STANFORD

Lecture 13

MAC GDFEs and Design Measures

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Announcements & Agenda

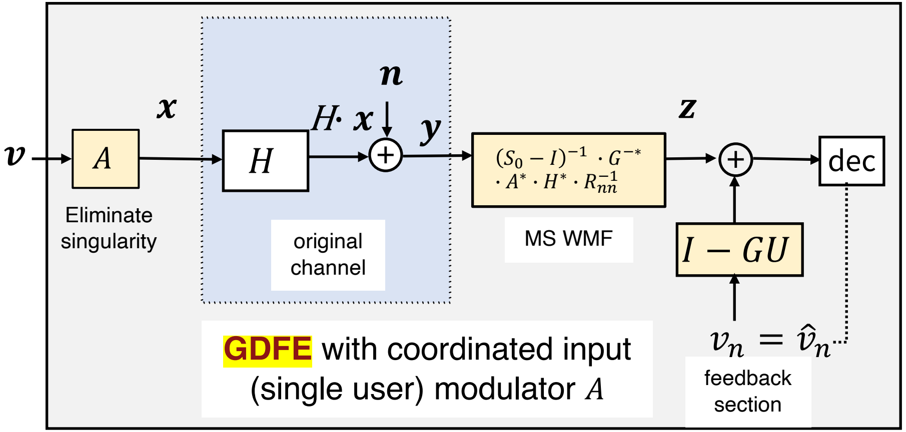
- Announcements
  - Section 5.4
  - PS7 – last homework, 2 weeks, double weight.

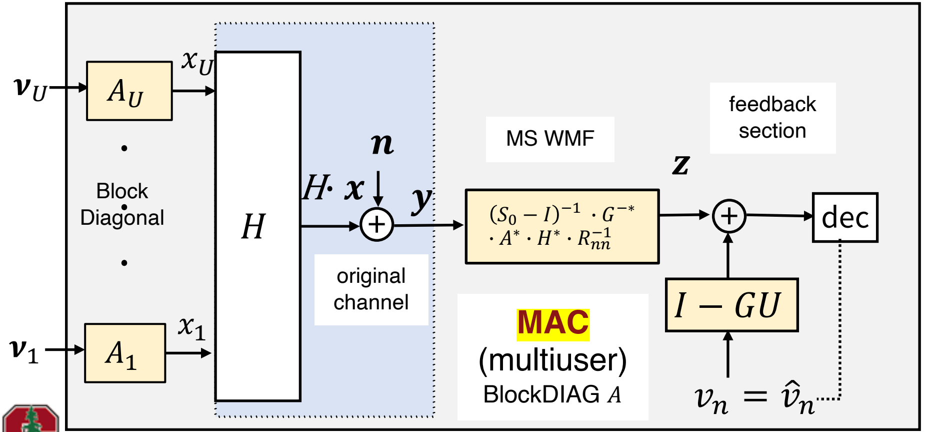
- Problem Set 7 = PS7 (due June 5 or 7)
  1. 5.16 A tonal channel
  2. 5.17 GDFE MAC Design
  3. 5.18 Dual computations
  4. 5.19 GDFE BC design via duality
  5. 5.20 IC with/without GDFE

- Agenda
  - MAC and GDFE Comparison (Sec 5.4.1)
  - Tonal MAC with DMT (Section 5.4.2)
    - Tonal GDFE
    - SWF (simultaneous water-filling returns)
  - Designs with weighted sums (Section 5.4.3)

MAC and GDFE Comparison

The MMSE MAC vs MMSE GDFE





- **GDFE** is:
  - Designed for "single-user" H ; $A = R_{\boldsymbol{xx}}^{1/2}$,
  - MMSE: $R_b^{-1} = R_{\boldsymbol{xx}}^{-*/2} \cdot H^* \cdot R_{\boldsymbol{nn}}^{-1} \cdot H \cdot R_{\boldsymbol{xx}}^{-1/2} + I = G \cdot S_0 \cdot G^*$,
  - Canonical performance (decisions correct),
  - Input has only $trace\{R_{\boldsymbol{xx}}\} \leq \mathcal{E}_x$ energy constraint, &
  - Reliable rate is independent of dimensional order.

- **MAC** has:
  - **block-diag** $R_{\boldsymbol{xx}}, , and\ A,$ with $trace\{R_{\boldsymbol{xx}}\} \leq \mathcal{E}_x$,
    - only in energy-sum case, and otherwise
  - input energies $trace\{R_{\boldsymbol{xx}}(u)\} \leq \mathcal{E}_u$,
  - separated locations so $A_u = R_{\boldsymbol{xx}}^{1/2}(u)$; $A = R_{\boldsymbol{xx}}^{1/2}$,
  - MMSE: $R_b^{-1} = R_{\boldsymbol{xx}}^{-*/2} \cdot H^* \cdot R_{\boldsymbol{nn}}^{-1} \cdot H \cdot R_{\boldsymbol{xx}}^{-1/2} + I = G \cdot S_0 \cdot G^*$
  - Canonical performance (decisions correct)
    - Rates per user – order shifts rate sum's allocation to users.

