STANFORD

Lecture 16

Duality and MAC-dual Basis

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Announcements & Agenda

- Announcements
  - PS7 – last homework 6/4
  - Section 5.5
  - admMAC nominally works, but can run very long time
    - Use minPMAC if you experience this.

- Problem Set 7 = PS7 (due 5/30. extend to 6/4)
  1. 5.16 A tonal channel
  2. 5.17 GDFE MAC Design
  3. 5.18 Dual computations
  4. 5.19 GDFE BC design via duality
  5. 5.20 IC with/without GDFE

- Agenda
  - Finish admMAC
  - MAC/BC Duality Basics
    - Input deflection
    - Mappings
  - Vector MAC/BC Duality
  - MAC-dual Design

Finish AdmMAC

Section 5.5

admMAC

- There can be multi-solution vertex-sharing when in the interior of $\mathcal{C}(\boldsymbol{b})$.
  - Slope (or hyperplane normal vector) is not necessarily = -1 $(\cdot\boldsymbol{1})$.
- minPMAC can use any $\boldsymbol{w}$, including all equal (so energy sum), but does not guarantee a point in $\mathcal{C}(\boldsymbol{b})$.
- minPMAC may be preferred design method as it tends to produce larger margin.
  - Unless any user's energy is too large.
  - Then use admMAC , judiciously with the energies found – limiting any user energies that exceed allowed amounts.
- If admMAC does not work, use it's $\boldsymbol{w}$ in minPMAC, try again the cycle.
- If the first admMAC run does produces FEAS_FLAG = 0 , at least one user data rate is too high.
- Could do similar cycle with maxRMAC, and use admMAC's $\boldsymbol{\theta}$ as admMAC input for the generated $\boldsymbol{b}$.

The admMAC program can be finicky, and it can run very long time.
(We work to improve both admMAC and minPMAC and those will be at future 379B website.)

Two users, high pass and low pass

- A past 2-user MAC with memory (user 2 at 1+.9D and user 1 at 1-D)

```
>> H=zeros(1,2,8);
>> H(1,1,:)=fft([1 .9],8);
>> H(1,2,:)=fft([1 -1],8);
>> H=(1/sqrt(.181))*H;
>> b=[1 ; 1];
>> energy=[8 ; 8];
```

```
[FEAS_FLAG, bu_a, Rxxs, Eun, theta, w, info] = admMAC(H,[1 1], 18*b,
(8/9)*energy,1)
FEAS_FLAG =     2
bu_a' = 19.6385           19.6385
info =  2x7 table
        bu_v            frac
______________   ______
 21.698    17.579    0.5814
16.778    22.499    0.4186
>> buntop=info.bun{1,:,:}; bunbot=info.bun{2,:,:};
>> reshape(buntop,2,8) =
 5.2089    4.9796    3.2601    0.0047    0.0000    0.0047    3.2601    4.9796
      0    0.0001    1.0629    5.0737    5.3058    5.0737    1.0629    0.0001
>> reshape(bunbot,2,8) =
5.2089    4.9768    0.8077    0.0001    0.0000    0.0001    0.8077    4.9768
     0    0.0028    3.5152    5.0783    5.3059    5.0783    3.5152    0.0028
>> Eun =
   1.8043    1.7937    0.8580    0.0011    0.0006    0.0011    0.8580    1.7937
   0.0005    0.0006    0.9443    1.7381    1.7447    1.7381    0.9443    0.0006
>> sum(Eun,2)' =    7.1105    7.1111
```

```
[Eun, theta, bun, FEAS_FLAG, bu_a, info]=minPMAC(H,18*b, [1 1]',1)
Eun =
   1.3437    1.3352    0.3090    0.0000    0.0000    0.0000    0.3090    1.3352
   0.0000    0.0000    0.9849    1.3020    1.3098    1.3020    0.9849    0.0000
theta =    2.7878
           2.7100
bun =
   4.7970    4.5693    2.0322    0.0000    0.0000    0.0000    2.0322    4.5693
        0    0.0000    1.8721    4.6758    4.9043    4.6758    1.8721    0.0000
FEAS_FLAG =    1
bu_a =  18.0000
        18.0000
>> sum(info.Eun{:,:},2)' = 4.6321    5.8835
>> 64/9 =     7.1111
>> [Eun, theta, bun, FEAS_FLAG, bu_a, info]=minPMAC(H,18*b+[1 ;1], [1
1]',1)
sum(info.Eun{:,:},2)' =    5.4922    7.1035
```

Margin is 3dB
For b=[18 18]'

64-tone Example (3 users)

```
>> [FEAS_FLAG, bu_a, Rxxs, Eun, theta, w, info] = admMAC(H64, [1 1 1], [410 390 210]', 64^2/67*[1 1 1], 1)
flag =     2
bu_achieved =  410.3922  390.3730  210.2009
info =  3×8 table
            bu_v                    Rxxs     Eun         bun          theta    order    frac      clusterID
   481.11   386.38   143.47   {1×64}  {1×3x64}  {1×3×64}  {1×3}  {1×3}  0.72759     1
   385.85   207.15   417.96   {1×64}  {1×3x64}  {1×3×64}  {1×3}  {1×3}  0.021875    1
   207.15   417.96   385.85   {1×64}  {1×3x64}  {1×3×64}  {1×3}  {1×3}  0.25054     1
>> info.bu_v'*info.frac =  410.3922  390.3730  210.2009
>> sum(cell2mat(info.Eun),3) =
  61.1343   61.1343   61.1343
  61.1343   61.1343   61.1343
  61.1343   61.1343   61.1343
```

Takes LONG time

```
[Eun, theta, bun, FEAS_FLAG, bu_a, info]=minPMAC(H,[410 390 210]', [1 1 1], 1);
theta =   2.8865    2.8865    2.8739
FEAS_FLAG = 2
bu_a =  410.0000  390.0000  210.0002
theta =   2.9830    2.9830    2.9830
info =  2x6 table
          bu_v                 Eun                  bun                  theta           frac       clusterID
   410.24   389.76   210   {1×3×64 double}  {1×3×64 double}  {1×3 double}  0.98818     1
   390.02   409.98   210   {1×3×64 double}  {1×3×64 double}  {1×3 double}  0.011823    1
>> Eun=info.Eun{1,:,:};  >> sum(Eun(1,:,:),3) =   59.5706   56.2791   66.2707
>> >> [Eun, theta, bun, FEAS_FLAG, bu_a, info]=minPMAC(H,[410 390 210]', [1.03 1 1.08], 1) bu_a' =  410
390  210
info =  2x7 table
          bu_v                 Eun        bun        theta    order    frac      clusterID
  410   394.28   205.72   {1×3×64}  {1×3×64}  {1×3}  {1×3}  0.98208     1
  410   155.42   444.58   {1×3×64}  {1×3×64}  {1×3}  {1×3}  0.017919    1

>> sum(Eun(1,:,:),3) = 60.9146   60.3152   61.0854 😀
```

- Design might ignore one of the vertices – do ¾ share, other 2.

- Solution is close to admMAC solution,
- and maybe just 1 vertex.

MAC Capacity Region

- Use admMAC in U-loop that gradually increases b_u's (for all $u = 1, \ldots, U$) until admMAC returns negative value and record the last rate pair (the last before that violation happens).

- This is the exterior of $\mathcal{C}_{MAC}(\boldsymbol{b})$.

