



Lecture 5

Adapting Modulation Coding Scheme

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Announcements & Agenda

■ Announcements

- Problem Set #1 Solutions posted – see [PS1](#) (also link at course page and connects to canvas where actual solution is stored)
- Problem Set #2 Wed April 19 at 17:00
- Sections **1.6**, 4.4
- Problem Set #3 due April 26 at 17:00

■ Agenda

- Statistical Loading Examples
- Ergodic Coded-OFDM Loading
- Spatial Modulation

■ Problem Set 3 = PS3, due 4/26

1. 4.13 basic C-OFDM design
2. 4.14 ergodic water-fill
3. 4.22 wireless spatial loading
4. 4.16 estimating gain distribution
5. 4.15 Simple Wi-Fi Loading



Statistical Channel Loading Examples

Subsection 4.4.2

PS3.1 (Prob 4.13), PS3.3 (Prob 4.22) and PS3.4 (Prob 4.16)

Ergodic rate-adaptive loading

Definition 4.4.5 [Ergodic Rate-Adaptive Coded MT Loading with constant energy] Ergodic Rate-adaptive coded MT loading with constant energy solves

$$\text{objective: } \max_{r, |C|, g_{out}} r \cdot \log_2 |C| \quad (4.155)$$

$$(4.156)$$

$$\text{subject to: } \langle \bar{P}_e \rangle = \sum_{g > g_{out}} p_g \cdot \bar{N}_e \cdot Q \left[\sqrt{\frac{3 \cdot \bar{\mathcal{E}}_x \cdot \boxed{g \cdot d_{free}(r)}}{|C| - 1}} \right] \leq \bar{P}_e \quad (4.157)$$

$$r \leq 1 - \sum_{g \leq g_{out}} p_g, \quad (4.158)$$

where the algorithm selects the code rate $0 < r \leq 1$ from among the allowed code rates, and $|C|$ is the selected (usually square) QAM constellation size in (4.155) - (4.158). The outage threshold g_{out} characterizes the two sums that are computed for each candidate ordered pair of $[r, |C|]$. The fraction $\frac{3}{|C|-1}$ can be adjusted to κ with non-square constellations, but the concept is the same.

Q-func is correct with quasi-static assumption.

Also BICM is presumed so that this formula essentially assumes $\log_2 |C|$ parallel bit channels.

Definition 4.4.3 [Geometric-Average Channel-Gain] The geometric-average channel gain is

$$\gamma_{geo}^* \triangleq \prod_{g \in \mathcal{G}^*} (g)^{\left[\frac{p_g}{\sum_{g \in \mathcal{G}^*} p_g} \right]}. \quad (4.143)$$

- Finds the cut-off channel gain g_0 that maximizes data rate ($\tilde{b} = r \cdot |C|$).
 - The $[r, |C|]$ choice is over allowed constellations and codes with $(d_{free}(r))$, for the average error probability.
 - Assumes $r \leq 1 - P_{out}$. So applied code with rate r can correct the subsymbols lost to outage (Reed Solomon can at least do this, soft-decoded codes may be yet better).



Loading on Actual Wireless Channel with Flat Energy

- Compute data rate for both average and max (over all tones) with rough gap estimate.

$$b_{flat-geo} = \frac{1}{N} \cdot \sum_{n=1}^N \log_2 \left(1 + \frac{\bar{\mathcal{E}}_x \cdot g_n}{\Gamma} \right) \quad b_{max} = \log_2 (\text{max constellation size}) \leq \max_n \left\{ \log_2 \left(1 + \frac{\bar{\mathcal{E}}_x \cdot g_n}{\Gamma} \right) \right\}$$

- Compute code rate for “good code” (low gap) with this flat energy.

- Then this code rate r then leads to a d_{free} for the selected code.

$$r = \frac{b_{flat-geo}}{b_{max}} \leq 1$$

- Compute outage probability from r as $\bar{P}_{out} = 1 - r$.

- Estimate p_g by **binning** or counting of g values in $6 \text{ dB} - 10 \log_{10} (d_{free-new}/d_{free-old})$, over which the b value does not change the constellation-size choice (see example L5:13).

- Binning counts FEQ noise-estimate values in each range (current symbol or over many symbols).

- Solve for g_0 -- tones with $g < g_0$ will be “erased.”

- Indicate “erasure” (delete from sum is one way) in ML detector.