Revisit Chapter 2's Scalar MAC

- **MAC** 80/60 channel

```
>> H=[80 60];
>> Rxx=0.5*eye(2); %(equal energy both dim/users)
>> A=[sqrt(.5) 0' ; 0 sqrt(.5) ];
>> Lxu=[1 1];
>> cb=2;

>> [BU, GU, WU, S0, MSWMFU] = mu_mac(H, A, Lxu , cb);

BU =   5.8222   0.3218
GU =   1.0000   0.7500     MSWMFU =   0.0177
            0   1.0000                0.0236
S0 =  1.0e+03 *
       3.2010        0
            0   0.0016

>> sum(BU) =   6.1440
>> 10*log10(2^(6.1440)-1) =  18.4953 dB
```

(shares 2 input dim that cannot coordinate modulator)

- **GDFE** – remove singularity

```
>> [F,L,M]=svd(H);
>> [F,L,M]=svd(H)
F = 1
L =  100     0
M =
   0.8000   -0.6000
   0.6000    0.8000
>> 0.5*log2(1+ 0.5*L(1)^2) =  6.1440

All energy on pass space
>> 0.5*log2(1+ 1*L(1)^2) =   6.64 > 6.144
```

- For 6.64, (single-user) input is: $\boldsymbol{x} = \begin{bmatrix} .800 \\ .600 \end{bmatrix} \cdot v$
- v goes to both channel input dimensions (not MAC).
- All GDFE's with this input $R_{xx} = [\mathbf{1}\ \mathbf{0}\ ;\mathbf{0}\ \mathbf{0}]$ perform same
  - and trivially have $G = 1$.
  - Modulator coordinates dimensions

Or use computeGDFE.m

```
>> [snrGDFEu, GU, WU, S0, MSWMFU, b, bbar,
snrGLEu] = computeGDFE(H, A, cb)

snrGDFEu =   18.4334 dB
GU =
    1.0000    0.7500
         0    1.0000
WU =
    0.0003         0
   -1.3333    1.7783
S0 =  1.0e+03 *
    3.2010         0
         0    0.0016
MSWMFU =
    0.0177
    0.0236
b =
    5.8222
    0.3218
bbar =    3.0720
snrGLEu =  16.8125 dB
>> sum(b) =   6.1440
```

No Lx input

```
>> [F,L,M]=svd(H);  % Vector Coding Example
>> Lx=2;
[snrGDFEu, GU, WU, S0, MSWMFU, b, bbar, snrGLEu] =
computeGDFE(H, M(:,1), cb, Lx)
snrGDFEu =   19.9566
GU =     1
WU =   1.0000e-04
S0 = 10001
MSWMFU =   0.0100
b =    6.6439 > 6.144 (MAC is worse than single user).
bbar =    3.3220
snrGLEu =   19.9566(dB) – linear is same for VC

(same energy, but more going to best mode)
```

Not a MAC, A is not diag.

Better to use mu_mac with a MAC, than to play with cb & Lx on computeGDFE, which is really for single user GDFEs.

Similarly: use computeGDFE on single user. ComputeGDFE also provides linear output.

Correct comparison with GDFE notes the A input has 2 real dimensions

Vector Coding resets the Lx to 2 as optional 4th computeGDFE input.

MAC Loss

- MAC Loss – ratio of single-user capacity SNR to MAC maximum-rate-sum SNR (for $[H \quad R_{nn}]$).

$$\gamma_{MAC} \triangleq \frac{2^{2\cdot\bar{\mathcal{C}}} - 1}{2^{2\cdot\bar{\mathcal{C}}_{e-sum}} - 1}$$

```
>> [~, btsum , ~]=macmax(1,[80 60],[1 1],1,2)
btemp =    6.3220
```