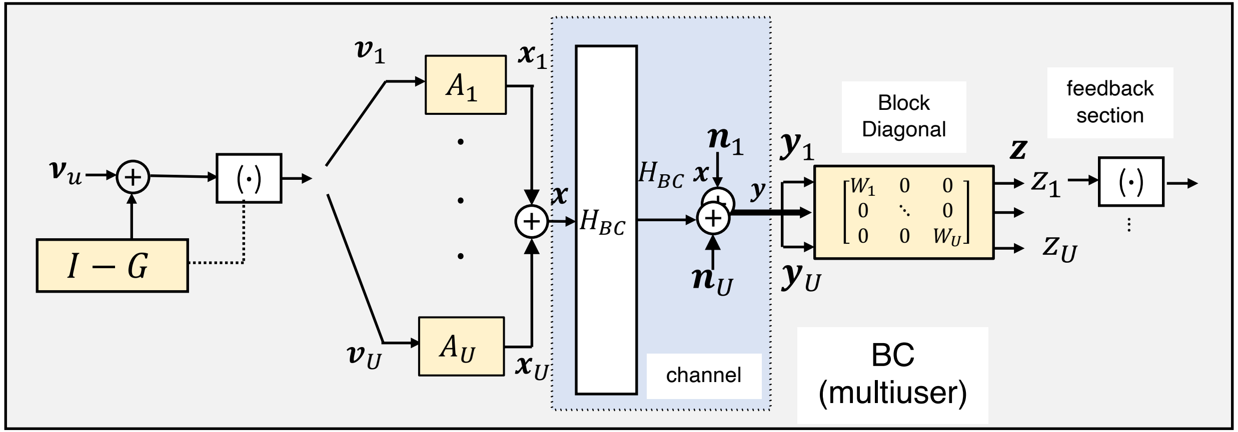
- Check in at 379B website in future as I believe the minPMAC, admMAC, and maxRMAC will all run much faster and more smoothly in a few months.
  - CVX approach with bounding faces (Sagnik and Kami)
  - Ordered Water-Fill

MAC / BC Duality Basics

Section 5.5

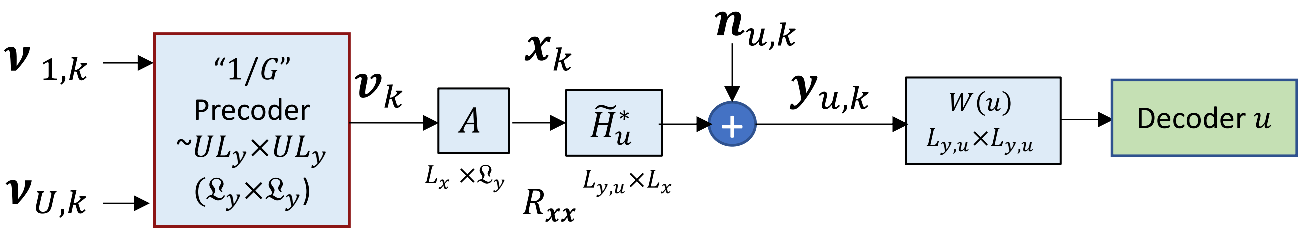
BC Input Addition & Design

- The matrix-AWGN BC design adds user input symbols $\boldsymbol{x}_u$ before transmission.
  - $\boldsymbol{x} = \sum_{u=1}^{U} \boldsymbol{x}_u$ tends to "hide" independent input contributions.
    - Remember the "secondary-user component" precoder – "freeloading" compounds the hidden-subuser complexity.
    - Primary users (or really any ϱ_H users) more productively separate.
  - $R_{\boldsymbol{xx}} = \sum_{u=1}^{U} R_{\boldsymbol{xx}}(u)$ - the contributions get "mixed" on the dimensions.
  - Key result: $R_b^{-1} = H_{BC}^* \cdot R_{\boldsymbol{nn}}^{-1} \cdot H_{BC} + I = G \cdot S_0 \cdot G^*$.



- Design:
  - would prefer single precoder, with
    - no concern for primary / secondary,
  - that is MMSE based,
  - that finds A's and G, (and W's), &
  - derives from **dual** MAC's design.

GDFE per user (useful for BC), mu_bc.m



k is subsymbol index.

Up to $\mathfrak{L}_y$ subuser components can affect decoder u (if $\mathfrak{L}_y > U^2$, design simplifies).

- Design achieves canonical performance for user u *for the given set of inputs and $\{\tilde{H}_u^*\}$ that is:*
  - *only a max rate sum if all- primary users with a corresponding special worst-case-noise-designed square root A.*
  - *But, this design is for specific $\{R_{\boldsymbol{xx}}(u)\}$.*

- The receiver $W(u)$ is indeed MMSE for these inputs and $\tilde{H}_u$.

- Max rate is $\mathcal{I}_{BC}(u) = \log_2 \left(\frac{\left| I + \sum_{i=1}^{u} H_u^* \cdot R_{\overline{\boldsymbol{xx}}}(i) \cdot H_u \right|}{\left| I + \sum_{i=1}^{u-1} H_u^* \cdot R_{\overline{\boldsymbol{xx}}}(i) \cdot H_u \right|} \right)$ and corresponds to the individual- $\tilde{H}_u$ MMSE GDFE.

- This is NOT a chain rule form because the H subscript is u, not i (but is a mini chain rule for rcvr u).
  - Thus, $\mathcal{I}(\boldsymbol{x}, \boldsymbol{y}) \neq \sum_{u=1}^{U} \mathcal{I}_{BC}(u)$; indeed $\mathcal{I}(\boldsymbol{x}, \boldsymbol{y}) \geq \sum_{u=1}^{U} \mathcal{I}_{BC}(u)$.

Order Reversal & The Dual

- Reverse MAC order so that BC has user 1 at top (still best position).

- $\mathcal{J}_x$ applies to $\boldsymbol{x}$ so that $\mathcal{J}_x \cdot \boldsymbol{x}$ reverses the input vector $\boldsymbol{x}$'s user order.

$$\mathcal{J}_x \triangleq \begin{bmatrix} 0 & 0 & I_{L_{x,1}} \\ 0 & \iddots & 0 \\ I_{L_{x,U}} & 0 & 0 \end{bmatrix}$$

- Thus, reversing a MAC's input order corresponds to $\tilde{H} \cdot \mathcal{J}_x$.
  - Can also have a $\mathcal{J}_y$ on BC input side (but since BC has one joint input, less important).

$$H_{dual} = \mathcal{J}_x \cdot \tilde{H}^* \cdot \mathcal{J}_y$$

- The channel $\tilde{H}$'s dual is its order-reversed transpose.

- SVDs: $\tilde{H} = F \cdot \Lambda \cdot M^*$ $\quad H_{dual} = \mathcal{J}_x \cdot M \cdot \Lambda \cdot F^* \cdot \mathcal{J}_y$

Still an SVD (singular values the same); L_y **inputs**, L_x **outputs**

This includes the $\mathcal{J}_y$ to be consistent with text, but probably will eliminate them in future lectures and text.

Specific to MAC and BC

- MAC Channel has normal notation:

$$\tilde{H}_{MAC} = [\tilde{H}_U \quad \cdots \quad \tilde{H}_1]$$

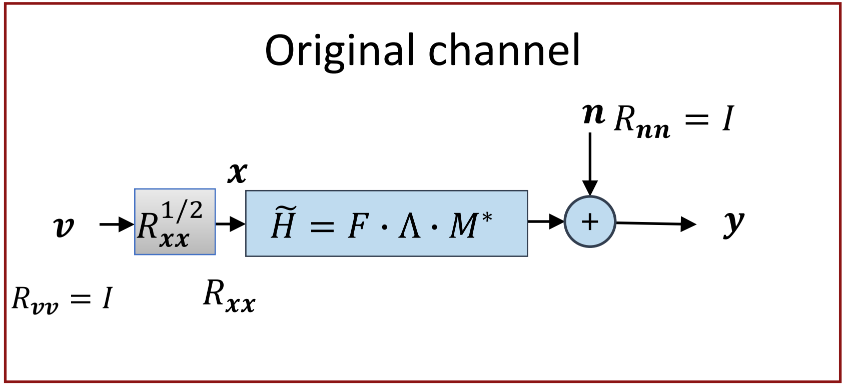
- Dual BC Channel transposes each user channel and reorders outputs and inputs so 1 is at top/left:

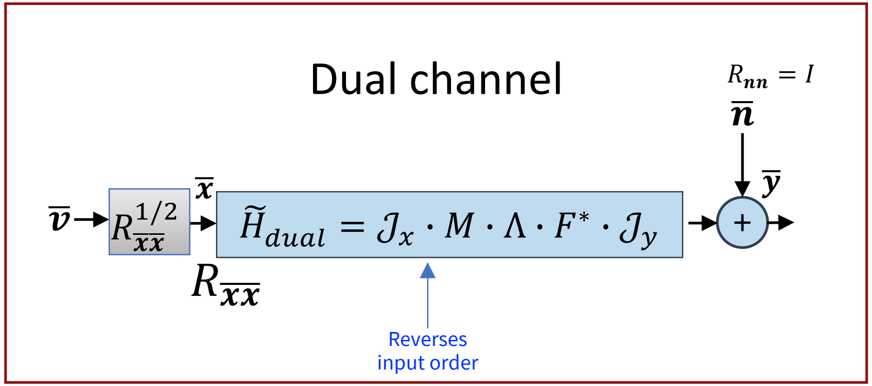
$$\tilde{H}_{BC} = \mathcal{J}_x \cdot \tilde{H}_{MAC}^{*} \cdot \mathcal{J}_y = \mathcal{J}_x \cdot \begin{bmatrix} \tilde{H}_U^* \\ \vdots \\ \tilde{H}_1^* \end{bmatrix} \cdot \mathcal{J}_y = \begin{bmatrix} \tilde{H}_1^* \\ \vdots \\ \tilde{H}_U^* \end{bmatrix} \cdot \mathcal{J}_y$$

- The reversal allows some simplification of notation (its worse without the reversal).

The (single-user & overall-channel) Dual

- Original and dual have the same mutual information equal:





$$\mathcal{I}(\boldsymbol{x}, \boldsymbol{y}) = \log_2 \left| \tilde{H} \cdot R_{xx} \cdot \tilde{H}^* + I \right| = \log_2 \left| \mathcal{J}_x \cdot \tilde{H}^* \cdot R_{\bar{x}\bar{x}} \cdot \tilde{H} \cdot \mathcal{J}_x + I \right|$$

- A solution is $\bar{\boldsymbol{x}} = \mathcal{J}_y \cdot F \cdot M^* \cdot \mathcal{J}_x \cdot \boldsymbol{x}$, which yields

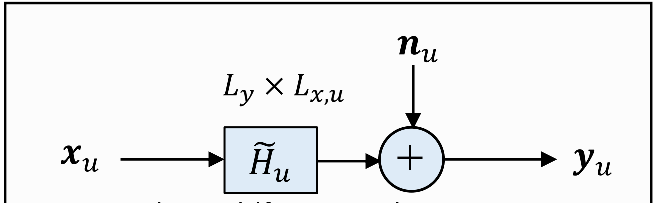
- $R_{\bar{x}\bar{x}} = \mathcal{J}_y \cdot F \cdot M^* \cdot \mathcal{J}_x \cdot R_{xx} \cdot \mathcal{J}_x \cdot M \cdot F^* \cdot \mathcal{J}_y$

  The $\mathcal{J}_{x,y}$'s don't change rate sums/determinants

- SNR's also equal because $R_{vv} = I = R_{\bar{v}\bar{v}}$

Proof in Sec 5.5

Each user's dual with crosstalk as noise:

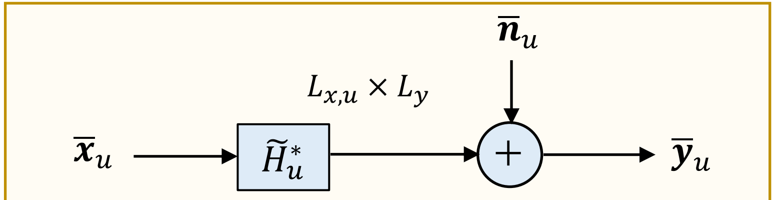


Per-user channel (from MAC)

$$\underset{L_y \times L_y}{\mathcal{R}_{\boldsymbol{nn}}(u)} = I + \sum_{i=u+1}^{U} \tilde{H}_i \cdot R_{\boldsymbol{xx}}(i) \cdot \tilde{H}_i^*$$ implies a user order.

Noise + Xtalk not white

$$\tilde{H} = R_{\boldsymbol{nn}}^{-1/2} \cdot H = [\tilde{H}_U \quad \ldots \quad \tilde{H}_1]$$



Per-user dual channel for BC

$$\underset{L_x \times L_x}{\mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)} = I + \sum_{i=1}^{u-1} \tilde{H}_u^* \cdot R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(i) \cdot \tilde{H}_u$$ reverses implied order.

- For equal individual-user mutual information:

$$\mathcal{I}_u = \frac{\left|\tilde{H}_u \cdot R_{\boldsymbol{xx}}(u) \cdot \tilde{H}_u^* + \mathcal{R}_{\boldsymbol{nn}}(u)\right|}{|\mathcal{R}_{\boldsymbol{nn}}(u)|} = \frac{\left|\tilde{H}_u^* \cdot R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) \cdot \tilde{H}_u + \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)\right|}{|\mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)|}$$

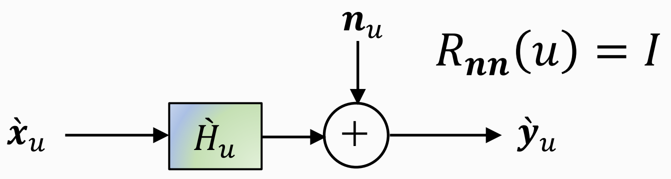
Suggests an input adjustment (so dimensionality is consistent).

- Duality design follows MAC's independent ($\boldsymbol{x}_u$ and $\boldsymbol{x}_{i\neq u}$) & causes BC's ($\bar{\boldsymbol{x}}_u$ and $\bar{\boldsymbol{x}}_{i\neq u}$) 's to be independent.

- Duality **deflects input** so that it offsets the channel shaping (same dim as "dual's xtalk):

$$\grave{\boldsymbol{x}}_u = \mathcal{R}_{\boldsymbol{nn}}^{1/2}(u) \cdot \boldsymbol{x}_u$$

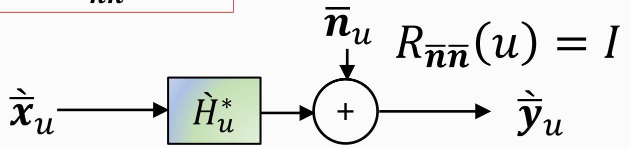
$$\grave{\bar{\boldsymbol{x}}}_u = \mathcal{R}_{\boldsymbol{nn}}^{*/2}(u) \cdot \bar{\boldsymbol{x}}_u$$



$$R_{\boldsymbol{xx}}(u) = \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-1/2}(u) \cdot R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-*/2}(u)$$

$$\grave{H}_u = \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot \tilde{H}_u \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-1/2}(u)$$

$$R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) = \mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u) \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u)$$



- This then **whitens both channels' noises** and is 1-to-1 on input deflection, so $\mathcal{I}(\grave{\boldsymbol{x}}_u;\grave{\boldsymbol{y}}_u) = \mathcal{I}(\grave{\bar{\boldsymbol{x}}}_u;\grave{\bar{\boldsymbol{y}}}_u) = \mathcal{I}(\boldsymbol{x}_u;\boldsymbol{y}_u)$.

$$2^{\mathcal{I}_u} = \frac{\left|\tilde{H}_u \cdot R_{\boldsymbol{xx}}(u) \cdot \tilde{H}_u^* + \mathcal{R}_{\boldsymbol{nn}}(u)\right|}{|\mathcal{R}_{\boldsymbol{nn}}(u)|} = \frac{\left|\tilde{H}_u^* \cdot R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) \cdot \tilde{H}_u + \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)\right|}{|\mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)|}$$

$$= \left|\grave{H}_u \cdot R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) \cdot \grave{H}_u^* + I\right| = \left|\grave{H}_u^* \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot \grave{H}_u + I\right|$$