- This could be an erasure in Reed-Solomon.
- It can also be zero LLR in iterative decoder for bits corresponding to that tone.

$$\bar{P}_{out} = \sum_{g < g_0} p_g$$



Error Correction Code (Sections 2.2, 8.6)

- Example:
 - Use Reed Solomon Block FEC byte wise ($N < 256$):
 - Parity P is up to 32 bytes.
 - It corrects up to $P/2$ bytes if in random codeword positions, or up to P if “erasures” (locations of likely byte error) are determined.
 - Suppose $P_{out} = 5\%$ and $N=200$,
 - then 10 to 20 parity bytes are needed.
 - Suppose $P_{out} = 50\%$, then $N=40$ and $P > 20$ might be needed (low rate code $< 1/2$)
- Designs can also use soft iterative decoding with BICM on binary code use.



Wireless Example

- Extension of well-known code as example (64-state rate-1/2 code with puncturing) has:

- >> r = 0.9 0.8 0.75 0.67 0.5 0.25 0.2
- >> dfree = 2 4 6 7 10 20 25

- Given (measured) channel-gain distribution

- >> g = 3 30 300 600 1200 2400 4800 10000
- >> pg=[.11 .1 .03 .05 .35 .2 .1 .06];

```
>> prob2 = kron(ones(7,1),pg);
```

```
Pout=cumsum(prob2(1,1:8)) % =
```

```
0.1100 0.2100 0.2400 0.2900 0.6400 0.8400 0.9400 1.0000
```

```
>> ones(1,8)-Pout =
```

```
0.8900 0.7900 0.7600 0.7100 0.3600 0.1600 0.0600 0
```

```
>> r =
```

```
0.9000 0.8000 0.7500 0.6700 0.5000 0.2500 0.2000
```

Not correctable correctable Not correctable



Wireless Example continued for $\langle P_e \rangle$

- The SNR (with $\tilde{\mathcal{E}}_x = 1$) is

```
>> SNR=kron(dfree',g);
>> 10*log10(SNR) = (in dB)
  7.7815  17.7815  27.7815  30.7918  33.8021  36.8124  39.8227  43.0103
 10.7918  20.7918  30.7918  33.8021  36.8124  39.8227  42.8330  46.0206
 11.7609  21.7609  31.7609  34.7712  37.7815  40.7918  43.8021  46.9897
 12.5527  22.5527  32.5527  35.5630  38.5733  41.5836  44.5939  47.7815
 14.7712  24.7712  34.7712  37.7815  40.7918  43.8021  46.8124  50.0000
 17.7815  27.7815  37.7815  40.7918  43.8021  46.8124  49.8227  53.0103
 18.7506  28.7506  38.7506  41.7609  44.7712  47.7815  50.7918  53.9794
```

$$\tilde{\mathcal{E}}_x \cdot d_{free} \cdot g$$

Only rows 3-4 are eligible.

- Compute $\langle P_e \rangle$ for several SNRs and SQ QAM Constellations

- 4QAM -- \gg prob1=q(sqrt(SNR(1:4,1:4)))
- 16 QAM -- \gg prob1=2*q(sqrt((3/15)*SNR(1:4,1:4)))
- etc

Compute $\langle P_e \rangle$ tables

4SQ QAM

16SQ QAM

```
>>> prob1=qfunc(sqrt(SNR(3:4,:)))
prob1 = 1.0e-04 *
  0.1105  0.0000  0  0  0  0  0  0  0  0  r = 3/4
  0.0230  0.0000  0  0  0  0  0  0  0  0  r = 2/3
>> avePe=cumsum((prob2(3:4,1:8).*prob1)',reverse')
avePe = 1.0e-05 *
  0.1215  0.0000  0  0  0  0  0  0  0
  0.0253  0.0000  0  0  0  0  0  0  0
```

```
>> prob1=qfunc((3/15)*sqrt(SNR(3:4,:))) % =
  0.1981  0.0036  0.0000  0.0000  0.0000  0.0000  0.0000  0
  0.1797  0.0019  0.0000  0.0000  0.0000  0.0000  0.0000  0
>> 100*avePe=cumsum((prob2(3:4,1:8).*prob1)',reverse') % =
  2.2152  0.0365  0.0000  0.0000  0.0000  0.0000  0.0000  0
  1.9954  0.0188  0.0000  0.0000  0.0000  0.0000  0.0000  0
```