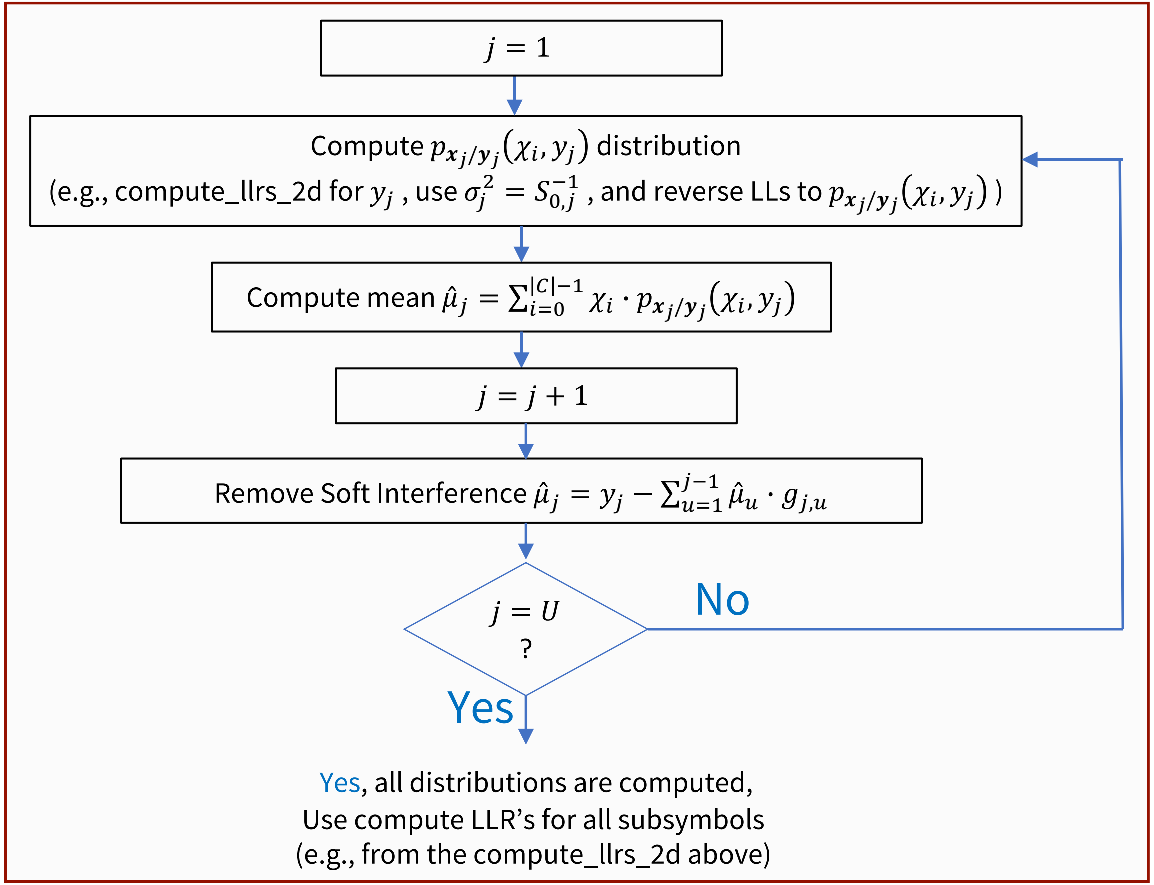
- For the previous example: $\gamma_{MAC} = \frac{2^{6.64} - 1}{2^{6.322} - 1} = 1.5$ dB.

- Clearly $0 \leq \gamma_{MAC} \leq 1$.

See also *split-dimensionality* example in Section 5.4.1.

Soft Canceller Flow Chart (379A-L16:32)

- If decision is likely correct, $= \mathbb{E}[x_u/y_u]$.
- Then soft canceller = GDFE.
- Mean (user) value when not so certain, essentially allows reuse of the other user's soft information with a channel output adjusted by removal of crosstalk.
- The use in this case is to reduce any error propagation from one user to another in the MAC GDFE, instead of single-user error propagation in 379A, L16.



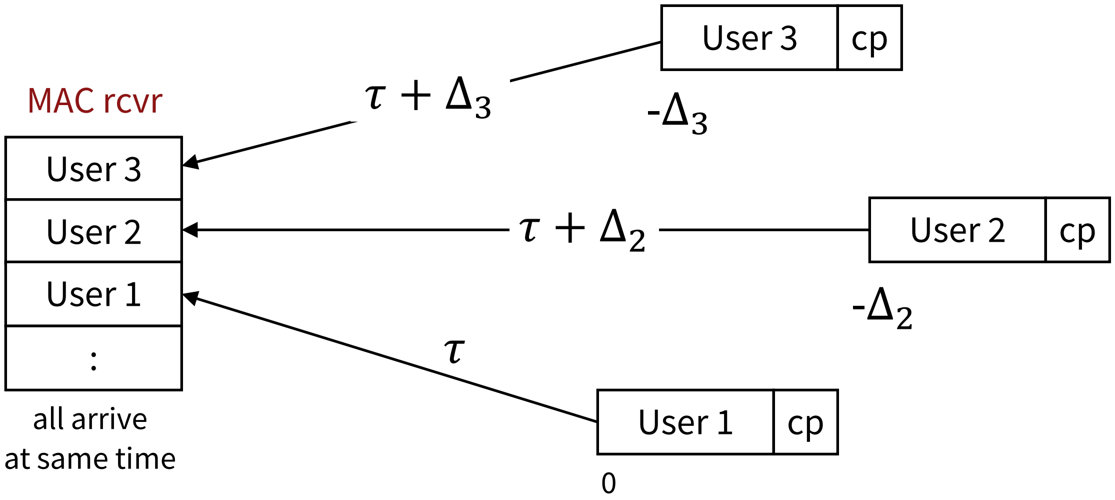
Tonal MAC (with DMT)