Scalar duality revist and example

- Rewrite the scalar-duality input-deflection equations (follow from L10 scalar-energy duality):

$$
\begin{aligned}
x_1^{BC} \cdot \sqrt{1 + \mathcal{E}_2^{MAC} \cdot g_2 + \ldots + \mathcal{E}_U^{MAC} \cdot g_U} &= x_1^{MAC} \\
x_2^{BC} \cdot \sqrt{1 + \mathcal{E}_3^{MAC} \cdot g_3 + \ldots + \mathcal{E}_U^{MAC} \cdot g_U} &= x_2^{MAC} \cdot \sqrt{1 + \mathcal{E}_1^{BC} \cdot g_2} \\
\vdots &= \vdots \\
x_U^{BC} &= x_U^{MAC} \cdot \sqrt{\left(1 + \left[\mathcal{E}_1^{BC} + \ldots + \mathcal{E}_{U-1}^{BC}\right] \cdot g_U\right)}
\end{aligned}
$$

$\mathcal{R}_{\bar{n}\bar{n}}^{-1/2}(u)$

- Recognize these as a simple form of input deflection.
- $g_u = \frac{|h_u|^2}{\sigma_u^2}$, above equations do not depend on (MAC's) g_1, which is BC's g_2.
- L10's duality example was for $H_{MAC} = [50 \quad 80]$ (user 2, user 1)

$$H_{BC} = \begin{bmatrix} 80 \\ 50 \end{bmatrix} \begin{matrix} \text{user 1} \\ \text{user 2} \end{matrix}$$

$$\mathcal{E}^{BC} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1/7504 \\ 7503/7504 \end{bmatrix} = \mathcal{E}^{MAC}$$

Simple example – treat BC as reversed MAC

$$\mathcal{E}^{BC} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1/7504 \\ 7503/7504 \end{bmatrix} = \mathcal{E}^{MAC}$$

- Direct Design with MAC produces:

```
>> Hmac=[50 80];
>> Rxx=diag([1/7504 7503/7504]);
>> A=sqrtm(Rxx);
>> [Bu, GU, WU, S0, MSWMFU] = mu_mac(Hmac, A, [1 1] , 2)
Bu =
   0.2074    6.1146
GU =
   1.0000  138.5918
        0         1.0000
WU =
   3.0016        0
  -0.0072    0.0002
S0 =   1.0e+03 *
   0.0013        0
        0    4.8010
MSWMFU =
   1.7325
   0.0125
>> sum(Bu) =     6.3220
>> MSWMFU*Hmac*A  =
   1.0000  138.5918
   0.0072      1.0000
```

- Recall (sec 2.8) 6.322 is max sum rate (for this BC).

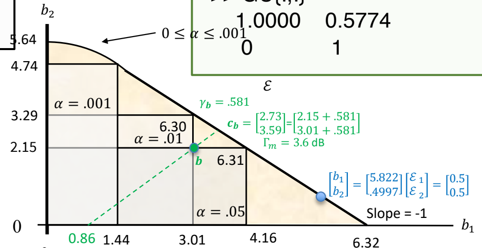
- Direct Design with BC produces:

```
>> Hbc=[80 ; 50];
>> AU=[sqrt(.75) sqrt(.25)];
>> Lyu=[1 1];
>> cb=2;
>> [Bu, GU, S0, MSWMFunb , B] = mu_bc(Hbc, AU, Lyu , cb)
Bu =
   6.1146    0.2074
GU =  2x1 cell array
S0 =  2x1 cell array
   {[4.8010e+03]}
   {[    1.3332]}
MSWMFunb = 2x1 cell array
   {[0.0144]}
   {[0.0400]}
B =  2x1 cell array
   {[6.1146]}
   {[0.2074]}
>> sum(Bu) =     6.3220
>> MSWMFunb{1}*Hbc(1)*AU(1) =     1
>> MSWMFunb{2}*Hbc(2)*AU(2) =     1.0000
>> GU{:,:} =
   1.0000    0.5774
   0              1
```

$[\sqrt{user\ 1} \quad \cdots \quad \sqrt{user\ 2}]$

Crosstalk in BC scales @ xmit versus in MAC @ rcvr, but same cancellation.

MAX single user is $b = 6.56 > 6.322$.



BC Loss

- BC Loss - ratio of single-user capacity SNR to BC maximum-rate-sum SNR (for $[H \quad R_{\boldsymbol{nn}}]$):

$$\gamma_{BC} \triangleq \frac{2^{2\cdot\bar{\mathcal{C}}} - 1}{2^{2\cdot\bar{\mathcal{C}}_{E-sum,dual-MAC}} - 1} = \gamma_{E-sum,dual-MAC} \geq 1$$

- This equality is assured by duality.

- For slide L16:12's example:

$$\gamma_{BC} = \frac{2^{2\cdot 6.556} - 1}{2^{2\cdot 6.322} - 1} = 1.5\text{ dB.}$$

Try a different input on each dual

$$\mathcal{E}^{MAC} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \rightarrow \begin{bmatrix} 1/2502 \\ 2501/2502 \end{bmatrix} = \mathcal{E}^{BC}$$

special case with scalar and $U = 2$

- Direct Design with BC

```
>> Hmac=[50 80];
>> Rxx=diag([1/2 1/2]);
>> A=sqrtm(Rxx);
>> [Bu, GU, WU, S0, MSWMFU] = mu_mac(Hmac, A, [1 1] , 2)

Bu =
   5.1444    0.9155

GU =
   1.0000    1.6000
        0    1.0000

S0 =   1.0e+03 *
   1.2510         0
        0    0.0036

MSWMFU =
   0.0283
   0.0177

>> MSWMFU*Hmac*A  =
   1.0000    1.6000
   0.6250    1.0000

>> sum(Bu)  =    6.0600
```

```
>> Hbc=[80 ; 50];
>> AU=[sqrt(1/2502) sqrt(2501/2502)];
>> Lyu=[1 1]; cb=2;
>> [Bu, GU, S0, MSWMFunb , B] = mu_bc(Hbc, AU, Lyu , cb)

Bu =
   0.9155    5.1444

>> GU{:,:} =
   1.0000   50.0100
    0            1

>> diag(cell2mat(S0)) =   1.0e+03 *
   0.0036         0
        0    1.2510

>> MSWMFunb{:,:} =
   0.6252
   0.0200

>> diag(cell2mat(MSWMFunb))*Hbc*AU =
   1.0000   50.0100
   0.0200    1.0000

>> sum(Bu) =    6.0600
```

Data rates & SNRs reverse in order.

Filters are not same.

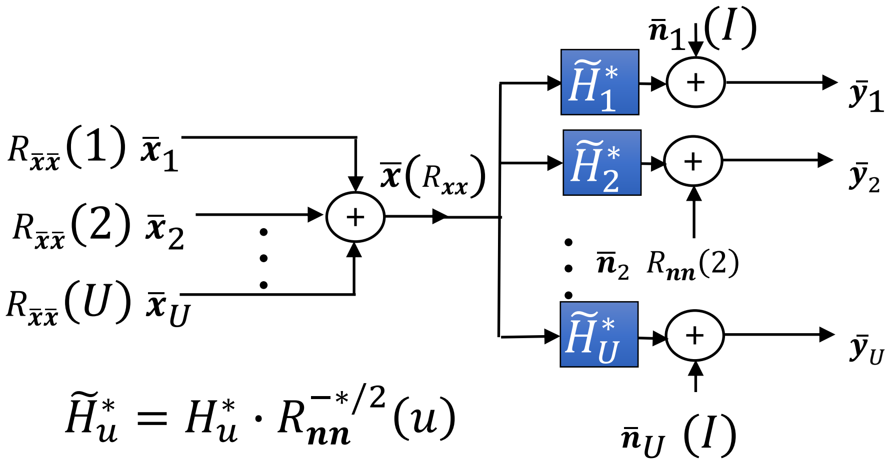
Crosstalk cancels.

6.06 < 6.322 = previous rate sum because different input energy of [0.5 0.5] in this sum.