$$\bar{b} = \frac{3}{4} \cdot 1 = .75$$

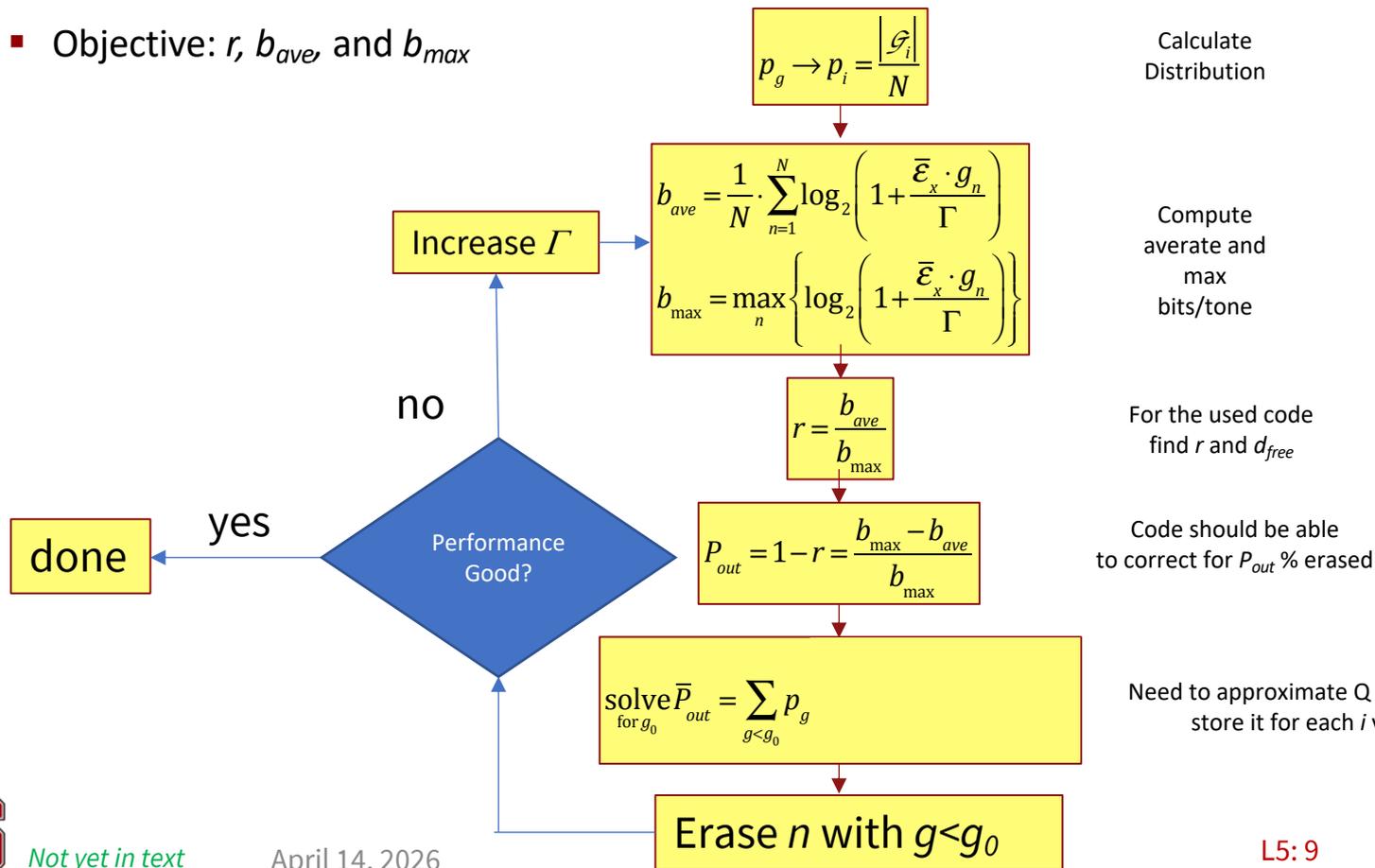
$$\bar{b} = \frac{3}{4} \cdot 2 = 1.5 \text{ bits/dim}$$

- 64 QAM doesn't quite work with anything, so 16SQ with $r=3/4$ is best design, see Section 4.4.2.3.