Section 5.4.2

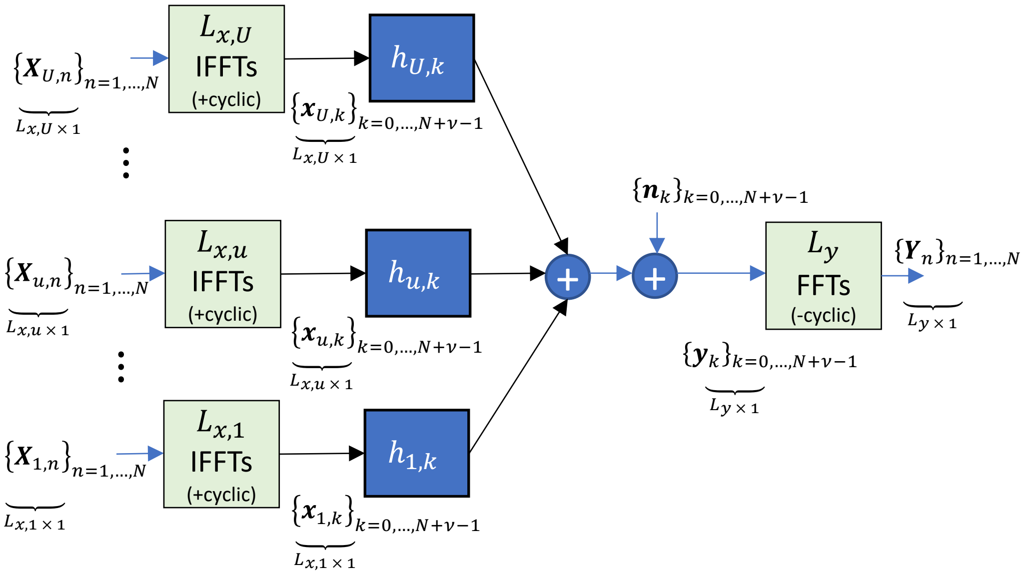
Note Section 5.4.1’s two-user ISI-GDFE is interesting, but largely becomes superfluous with the tonal vector-DMT system.

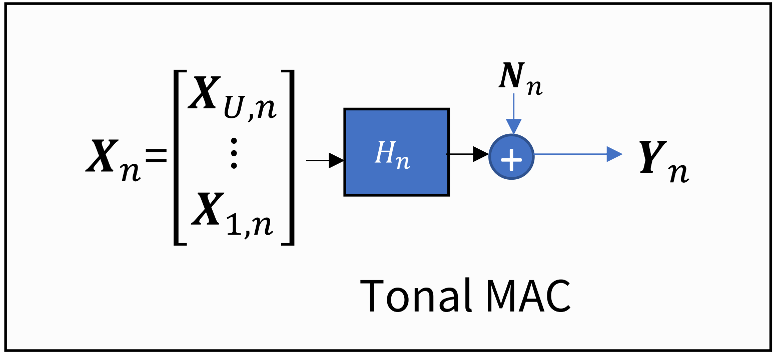
Align Receiver DMT Symbols for MAC

- So far, examples have largely been space time (with a few antennas).
- In practice, there usually is a temporal (time-freq) C-OFDM or DMT system **also** present.



Vector DMT/OFDM with MAC





$$R_{XX}(u,n) = \mathbb{E}\left[X_{u,n} \cdot X_{u,n}^*\right]$$

$$\sum_n trace\,\{R_{XX}(u,n)\} \leq \mathcal{E}_u \;\forall\, u = 1, \ldots, U$$

Esum−MAC: $\sum_{u=1}^{U} \sum_{n=0}^{\bar{N}} trace\,\{R_{XX}(u,n)\} \leq \mathcal{E}_x$