Vector MAC / BC Duality

Section 5.5.2

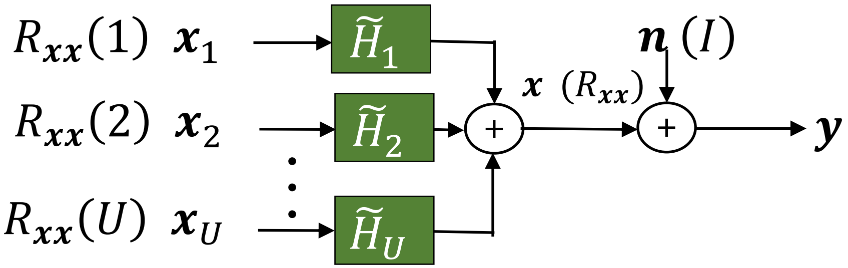
Vector MAC/BC Duals



$$\tilde{H}_u^* = H_u^* \cdot R_{nn}^{-*/2}(u)$$

Broadcast – input is $\bar{x}$

$$\mathcal{R}_{nn}(u+1) = \sum_{i=1}^{u} \tilde{H}_i^* \cdot R_{xx}(i) \cdot \tilde{H}_i + I$$



Multiple Access input is x

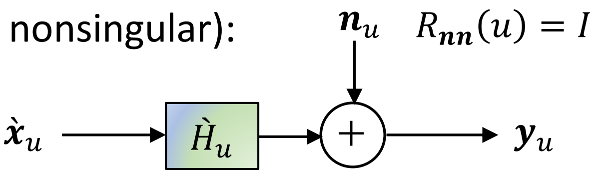
Order reversed

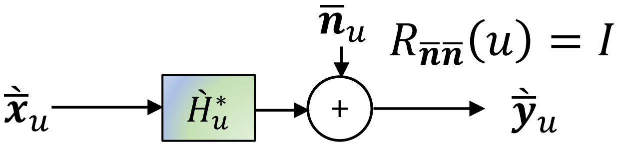
$$\mathcal{R}_{nn}(u-1) = \sum_{i=u}^{U} \tilde{H}_i \cdot R_{xx}(i) \cdot \tilde{H}_i^* + I$$

- Esum-MAC $\mathcal{E}_x = \sum_{u=1}^{U} \mathcal{E}_u = \sum_{u=1}^{U} trace\{R_{xx}(u)\} = \sum_{u=1}^{U} trace\{R_{\bar{x}\bar{x}}(u)\}$

Summary: Input Deflection – 2 new channels

- Deflect input to offset the channel shaping (deflectors are nonsingular):
  - $\grave{\boldsymbol{x}}_u = \mathcal{R}_{\boldsymbol{nn}}^{1/2}(u) \cdot \boldsymbol{x}_u$,
  - $\grave{\bar{\boldsymbol{x}}}_u = \mathcal{R}_{\boldsymbol{nn}}^{*/2}(u) \cdot \bar{\boldsymbol{x}}_u$.
- Also , re-whiten the noise:
  - $\grave{H}_u = \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot \widetilde{H}_u \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-1/2}(u)$.
- Relate autocorrelation matrices
  - $R_{\boldsymbol{xx}}(u) = \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-1/2}(u) \cdot R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-*/2}(u)$,
  - $R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) = \mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u) \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u)$.





- Whitens both channels' distortion and is 1-to-1 on input deflection, so $\mathcal{I}(\grave{\boldsymbol{x}}_u; \boldsymbol{y}_u) = \mathcal{I}(\grave{\bar{\boldsymbol{x}}}_u; \grave{\bar{\boldsymbol{y}}}_u) = \mathcal{I}(\boldsymbol{x}_u; \boldsymbol{y}_u)$.

$$2^{\mathcal{I}_u} = \frac{\left|\widetilde{H}_u \cdot R_{\boldsymbol{xx}}(u) \cdot \widetilde{H}_u^* + \mathcal{R}_{\boldsymbol{nn}}(u)\right|}{|\mathcal{R}_{\boldsymbol{nn}}(u)|} = \frac{\left|\widetilde{H}_u^* \cdot R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) \cdot \widetilde{H}_u + \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)\right|}{|\mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)|}$$

$$= \left|\grave{H}_u \cdot R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) \cdot \grave{H}_u^* + I\right| = \left|\grave{H}_u^* \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot \grave{H}_u + I\right|$$

Finish Vector Duality

- Write equality for deflected with actual autocorrelation matrices explicitly:

$$\left|\grave{H}_u \cdot \mathcal{R}_{\boldsymbol{nn}}^{*/2}(u) \cdot R_{\boldsymbol{xx}}(u) \cdot \mathcal{R}_{\boldsymbol{nn}}^{1/2}(u) \cdot \grave{H}_u^* + I\right| = \left|\grave{H}_u^* \cdot \mathcal{R}_{\boldsymbol{nn}}^{*/2}(u) \cdot R_{\overline{\boldsymbol{xx}}}(u) \cdot \mathcal{R}_{\boldsymbol{nn}}^{1/2}(u) \cdot \grave{H}_u + I\right|$$

- SVD: $F_u \cdot \Lambda_u \cdot M_u^* = svd\left(\grave{H}_u\right)$
  - use "economy mode SVD" so that F_u and M_u may be non-square and the multiplication $F_u \cdot M_u^*$ is dimensionally ok.
- Inside determinant above is $F_u \cdot \Lambda_u \cdot M_u^* \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{*/2} \cdot R_{\boldsymbol{xx}}(u) \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{1/2} \cdot M_u \cdot \Lambda_u \cdot F_u^* + I$.
- Pre/post multiply by F's causes no change (nor by M's in 2nd determinant).
- Data Rate Equality occurs if $R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) = M_u \cdot F_u^* \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot F_u \cdot M_u^*$.

- **MAC2BC:** $R_{\overline{\boldsymbol{xx}}}(u) = \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot F_u \cdot M_u^* \cdot \mathcal{R}_{\overline{\boldsymbol{nn}}}^{-*/2}(u) \cdot R_{\boldsymbol{xx}}(u) \cdot \mathcal{R}_{\overline{\boldsymbol{nn}}}^{-1/2}(u) \cdot M_u \cdot F_u^* \cdot \mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u)$

- **BC2MAC:** $R_{\boldsymbol{xx}}(u) = \mathcal{R}_{\overline{\boldsymbol{nn}}}^{-*/2}(u) \cdot M_u \cdot F_u^* \cdot \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot R_{\overline{\boldsymbol{xx}}}(u) \cdot \mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u) \cdot F_u \cdot M_u^* \cdot \mathcal{R}_{\overline{\boldsymbol{nn}}}^{-1/2}(u)$.

MAC2BC Full Algorithm

$R_{\boldsymbol{xx}}(u)$ from minPMAC

Given:	$R_{\boldsymbol{xx}}(u)$ for $u=1,\ldots,U$; $R_{\overline{\boldsymbol{xx}}}(u)=R_{\overline{\boldsymbol{xx}}}=0$ $\mathcal{R}_{\boldsymbol{nn}}(U)=\mathcal{R}_{\overline{\boldsymbol{nn}}}(1)=I$ BC: $\tilde{H}_u^*, R_{\boldsymbol{nn}}(u)$ MAC: $\tilde{H}_u=R_{\boldsymbol{nn}}^{-1/2}(u)\cdot H_u$

For $u=U,\ldots,2$; $\mathcal{R}_{\boldsymbol{nn}}(u-1)=\mathcal{R}_{\boldsymbol{nn}}(u)+\tilde{H}_u\cdot R_{\boldsymbol{xx}}(u)\cdot\tilde{H}_u^*$

For $u=1,\ldots,U$

$$\grave{H}_u=\mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u)\cdot\tilde{H}_u\cdot\mathcal{R}_{\overline{\boldsymbol{nn}}}^{-1/2}(u)$$

$$F_u\cdot\Lambda_u\cdot M_u^*=svd\left(\grave{H}_u\right)$$

$$R_{\overline{\boldsymbol{xx}}}(u)=\mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u)\cdot F_u\cdot M_u^*\cdot\mathcal{R}_{\overline{\boldsymbol{nn}}}^{*/2}(u)\cdot R_{\boldsymbol{xx}}(u)\cdot\mathcal{R}_{\overline{\boldsymbol{nn}}}^{1/2}(u)\cdot M_u\cdot F_u^*\cdot\mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u)$$

$$R_{\overline{\boldsymbol{xx}}}=R_{\overline{\boldsymbol{xx}}}+R_{\overline{\boldsymbol{xx}}}(u)$$

$$\mathcal{R}_{\overline{\boldsymbol{nn}}}(u+1)=\mathcal{R}_{\overline{\boldsymbol{nn}}}(u)+\tilde{H}_u^*\cdot R_{\overline{\boldsymbol{xx}}}\cdot\tilde{H}_u \;;\; \text{skip } u=U$$

- So, find the $R_{\boldsymbol{xx}}(u)$ for the dual-BC's original MAC that has necessary data rate/energy.