Project: Flow Chart for C-OFDM Loading

- Objective: r , b_{ave} , and b_{max}



**untested
by instructor**

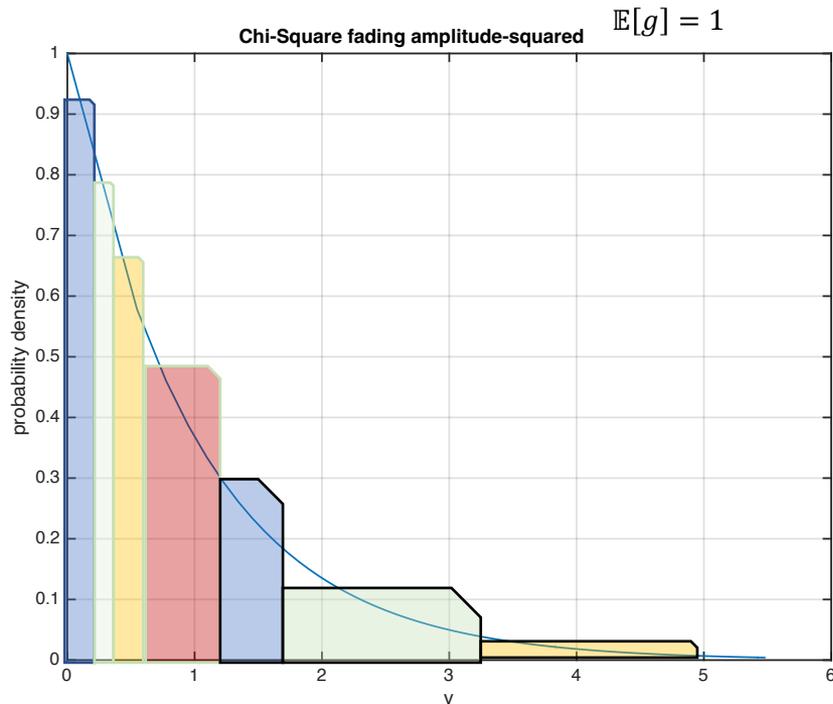


Ergodic Coded-OFDM Loading

Subsection 4.4.2

[See PS3.2 \(Prob 4.14\)](#)

Approximate the distribution as discrete



$$p_g(v) \cdot dv \rightarrow p_{g,n}$$

May care about
lower ranges more and
so more finely divide
distribution there into
samples

- This simplifies calculations for loading, and the Rayleigh model was only approximate anyway.



Discrete Distributions

- All distributions (Rayleigh, Rician, Log-Normal, etc) are gross approximations in wireless.

- Approximate by discrete distributions (which can be learned): $p_g(v) \cdot dv \rightarrow p_{g,n}$

- May need to renormalize so probabilities sum to 1

$$p_{g,n} = \frac{p_g(n \cdot \Delta) \cdot \Delta}{\sum_n p_g(n \cdot \Delta) \cdot \Delta}$$

- The probability $p_{g,n}$ represents the fraction of time that $g_n - \frac{\Delta_n}{2} < g \leq g_n + \frac{\Delta_n}{2}$.
 - Intervals Δ_n do not need to be the same size (may want to align with constellation-size choices).

- Simplify so that g is discrete set of center values $p_g(n\Delta)$ and discrete index $g \in \mathcal{G}$, size $|\mathcal{G}|$.
- g takes place of dimension index, almost (weight not 1 as with integer index).



Average bit rate and ergodic capacity ($\Gamma = 0$ dB)

- Average bit rate is: $\langle b \rangle = \sum_{g \in \mathcal{G}} p_g \cdot \log_2 \left(1 + \frac{\mathcal{E}_{x,g} \cdot g}{\Gamma} \right)$

Ave Mutual Info has $\Gamma=0$ dB.

- Average energy is: $\mathcal{E}_x = \sum_{g \in \mathcal{G}} \mathcal{E}_{x,g} \cdot p_g$

g takes n 's place in DMT RA.

- Rate Adaptive and Margin-Adaptive Water-fill have $\mathcal{E}_g = K - \Gamma/g$.

[See PS3.2 \(Prob 4.14\)](#)

$$\mathcal{E}_{x,g} = K_{ra} - \frac{\Gamma}{g},$$

$$\mathcal{E}_x = \sum_{g \in \mathcal{G}^*} p_g \cdot \left(K_{ra} - \frac{\Gamma}{g} \right)$$

$$= K_{ra} \cdot \sum_{g \in \mathcal{G}^*} p_g - \sum_{g \in \mathcal{G}^*} p_g \cdot \frac{\Gamma}{g}$$

$$K_{ra} = \frac{\mathcal{E}_x + \Gamma \cdot \sum_{g \in \mathcal{G}^*} \frac{p_g}{g}}{\sum_{g \in \mathcal{G}^*} p_g}$$