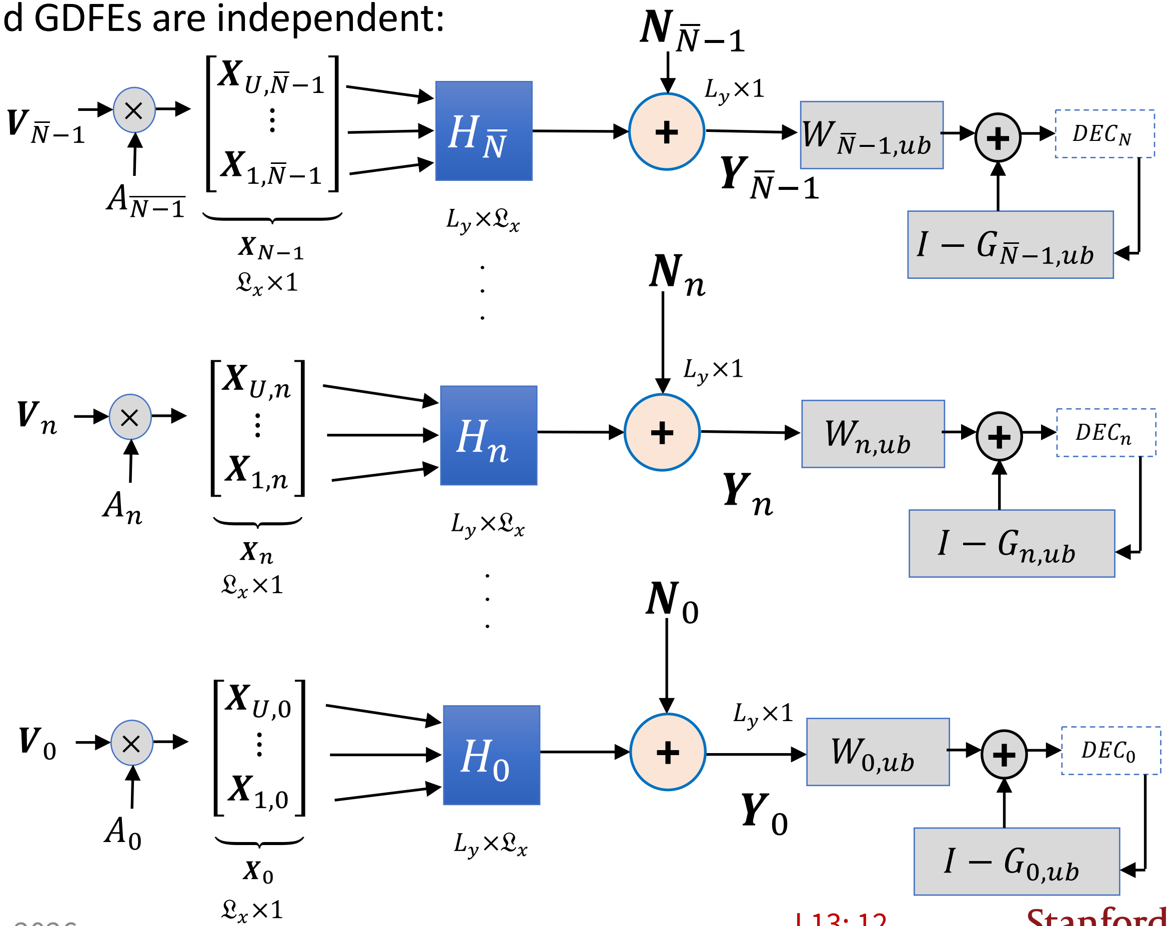
- Symbol boundaries align through cyclic-extension (guard period) uses (even with different channel delays, may increase ν).
- Basically, an IFFT per transmit-antenna-user – symbol boundary delayed/advanced so that receiver FFT captures all U.
- Discrete=time MT MAC, indeed all SVD's, Cholesky's, and QR factorizations become "frequency-dependent" (finite n replaces $\lim_{n\to\infty}[\ ]$).

Tonal GDFEs with MAC

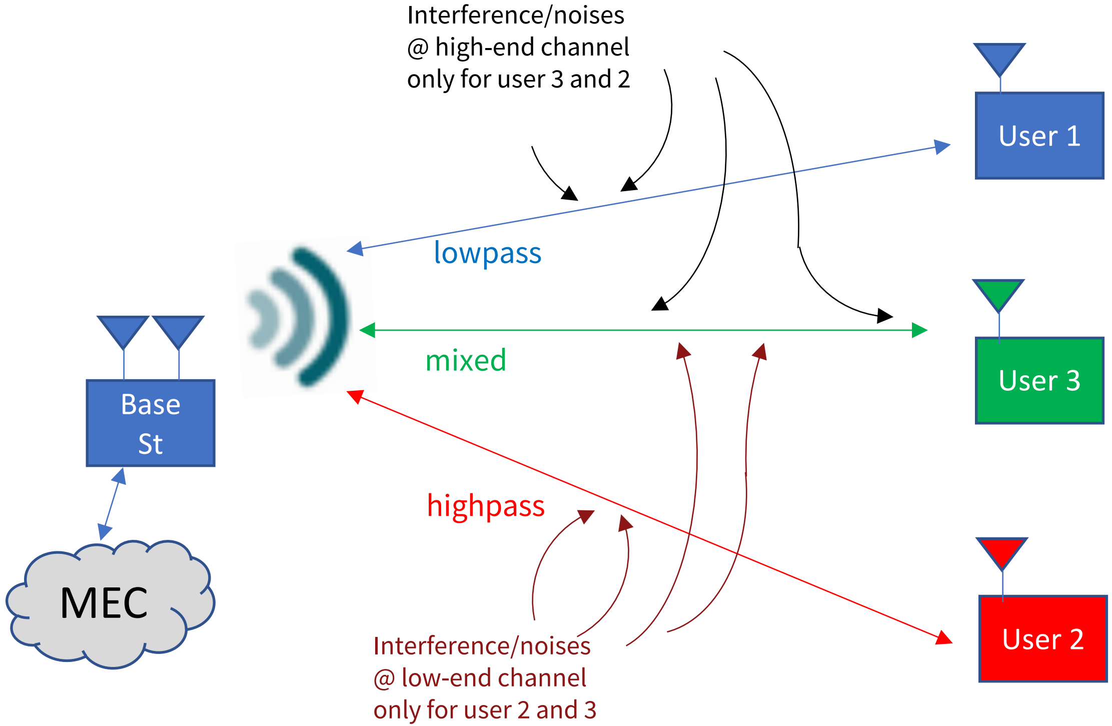
- Discrete tonal modulators and GDFEs are independent:

- Put an "n" index on all the GDFE design equations.

$$
\begin{aligned}
b_{u,\ell} &= \sum_{n=0}^{\bar{N}-1} b_{u,\ell,n} \\
b_{u,n} &= \sum_{\ell=1}^{L_{x,u}} b_{u,\ell,n} \\
b_u &= \sum_{\ell=1}^{L_{x,u}} \sum_{n=0}^{\bar{N}-1} b_{u,\ell,n} \\
b &= \sum_{u=1}^{U} \sum_{\ell=1}^{L_{x,u}} \sum_{n=0}^{\bar{N}-1} b_{u,\ell,n} \ .
\end{aligned}
$$



Example complex BB channel



More users than antennas

This allows up to 64 resource Blocks (tones) for each user, all in same channel.

- Illustrates many effects
  - ISI and crosstalk

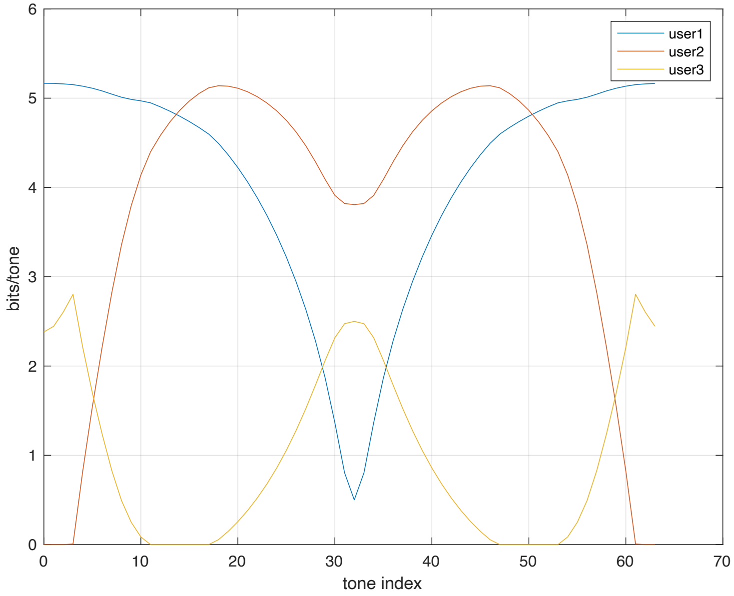
$$H(D) = \begin{bmatrix} 1 + .9 \cdot D & -.3 \cdot D + .2 \cdot D^2 & .8 \\ .5 \cdot D - .4 \cdot D^2 & 1 - D - .63 \cdot D^2 + .648 \cdot D^3 & 1 - D \end{bmatrix}$$

A bit oversimplified, as real only for cplex channel.

Example has ISI and MIMO together

- There are 3 user channels .
- Any tone will maximally have rank $\wp_H$=2; $\mathcal{N}_0 = .01$.

$$H(D) = \begin{bmatrix} 1+.9D & -.3D+.2D^2 & .8 \\ .5D-.4D^2 & 1-D-.63D^2+.648D^3 & 1-D \end{bmatrix}$$




```
h=cat(3,[1 0 .8 ; 0 1 1],[.9 -.3 0 ; .5 -1 -1],[0 .2 0 ; .4 -.63 0],[0 0 0 ; 0 .648 0])*10;
h(:,:,1) =
   10    0    8
    0   10   10
h(:,:,2) =
    9   -3    0
    5  -10  -10
h(:,:,3) =
        0   2.0000        0
   4.0000  -6.3000        0
h(:,:,4) =
        0        0        0
        0   6.4800        0
N=8;
H = fft(h, N, 3)
>> H = fft(h, N, 3)
H(:,:,1) =
  19.0000 + 0.0000i  -1.0000 + 0.0000i   8.0000 + 0.0000i
   9.0000 + 0.0000i   0.1800 + 0.0000i   0.0000 + 0.0000i
H(:,:,2) =
  16.3640 - 6.3640i  -2.1213 + 0.1213i   8.0000 + 0.0000i
   3.5355 - 7.5355i  -1.6531 + 8.7890i   2.9289 + 7.0711i
And 6 more values, see text
```

← Increase to 64 – look ahead at MAC with equal energy every dimension.