BC2MAC Full Algorithm

Given:

$$R_{\bar{x}\bar{x}}(u) \text{ for } u = 1, \ldots, U \;\; ; \mathcal{R}_{nn}(u) = \mathcal{R}_{\bar{n}\bar{n}}(u) = 0$$

$$\mathcal{R}_{nn}(U) = \mathcal{R}_{\bar{n}\bar{n}}(1) = I \; ; R_{\bar{x}\bar{x}} = 0$$

$$\text{BC: } \tilde{H}_u^*, R_{nn}(u)$$

$$\text{MAC: } \tilde{H}_u = R_{nn}^{-1/2}(u) \cdot H_u$$

↓

$$\text{For } u = 1, \ldots, U-1 \; ;$$

$$R_{\bar{x}\bar{x}} = R_{\bar{x}\bar{x}} + R_{\bar{x}\bar{x}}(u)$$

$$\mathcal{R}_{\bar{n}\bar{n}}(u+1) = \mathcal{R}_{\bar{n}\bar{n}}(u) + \tilde{H}_u^* \cdot R_{\bar{x}\bar{x}} \cdot \tilde{H}_u$$

↓

$$\text{For } u = U, \ldots, 1$$

$$\grave{H}_u = \mathcal{R}_{nn}^{-1/2}(u) \cdot \tilde{H}_u \cdot \mathcal{R}_{\bar{n}\bar{n}}^{-1/2}(u)$$

$$F_u \cdot \Lambda_u \cdot M_u^* = svd\left(\grave{H}_u\right)$$

$$R_{xx}(u) = \mathcal{R}_{\bar{n}\bar{n}}^{-*/2}(u) \cdot M_u F_u^* \cdot \mathcal{R}_{nn}^{1/2}(u) \cdot R_{\bar{x}\bar{x}}(u) \cdot \mathcal{R}_{nn}^{*/2}(u) \cdot F_u M_u^* \cdot \mathcal{R}_{\bar{n}\bar{n}}^{-1/2}(u)$$

$$\mathcal{R}_{nn}(u-1) = \mathcal{R}_{nn}(u) + \tilde{H}_u \cdot R_{xx}(u) \cdot \tilde{H}_u^* \;\; ; \;\; \text{skip } u = 1$$

- Reverse is less interesting, but provided for completeness.

Duality conversion program (Lx constant)

- Duality design uses the Rxxb = mac2bc(Rxxm , Hmac) program:
  - The input Rxxm is $L_x \times L_x \times U$ where L_x is for the MAC.
  - The input Hmac is $L_y \times L_x \times U$ where L_y AND Hmac are for the MAC
    - Hmac(:,:,1) is on the left and so equivalent to $\tilde{H}_U$ - so then Hmac(:,:,U) is on the right
  - The output Rxxb is $L_y \times L_y \times U$ for the BC.
- With appropriate tensors, this I/O set can be repeated for each tone $n = 1, \ldots, \bar{N}$.

```
Rxxm=zeros(1,1,2);
Rxxm(1,1,1)=1/7504;
Rxxm(1,1,2)=7503/7504;
Hmac=zeros(1,1,2);
Hmac(1,1,1)=50;
Hmac(1,1,2)=80;
0.5*log2(det([50 80]*diag([1/7504 7503/7504])*[50 80]'+1)) = 6.3220

Rxxb=mac2bc(Rxxm,Hmac)
Rxxb(:,:,1) =    0.7500
Rxxb(:,:,2) =    0.2500

Rxxm(1,1,1)=1/2;
Rxxm(1,1,2)=1/2;
Rxxb=mac2bc(Rxxm,Hmac)
Rxxb(:,:,1) =   .9998
Rxxb(:,:,2) = 1.5620e-04

>> [Rwcn , bsum]=wcnoise(1, [80 ; 50], 1)
Rwcn =
    1.0000    0.6250
    0.6250    1.0000
bsum =    6.3220
```

Matlab order is reverse of MAC (same as BC).

Note Hmac order corresponds to examples on slide 12

$b_2 = .2074 \quad b_1 = 6.116$

- What if variable $L_{x,u}$, and so then variable $L_{y.u}$, on the dual?
- Set $L_x = \max_u L_{x,u}$ and append $L_x - L_{x,u}$ zero columns to each H_u and columns/rows of each $R_{\boldsymbol{xx}}(u)$.
- There will be corresponding zeroed columns/rows on $R_{\overline{\boldsymbol{xx}}}(u)$ outputs.
- Duality algorithm's inverted/square-root matrices remain nonsingular.

Possible extra credit project
Variable-Lxu mac2bc / bc2mac

Reversal bc2mac program (Lx constant)

- Rxxm = bc2mac(Rxxb , Hmac) program has:
  - The input Rxxb is $L_x \times L_x \times U$ where L_x is for the dual MAC.
  - The input Hmac is $L_y \times L_x \times U$ is for the dual MAC (not the BC).
    - To reverse from mac2bc, input the mac2bc output (Rxxb) with Hbc= **conj**(permute(Hvec(:,:,end:-1:1), [order' 3])
  - The output Rxxm is $L_x \times L_x \times U$ for the dual MAC.
- With appropriate tensors, this I/O set can be repeated for each tone $n = 1, \ldots, \bar{N}$.

```
Rxxm=zeros(1,1,2);
Rxxm(1,1,1)=1/7504;   (1.3326e-04)    user 2
Rxxm(1,1,2)=7503/7504;  (0.9999)     user 1
Hmac=zeros(1,1,2);
Hmac(1,1,1)=50;
Hmac(1,1,2)=80;
Rxxb=mac2bc(Rxxm,Hmac)
Rxxb(:,:,1) =    0.7500
Rxxb(:,:,2) =    0.2500

bc2mac(Rxxb, Hmac)
ans(:,:,1) =   1.3326e-04
ans(:,:,2) =    0.9999
Checks reverses to original.

bsum = 6.322 on this channel.
```

```
Rxxm(1,1,1)=1/2;
Rxxm(1,1,2)=1/2;
Rxxb=mac2bc(Rxxm,Hmac)
Rxxb(:,:,1) =  3.9968e-04 (1/2502_
Rxxb(:,:,2) =    0.9996  (2501/2502)
>> bc2mac(Rxxb,Hmac)
ans(:,:,1) =    0.5000
ans(:,:,2) =    0.5000
```

```
conj( permute( Hmac(:,:,end:-1:1) , [2 1  3] ) )=
   80
   50

conj( permute( Hbc(:,:,end:-1:1) , [2 1  3] ) ) =
   50
   80
```

With constant Lx , and N=1, then the designer can find the Hbc by
Hbc=conj(**permute(** Hmac(:,:,end:-1:1) **, [2 1 3])**)
Or reverse from BC to MAC with
Hmac=conj(**permute(** Hbc(:,:,end:-1:1) **, [2 1 3])**)

- This last reverse step is usually not necessary because the designer optimizes for the dual MAC.
- This Rxxm then leads through mac2bc to Rxxb to complete the BC's needed input for design.