RA

$$g_{geo}^* \triangleq \prod_{g \in \mathcal{G}^*} (g) \left[\frac{p_g}{\sum_{g \in \mathcal{G}^*} p_g} \right]$$

$$\langle b \rangle = \log_2 \prod_{g \in \mathcal{G}^*} \left(1 + \frac{\mathcal{E}_{x,g} \cdot g}{\Gamma} \right)^{p_g}$$

$$= \log_2 \prod_{g \in \mathcal{G}^*} \left(\frac{K_{ma} \cdot g}{\Gamma} \right)^{p_g}$$

$$2^{\langle b \rangle} = \left(\frac{K_{ma}}{\Gamma} \right)^{\sum_{g \in \mathcal{G}^*} p_g} \cdot \prod_{g \in \mathcal{G}^*} g^{p_g}$$

$$K_{ma} = \Gamma \cdot \left(\frac{2^{\langle b \rangle}}{\prod_{g \in \mathcal{G}^*} g^{p_g}} \right)^{\frac{1}{\sum_{g \in \mathcal{G}^*} p_g}}$$

$$= \Gamma \cdot \frac{(2^{\langle b \rangle}) \left[\frac{1}{\sum_{g \in \mathcal{G}^*} p_g} \right]}{g_{geo}^*}$$

MA



Ergodic Water-filling - Goldsmith

- The amount of energy is either zero or a value given by the water-fill equation for $g \in \mathcal{G}^*$.

$$\mathcal{E}_{x,g} = \begin{cases} \Gamma \cdot \frac{2^{\langle b \rangle}}{g_{geo}} - \frac{\Gamma}{g} & g > \frac{\Gamma}{K_{ma}} \\ 0 & g \leq \frac{\Gamma}{K_{ma}} \end{cases} \quad \text{transmit if } g > g_0 = \frac{\Gamma}{K_{ma}} .$$

- Direct water-fill calc from time-domain g values is non-causal; can't really be made causal with delay.
- Instead, the stationary statistics have been exploited above.
 - Those statistics (p_g) may need to be estimated by counting g values though over time.
 - If the system is ergodic, this will get better and better.
- Instantaneous transmit energy often has limit (less than the $\mathcal{E}_{x,g}$ value above) for small g .

If so, reduce $\langle b \rangle$ -- this is a kind of margin also

This is not practical – use erasures instead, but provides bound.



Average and Outage Capacities

- **Average Capacity** is $\langle \tilde{c} \rangle = \sum_g p_g \cdot \log_2(1 + \epsilon_{x,g} \cdot g)$ bits/complex-subsymbol.
 - $\langle \tilde{c} \rangle$ depends on energy distribution, which would be ergodic water-fill for maximum value.

$$= \log_2 \prod_{g \in \mathcal{G}^*} \left(\frac{K_{ma} \cdot g}{\Gamma} \right)^{p_g}$$

- **Outage Capacity** is $\tilde{C}_{out} \triangleq (1 - P_{out}) \cdot \langle \tilde{c} \rangle$ bits/complex-subsymbol.
 - Basically it reduces by data that would have been transmitted during outage
 - This may need retransmission, so would then be lower yet by average number of retransmissions + 1.

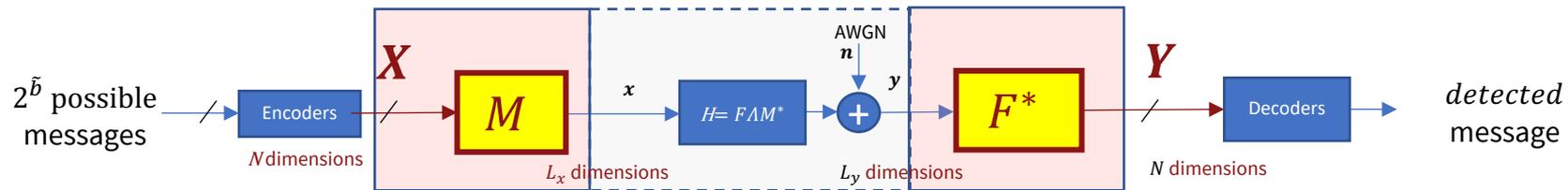


Spatial Modulation

“Space-Time Block Codes (STBC)”

Spatial Vector Coding is Optimal

- The symbols have crosstalk or inter-spatial-dimension interference.
- However, symbols otherwise have no intersymbol interference, $\nu = 0$ (or ISI is separately handled).
 - With Vector DMT/OFDM on all crosstalking channels, there is no ISI.
- Spatial Vector-Code channel partitioning remains for each tone n (index not shown).