(matlab commands on next page)

Actual MAC/GDFE calculations for L13:13

White-Input Tonal GDFE

```
Nmax=32;
U=3;
Ly=2;
cb=1;
Lxu=[1 1 1];
bsum=zeros(1,Nmax);
for index=1:Nmax
 i=2*index;
 H = fft(h, i, 3);
 GU=zeros(U,U,i);
 WU=zeros(U,U,i);
 S0=zeros(U,U,i);
 Bu=zeros(U,i);
 MSWMFU=zeros(U,Ly,i);
 AU=zeros(3,3,i);
 for n=1:i
  AU(:,:,n)=sqrt(i)/sqrt(i+3)*eye(3);
 end
 for n=1:i
  [Bu(:,n), GU(:,:,n) , WU(:,:,n),S0(:,:,n), MSWMFU(:,:,n)] = ...
   mu_mac(H(:,:,n), AU(:,:,n), Lxu, cb);
 end
bvec=sum(Bu');
Bsum(index) = sum(bvec);
End
bvec =  445.1264  412.8794  132.7477
sum(bvec) =  990.7535
```

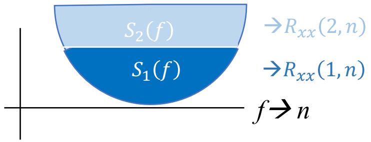
```
>> GU(:,:,23)  =
   1.0000 + 0.0000i  -1.3365 + 0.3447i  -0.3093 + 0.3130i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.8108 + 0.4218i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
>> MSWMFU(:,:,15) =
   0.0454 + 0.0341i  -0.0105 + 0.0249i
   0.0183 + 0.0179i   0.0275 - 0.0468i
   0.0589 + 0.0199i   0.0179 - 0.0416i
>> SNRs = 10*log10(diag(S0(:,:,29)).^(64/67)-eye(3)) =

   12.1946
   20.7691
    8.8939
```

Feedback Section G coeffients are large.

Simultaneous Water Filling with DMT

- Frequency f → tone index n.



See also Section 2.7.4.2 and Lecture 7.

1. Find all noise and crosstalk:
   - $R_{noise}(u,n) = \sum_{i \neq u} H_{i,n} \cdot R_{\boldsymbol{XX}}(i,n) \cdot H^*_{i,n} + R_{\boldsymbol{NN}}(n)$

2. Create a noise-equivalent that includes all other users as noise (no order, all):
   - $\widetilde{H}_{u,n} = R^{-1/2}_{noise}(u,n) \cdot H_{u,n} = F_{u,n} \cdot \Lambda_{u,n} \cdot M^*_{u,n}$

3. Water-fill each user:
   - $\mathcal{E}_{u,l,n} + \frac{1}{g_{u,l,n}} = K_u \;\; \forall\, n, l$ with $g_{u,l,n} = \lambda^2_{u,l,n}$

4. Form resulting input autocorrelation matrices (energy distribution with $L_{x,u} = 1$)
   - $R^o_{\boldsymbol{XX}}(u,n) = M_{u,n} \cdot \text{Diag}\{\mathcal{E}_{u,n}\} \cdot M^*_{u,n} \;\; \forall\, n = 0, \ldots, \overline{N} - 1$

With MT and $L_y = 1, n \to \infty$
There is always an FDM SWF solution.

SWF.m versus macmax.m

```
>> help SWF
 function [Rxx, bsum , bsum_lin] = SWF(Eu, H, user_ind, Rnn, cb)

 Simultaneous water-filling MAC max rate sum (linear and nonlinear GDFE)
 The input is space-time domain h, and the user can specify a temporal
 block symbol size N (essentially an FFT size).

 Inputs:
 Eu  U x 1 energy/SAMPLE vector. Single scalar equal energy all users
    any (N/N+nu) scaling should occur BEFORE input to this program.
 H   The FREQUENCY-DOMAIN Ly x sum(Lx(u)) x N MIMO channel for all users.
    N is determined from size(H) where N = # tones
    (equally spaced over (0,1/T) at N/T.
    if time-domain h, H = 1/sqrt(N)*fft(h, N, 3);
 user_ind  The start index for each user, in the same order as Eu
    The Lxu vector of each user's number of antennas is computed
    internally.  %    U is determined from user_ind
 Rnn The Ly x Ly x N noise-autocorrelation tensor (last index is per tone)
 cb  cb = 1 for complex, cb=2 for real baseband

 Outputs:
 Rxx A block-diagonal psd matrix with the input autocorrelation for each
    user on each tone. Rxx has size (sum(Lx(u)) x sum(Lx(u)) x N .
    sum trace(Rxx) over tones and spatial dimensions equal the Eu
 bsum the maximum rate sum.
 bsum  bsum_lin - the maximum sum rate with a linear receiver
    b is an internal convergence sum rate value, not output

 This program is modified version of one originally supplied by student
 Chris Baca
```