Example with 2 dimensions/user

```
Rxxm=zeros(2,2,2);
Rxxm(:,:,1)=eye(2);
Rxxm(:,:,2)=[2 1
1 2]
Hmac=zeros(2,2,2);
Hmac(:,:,1)=[80 70
50 60];
Hmac(:,:,2)=[80 -50
40 -25]
Rxxb=mac2bc(Rxxm,Hmac)

Rxxb(:,:,1) =

   0.0036   -0.0050
  -0.0050    0.0074

Rxxb(:,:,2) =

   2.7951    0.9843
   0.9843    3.1939
```

```
>> bc2mac(Rxxb, Hmac)
ans(:,:,1) =
   1.0000    0.0000
   0.0000    1.0000

ans(:,:,2) =
   2.0000    1.0000
   1.0000    2.0000  (checks)
```

64-tone 3-user channel dual

```
N=64;
nu=3;
h=cat(3,[1 0 .8 ; 0 1 1],[.9 -.3 0 ; .5 -1 -1],[0 .2 0 ; .4 -.63 0],[0 0 0 ; 0 .648 0])*10;
H = fft(h, N, 3);
Hbc=zeros(3,2,N);

Rxxm=zeros(1,1,3);
Rxxm(1,1,:)= N/(N+nu)*[1 1 1];
Rxxb=zeros(2,2,3,N);
bbc=zeros(3,N);
Hbc=zeros(3,2,N);
 for n=1:N
Rxxb(:,:,:,n)=mac2bc(Rxxm, reshape(H(:,:,n),2,1,3)); % input needs to be Ly x Lxu x U
Hbc(:,:,n)=H(:,end:-1:1,n)';
bbc(1,n)=real(log2(1+Hbc(1,:,n)*Rxxb(:,:,1,n)*Hbc(1,:,n)'));
bbc(2,n)=real(log2((1+Hbc(2,:,n)*(Rxxb(:,:,2,n)+Rxxb(:,:,1,n))*Hbc(2,:,n)')/(1+Hbc(2,:,n)*Rxxb(:,:,1,n)*Hbc(2,:,n)')));
bbc(3,n)=real(log2((1+Hbc(3,:,n)*(Rxxb(:,:,3,n)+Rxxb(:,:,2,n)+Rxxb(:,:,1,n))*Hbc(3,:,n)')/(1+Hbc(3,:,n)*(Rxxb(:,:,2,n)+Rxxb(:,:,1,n))*Hbc(3,:,n)')));
end
bvec=sum(bbc') =  132.7477  412.8794  445.1264

>> bsum=sum(bvec) =  990.7535
```

- Equal energy used on MAC input here, so this example's dual and original MAC are not max rate sum.
  - But they are equal -- & close to max sum rate as well.
  - Note order reversal – user 1 is in best position on BC, but worst position in MAC – a feature of duality.

MAC-dual Design

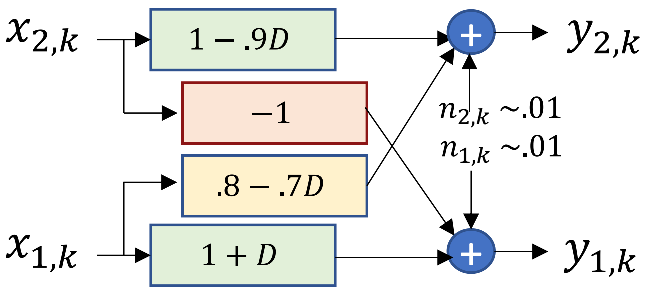
Section 5.5.4

Two-user channel with memory (Low/high pass)

$\mathcal{E}_2 = 1$

Per symbol

$\mathcal{E}_1 = 1$



$H(D) = \begin{bmatrix} 1 - .9D & .8 - .7D \\ -1 & 1 + D \end{bmatrix}$ complex baseband

Use 2×2 Vector DMT

```
h = cat(3, [1   .8 ;  -1    1], [-.9   -.7 ;  0   1]  )*10;
He = fft(h, 8, 3);         % (the matlab FFT increases energy)
```

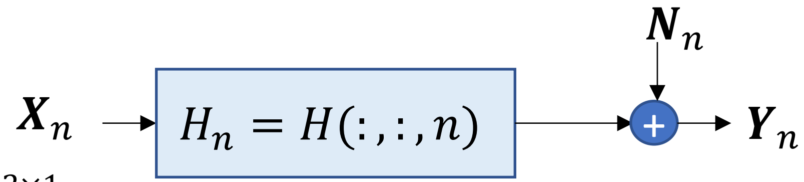
Cyclic prefix $\nu = 1$, energy loss 8/9

So far, just H →

8 tonal 2×2 channels

- Table

```
>> H
H(:,:,1) =
   1.0000 + 0.0000i   1.0000 + 0.0000i
 -10.0000 + 0.0000i  20.0000 + 0.0000i
H(:,:,2) =
   3.6360 + 6.3640i   3.0503 + 4.9497i
 -10.0000 + 0.0000i  17.0711 - 7.0711i
H(:,:,3) =
  10.0000 + 9.0000i   8.0000 + 7.0000i
 -10.0000 + 0.0000i  10.0000 -10.0000i
H(:,:,4) =
  16.3640 + 6.3640i  12.9497 + 4.9497i
 -10.0000 + 0.0000i   2.9289 - 7.0711i
H(:,:,5) =
  19.0000 + 0.0000i  15.0000 + 0.0000i
 -10.0000 + 0.0000i   0.0000 + 0.0000i
H(:,:,6) =
  16.3640 - 6.3640i  12.9497 - 4.9497i
 -10.0000 + 0.0000i   2.9289 + 7.0711i
H(:,:,7) =
  10.0000 - 9.0000i   8.0000 - 7.0000i
 -10.0000 + 0.0000i  10.0000 +10.0000i
H(:,:,8) =
   3.6360 - 6.3640i   3.0503 - 4.9497i
 -10.0000 + 0.0000i  17.0711 + 7.0711i
```



2×1

$= \begin{bmatrix} X_{2,n} \\ X_{1,n} \end{bmatrix}$ $= \begin{bmatrix} Y_{2,n} \\ Y_{1,n} \end{bmatrix}$

Single-User Upper Bound

- The highest data rate is the single-user capacity for the matrix AWGN – need SVD of big H.

```
Hblock=blkdiag(H(:,:,1),H(:,:,2),H(:,:,3),H(:,:,4),H(:,:,5),H(:,:,6),H(:,:,7),H(:,:,8));
g=svd(Hblock);
gains=(g.*g)' =
  651.4623  567.9619  567.9619  500.2007  447.4561  447.4561  405.2081  405.2081
  188.7919  188.7919  91.0919   91.0919   81.4901   81.4901   34.5377   1.7993
```

- Water-fill on these singular-value-squared parallel channels ($\Gamma = 0$ dB):

```
>> En = waterfill(128/9, gains', 1);
>> En' =
 0.9285   0.9283   0.9283   0.9280   0.9278   0.9278   0.9276   0.9276
 0.9247   0.9247   0.9191   0.9191   0.9178   0.9178   0.9011   0.3743
```

Single-user waterfill is close to equal energy on all tones and spatial dimensions – very high SNR.

```
>> bvec=log2(ones(16,1)+En.*gains')'
9.2429   9.0450   9.0450   8.8617   8.7010   8.7010   8.5579   8.5579
7.4560   7.4560   6.4046   6.4046   6.2439   6.2439   5.0055   0.7428

>> sumrate = sum(bvec) = 116.6695
>>sum(bvec)/ 9 = 12.9633
```

There are now 8 MMSE-MAC GDFEs, 1 for each tone

- Repetitive process for each n, initialize/size:

```
GU=zeros(2,2,8);
WU=zeros(2,2,8);
S0=zeros(2,2,8);
Bu=zeros(2,8);
MSWMFU=zeros(2,2,8);
AU=zeros(2,2,8);
for n=1:8  AU(:,:,n)=(sqrt(8)/3)*eye(2);  end
```

Note the sqrt(8)/3 on each of 8 dim's for each of 2 users. So 1/3 = 1/sqrt(9) for each Au a second factor of 8 matches unit-noise-whitening on 8 tones.

