldsymbol{U}\backslash\boldsymbol{u}};\boldsymbol{y}\right)$.
    - With independent input messages, these are separable and can be separately maximized.
    - The second term is a "single-user," $\boldsymbol{U}\backslash\boldsymbol{u}$, channel, and this channel thus has optimum Gaussian input.
    - The uncancelled users' crosstalk may contribute in MMSE sense to noise, which then is sum of Gaussians that is also Gaussian.
  - (Proof by induction: last user is single-user channel, which has Gaussian; then next to last has Gaussian xtalk and noise, so it also is Gaussian …), the optimum $\boldsymbol{u}$ is also Gaussian. This also works for any user subset $\boldsymbol{u}$. **QED.**

In general, with user components, treat $\boldsymbol{U} \to \boldsymbol{U}'$.