- Parallel spatial channels index as:
 - $l = 1, \dots, L \leq \wp_H \leq \min(L_x, L_y)$.

$$y_l = \lambda_l \cdot x_l + n_l$$

$$SNR_l = \frac{\lambda_l^2 \cdot \bar{\epsilon}_l}{\sigma^2}$$

$$\mathbb{E}[\mathbf{n} \cdot \mathbf{n}^*] = R_{nn} = R_{nn}^{1/2} \cdot R_{nn}^{*/2}$$

Noise-Equivalent Channel

$$\mathbf{y} \leftarrow R_{nn}^{-1/2} \mathbf{y} = \left(R_{nn}^{-1/2} \cdot H \right) \cdot \mathbf{x} + \tilde{\mathbf{n}}$$

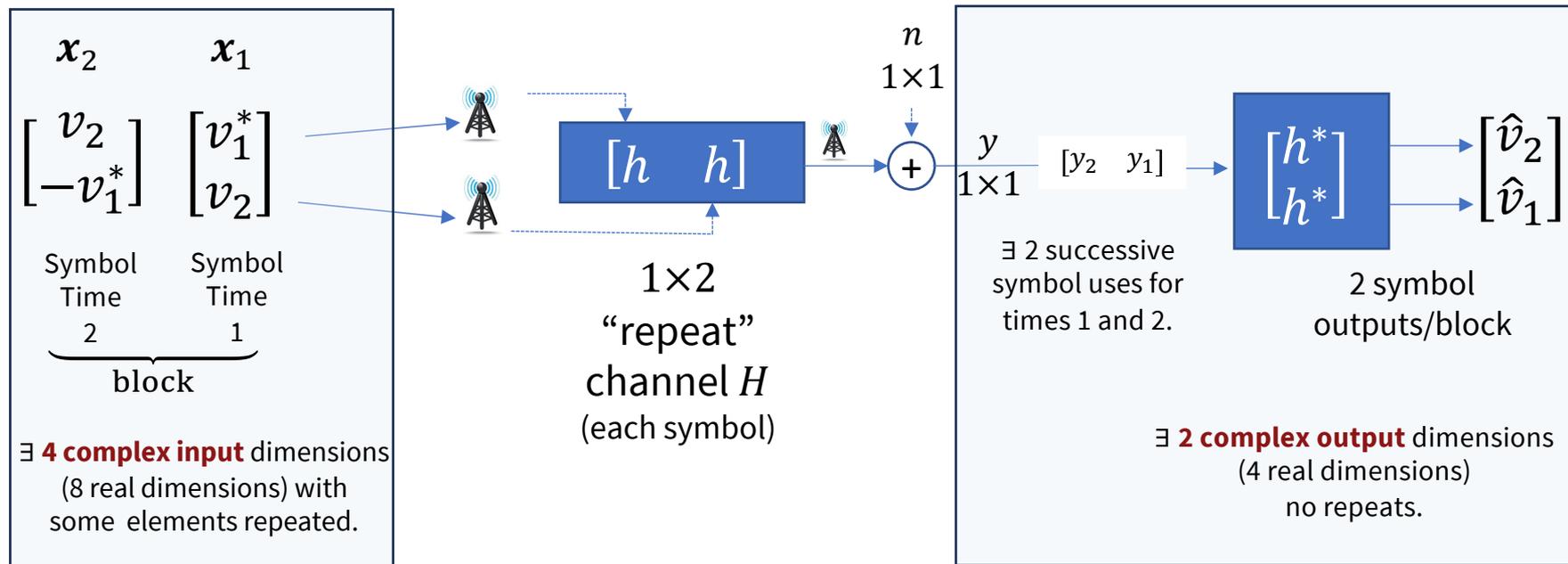


Spatial Modulator (Matrix)

- Best matrix modulator is M for any energy distribution.
- For this choice of M , it follows easily that the unbiased MMSE receiver matrix is F^* .
- In practice, it may be difficult for the transmitter to know M , since only the receiver can measure it.
 - This requires a reverse control channel.
 - The channel may change by the time it is reversed communicated.
- This leads to a variety of spatial approximations, among them Space-Time Block Codes (STBC).



Alamouti's Code (1998)



- **The trivial 1×2 "repeat" channel is the ONLY channel** for which Alamouti's code is Vector Coding.
 - It obviously does better by 3 dB than a single channel use $2 \cdot [h]^2$ instead of $[h]^2$
- Basically, two line-of-sight paths to the same single-antenna receiver that must have the same gain.



Alamouti with more general 2x2 H ??