- Compute the MAC GDFE for each n,

```
>> for n=1:8
[Bu(:,n), GU(:,:,n) , WU(:,:,n),S0(:,:,n), MSWMFU(:,:,n)] = mu_mac(H(:,:,n), AU(:,:,n), [1 1], 1);
end
bvec=sum(Bu') = 62.3515   54.1393
Bsum =sum(bvec)   = 116.4908
```

- Bits/user/tone **10.1778**

```
Bu =
   6.5043   7.1048   7.9703   8.5075   8.6822   8.5075   7.9703   7.1048
   3.6736   7.7329   7.9256   6.8322   5.4843   6.8322   7.9256   7.7329
sum(Bu) =
  10.1778   14.8377   15.8959   15.3396   14.1665   15.3396   15.8959   14.8377

bvec =   62.3515   54.1393
Bsum  = 116.4908
```

so then bsum/9 = 12.9434 bits/tone or roughly 13 bits/Hz for both users

Why are single-user and MAC close?

(The flat high energy input on ALL dimensions means that SVD M might as well be I. Lots of square roots and one is I)

MAC Receiver Designs

- Unbiased total linear rcvr processing, each tone

```
MSWMFU
MSWMFU(:,:,1) =
  0.0105 + 0.0000i  -0.1050 + 0.0000i
  0.2364 + 0.0000i   0.0412 + 0.0000i
MSWMFU(:,:,2) =
  0.0251 - 0.0439i  -0.0690 - 0.0000i
  0.0397 - 0.0382i   0.0392 + 0.0116i
MSWMFU(:,:,3) =
  0.0377 - 0.0340i  -0.0377 + 0.0000i
  0.0374 - 0.0084i   0.0449 + 0.0254i
MSWMFU(:,:,4) =
  0.0425 - 0.0165i  -0.0260 + 0.0000i
  0.0456 + 0.0096i   0.0681 + 0.0449i
MSWMFU(:,:,5) =
  0.0437 + 0.0000i  -0.0230 + 0.0000i
  0.0707 + 0.0000i   0.1329 + 0.0000i
MSWMFU(:,:,6) =
  0.0425 + 0.0165i  -0.0260 + 0.0000i
  0.0456 - 0.0096i   0.0681 - 0.0449i
MSWMFU(:,:,7) =
  0.0377 + 0.0340i  -0.0377 + 0.0000i
  0.0374 + 0.0084i   0.0449 - 0.0254i
MSWMFU(:,:,8) =
  0.0251 + 0.0439i  -0.0690 + 0.0000i
  0.0397 + 0.0382i   0.0392 - 0.0116i
```

- Unbiased feedback sections for each tone

```
GU
GU(:,:,1) =
  1.0000 + 0.0000i  -1.9703 + 0.0000i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,2) =
  1.0000 + 0.0000i  -0.8335 + 0.4508i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,3) =
  1.0000 + 0.0000i   0.1530 + 0.3488i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,4) =
  1.0000 + 0.0000i   0.5244 + 0.1697i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,5) =
  1.0000 + 0.0000i   0.6182 + 0.0000i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,6) =
  1.0000 + 0.0000i   0.5244 - 0.1697i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,7) =
  1.0000 + 0.0000i   0.1530 - 0.3488i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,8) =
  1.0000 + 0.0000i  -0.8335 - 0.4508i
  0.0000 + 0.0000i   1.0000 + 0.0000i
```

What about the other vertex (puts user 1 at top)

- Design uses the same initialization because both users had same energy anyway

```
J=hankel([0 1]);
for n=1:8
Hflip(:,:,n)=J*H(:,:,n)*J;
end
>> for n=1:8
[Bu(:,n), GU(:,:,n) , WU(:,:,n),S0(:,:,n), MSWMF(:,:,n)] = mu_mac(Hflip(:,:,n), AU(:,:,n), [1 1], 1);
end
Bu
sum(Bu)
bvec=sum(Bu')
bsum=sum(bvec)
```

- The user bit rates are useful in duality n,

```
Bu =
   8.4816   8.3860   8.1253   7.8068   7.6511   7.8068   8.1253   8.3860
   1.6963   6.4517   7.7706   7.5328   6.5155   7.5328   7.7706   6.4517

sum(Bu) =
  10.1778  14.8377  15.8959  15.3396  14.1665  15.3396  15.8959  14.8377

bvec =  64.7687  51.7220
bsum =  116.4908 ,  so same (check)
```

10.1778

Rate sum is maintained on each tone, but decoding order is reversed

And the loss

- For this example, the data rate is already close to single-user WF.

- Answer for E=[1 1] using SWF:

```
Rnn=zeros(2,2,8);
for n=1:8
Rnn(:,:,n)=eye(2);
end
>> [Rxx, bsum , bsum_lin] = SWF((8/9)*[1 1], H, [1 1], Rnn, 1)

Rxx(:,:,1) =        Rxx(:,:,2) =        Rxx(:,:,3) =        Rxx(:,:,4) =
0.5500      0       0.9339      0       0.9401      0       0.9394      0
   0   0.8401       0      0.8991       0      0.8996       0      0.8954
Rxx(:,:,5) =        Rxx(:,:,6) =        Rxx(:,:,7) =        Rxx(:,:,8) =
0.9344      0       0.9394      0       0.9401      0       0.9339      0
   0   0.8829       0      0.8954       0      0.8996       0      0.8991
bsum =  116.5835

gamma_mac = 10*log10((2^(116.5835/9) -1)/(2^(116.4908/9)-1)) = 0.0310  dB

bsum_lin = 106.1991

Linear loss is
10*log10( (2^(116.4908/9)-1) / (2^(106.1991/9)-1) ) = 3.4430 dB
```

- Maximum Esum MAC rate sum:

```
[Rxx, bsum, bsum_lin] = macmax(16/9, h, [1 1], 8 , 1);
>> bsum =  116.5891  (so greater than Evec, but less than vector code 116.6695)

>> bsum_lin =  106.2249
```

- So, **116.5** is (also) best E-sum MAC rate sum (and also for Evec of [1 1] – pretty close already).

Order Reversal

- Semantics – alternate dual definition $\tilde{H}_{dual}=\left(\mathcal{J}_y \cdot \tilde{H} \cdot \mathcal{J}_x\right)^* = \mathcal{J}_x \cdot \tilde{H}^* \cdot \mathcal{J}_y$.

$$\mathcal{J}_y \triangleq \begin{bmatrix} 0 & 0 & I_{L_{y,U}} \\ 0 & \iddots & 0 \\ I_{L_{y,1}} & 0 & 0 \end{bmatrix}$$

- All information, SVD, energy, etc are preserved as without $\mathcal{J}_y$:
  - Single output on MAC corresponds to single input on BC.
  - $\mathcal{J}_y$ just re-indexes dimensions (not users) – but this is matlab's usual indexing.
  - But the definition said reverse order (so can do it explicitly with $\mathcal{J}_y$).
- The mac2bc and bc2mac programs basically do this tacitly in finding inputs:
  - L16's Hbc=conj(**permute(** Hmac(:,:,end:-1:1) **, [2 1 3])**) for $N = 1$ presumes the channel input has dimension 1 at top
  - So, this is essentially multiplying by $\mathcal{J}_y$ **tacitly**.
- Example has: $$H_{dual}(D) = \left(\mathcal{J}_y \cdot \underbrace{\begin{bmatrix} 1-.9D & .8-.7D \\ -1 & 1+D \end{bmatrix}}_{H(D)} \cdot \mathcal{J}_x\right)^* = \begin{bmatrix} 1+D^* & .8-.7D^* \\ -1 & 1-.9D^* \end{bmatrix}$$

$D^* \rightarrow e^{j\omega}$ on unit circle (FFT)

  - Essentially dual here causes:
    - User direct (magnitudes, - phase) to be the same, but priority is reversed.
    - Crosstalk flow to flip from transfer of $u \rightarrow u'$ on original to $u \leftarrow u'$.

End Lecture 16