- Symmetric matrices: $J_2 \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $J_1 \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $R \triangleq \begin{bmatrix} a & c \\ c & b \end{bmatrix}$

- Identity: $J_2 \cdot R \cdot J_2 + J_1 \cdot R \cdot J_1 = \begin{bmatrix} a + b & 0 \\ 0 & a + b \end{bmatrix}$

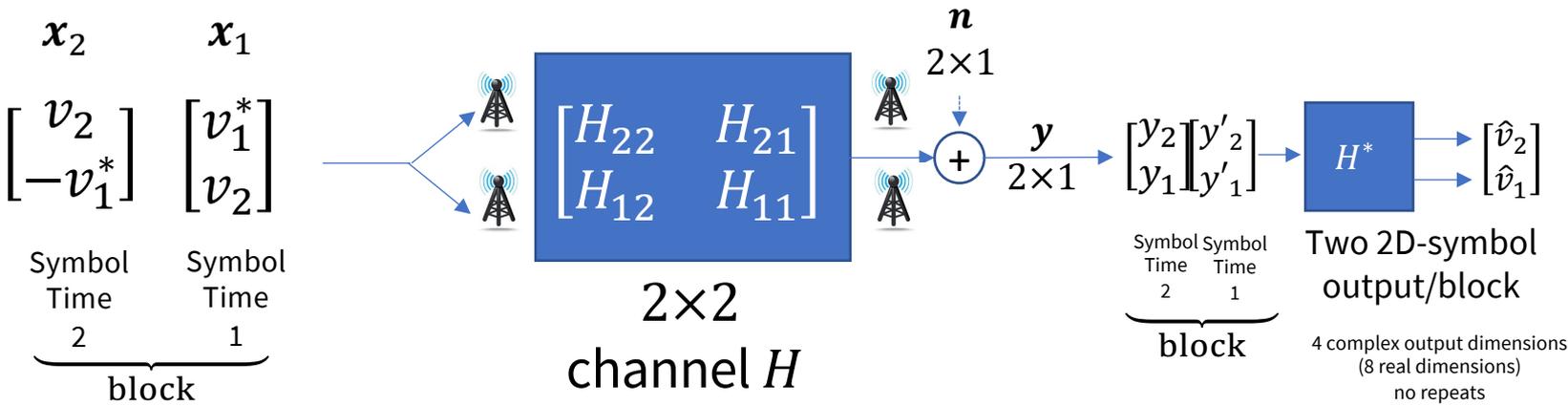
- Alamouti modulator: $\underbrace{\mathbf{x}}_{4 \times 1} = \underbrace{\begin{bmatrix} J_2 \\ J_1 \end{bmatrix}}_{4 \times 2} \cdot \underbrace{\begin{bmatrix} v_2 \\ v_1 \end{bmatrix}}_{\mathbf{v}, 2 \times 1}$ $\mathbf{y} = \underbrace{\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}}_{\mathbf{H}} \cdot \mathbf{x} + \mathbf{n} = \begin{bmatrix} H \cdot J_2 \\ H \cdot J_1 \end{bmatrix} \cdot \mathbf{v} + \mathbf{n}$

- Forward channel autocorrelation is the block diagonal $\mathbf{R}_f = \mathbf{H}^* \cdot \mathbf{H} = \begin{bmatrix} R_f & 0 \\ 0 & R_f \end{bmatrix}$.

- Each 2x2 sub-block diagonal is $R_f = J_2^* \cdot H^* \cdot H \cdot J_2 + J_1 \cdot H^* \cdot H \cdot J_1$
 $= \begin{bmatrix} H_{22}^2 + H_{21}^2 + H_{12}^2 + H_{11}^2 & 0 \\ 0 & H_{22}^2 + H_{21}^2 + H_{12}^2 + H_{11}^2 \end{bmatrix}$



SUBOPTIMAL Alamouti's Code Use, 2 x 2 channel



4 complex input dimensions
(8 real dimensions)
some symbol values repeated

$$\begin{aligned}
 R_f &= J_2^* \cdot H^* \cdot H \cdot J_2 + J_1 \cdot H^* \cdot H \cdot J_1 \\
 &= \begin{bmatrix} H_{22}^2 + H_{21}^2 + H_{12}^2 + H_{11}^2 & 0 \\ 0 & H_{22}^2 + H_{21}^2 + H_{12}^2 + H_{11}^2 \end{bmatrix}
 \end{aligned}$$

$$V_{repeat} = \frac{1}{2}$$

- The transmitter remains channel independent, and the receiver remains simple matched matrix.
- The SNR is higher, BUT the data rate is halved. **Vector coding is better.**



STBC Codebooks

- Variety of different sizes, but rate loss $\gamma_{repeat} \geq \frac{1}{2}$ (with more than one receive antenna).
- They perform a lot worse than vector-coding in practice.

Example: $\bar{\mathcal{E}}_x = 1$; $\sigma^2 = 1$

```
>> H
H =
    1.0000    0.9000
    0.8000    1.0000
>> [F,L,M]=svd(H)
F =
   -0.7245   -0.6892
   -0.6892    0.7245
L =
    1.8512     0
     0    0.1512
M =
   -0.6892   -0.7245
   -0.7245    0.6892
bvc=log2(det(L^2+eye(2))) = 2.1790
bstbc=log2(sqrt(norm(H,'fro')^2+1)) = 1.0769
10*log10((2^bvc-1)/(2^bstbc - 1)) = 5.0245 dB!
```

Sqrt same as $\frac{1}{2}$ in front,
must include the repeat
Factor on the two inputs.

**At higher data rates, the VC
advantage grows;**

Wireless Raleigh fading

$$\langle P_e \rangle \propto (SNR)^{d_{free}}$$

but still large VC advantage.



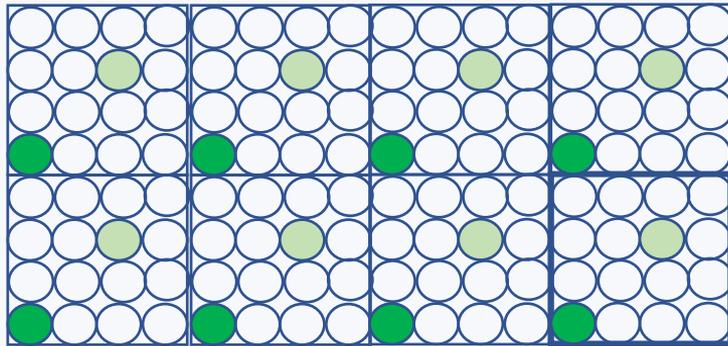
Wireless field use has STBC as “option”

- STBC will essentially repeat symbols, so if the channel is very poor and the codes (nonideal) are fixed, then it can increase SNR and create a reliable link at significant data-rate loss.
- This may be acceptable if no connection is otherwise reliably possible.
- For ideal outer codes ($\Gamma = 0$ dB), there is no such repeat-energy advantage theoretically.
- Most wireless systems (Cellular and Wi-Fi) allow use of STBC as an option, but also permit use of vector-coding approximations to avoid the STBC losses.



Cellular: Type 1 Precoders

$L_{x,2} \cdot O_2$ choices



$L_{x,1} \cdot O_1$ choices

O_i is oversampling factor (angle)

$L_x = L_{x,1} \cdot L_{x,2} =$ number used antennas

Each $O_1 \cdot O_2$ rectangle is a “subarray” of choices.

$Q_{L_{x,1} \cdot O_1} \otimes Q_{L_{x,2} \cdot O_2} =$ precoder factor is cartesian product of horizontal and vertical DFTs

$Q_{L_{x,i} \cdot O_i}$ is a DFT of size in subscript

- The precoder matrix A stacks the columns of $Q_{L_{x,1} \cdot O_1} \otimes Q_{L_{x,2} \cdot O_2}$ so each is $L_x \times 1$
- These columns can be multiplied by $1, j, -1, -j$ for different streams to form an $L_x \times ss$ matrix



Cellular: Type 2 Precoders

- Use the columns of Type 1, call them $\mathbf{w}_{l,ss}$ and weight them (not necessarily unity gain).
- Weights $a_{l,ss}$ can have amplitudes $2^{-i/2}$ for $i \in \{0, \dots, 6\}$ and phases $\frac{2\pi}{j} j \in \{0, \dots, L_x - 1\}$.

$$\mathbf{W}_{ss} = \sum_{\ell=1}^{L_x} \mathbf{w}_{\ell,ss} \cdot a_{\ell,ss}$$

- Receiver will also send back indices i and j along with indices for the Type 1 $\mathbf{w}_{l,ss}$.
- Factors $a_{l,ss}$ are often split into wideband slower-varying factor and narrow band (frequency-dependent) faster-varying factor.



Wi-Fi Spatial Modulators

- Wi-Fi Space-Time is much simpler than cellular.
 - But alas, also allows Alamouti on 2x2 antenna mix channels,
 - and will repeat them for several 2x2 groupings (ouch!).
- Specifies a series of 2x2 rotations (complex, 2 angles) and the spatial antenna indices to which they apply to attempt to approximate M (or exactly realize it if there are enough sent).
 - That's more like it!!





End Lecture 5