Energy-Vector MAC

```
function [Rxx, bsum , bsum_lin] = macmax(Esum, h, Lxu, N , cb)

 Simultaneous water-filling Esum MAC max rate sum (linear & nonlinear GDFE)
 The input is space-time domain h, and the user can specify a temporal
 block symbol size N (essentially an FFT size).

 This program uses the CVX package

 the inputs are:
 Esum The sum-user energy/SAMPLE scalar in time-domain.
     This will be increased by cb*N by this program.
     Each user energy should be scaled by N/(N+nu)if there is cyclic prefix
     This energy is the trace of the corresponding user Rxx (u)
     The sum energy is computed as the sum of the Eu components
     internally.
 h   The TIME-DOMAIN Ly x sum(Lx(u)) x N channel for all users
 Lxu  The number of antennas for each user 1 x U
 N   The FFT size (equally spaced over (0,1/T) at 1/(NT).
 cb   cb = 1 for complex, cb=2 for real baseband

 the outputs are:
 Rxx A block-diagonal psd matrix with the input autocorrelation for each
    user on each tone. Rxx has size (sum(Lx(u)) x sum(Lx(u)) x N .
    sum trace(Rxx) over tones and spatial dimensions equal the Eu
    FREQUENCY DOMAIN
 bsum the maximum rate sum, when cb=2, this is effectively over lower half
 of tones, or equivalently the 1/2*log2 form of data rates are summed
 bsum  bsum_lin - the maximum sum rate with a linear receiver

    b is an internal convergence (vector, rms) value, but not sum rate
```

Energy-Sum MAC

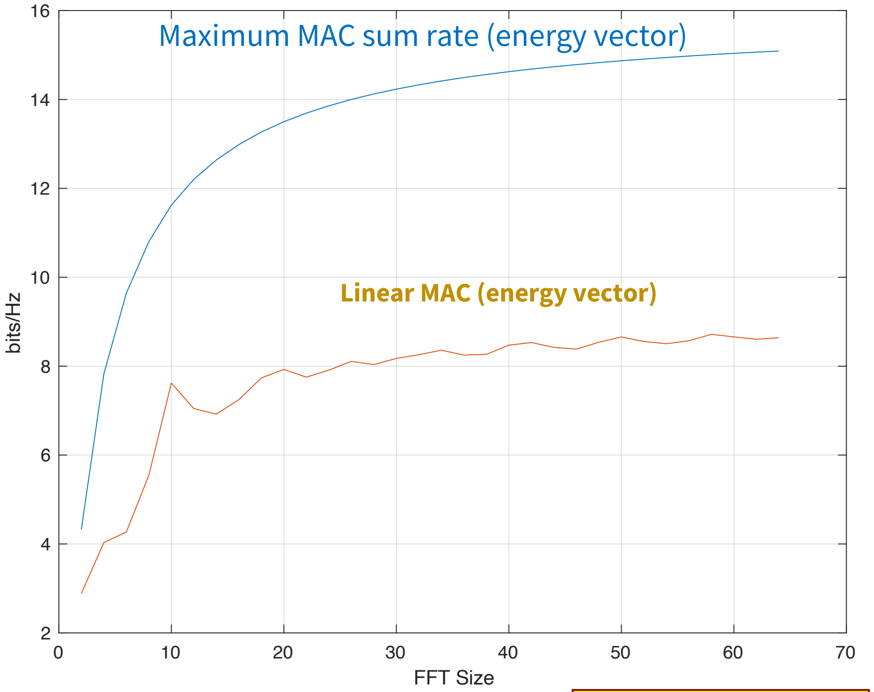
Up to N on h

- SWF is frequency domain input (useful with non-white noise psd), has separate user energies, and uses no CVX.
- macmax is time-domain (and uses Lxu instead of user_ind), has single sum energy, and uses CVX, must supply N.

Max rate-sum Example

```
h=cat(3,[1 0 .8 ; 0 1 1],[.9 -.3 0 ; .5 -1 -1],[0 .2 0 ; .4 -.63 0],[0 0 0 ; 0 .648 0])*10;
bsum=zeros(1,Nmax);
bsumlin=zeros(1,Nmax);

for index=1:Nmax
  i=2*index;   % (don't need to plot a point for every number of tones)
  H = fft(h, i, 3);
  Rnn=zeros(Ly,Ly,i);
    for n=1:i
    Rnn(:,:,n) = eye(2);
    end
  [Rxx, bsum(index), bsumlin(index)] = SWF(i/(i+3)*[1 1 1], H, [1  1 1], Rnn(:,:,:), 1);
  bsum(index)=bsum(index)/(i+3);
  bsumlin(index)=bsumlin(index)/(i+3);
end
bsum(32)*67= 1011.1 >  990.8
bsumlin(32)*67 = 578.8502
plot(2*[1:Nmax], bsum,2*[1:Nmax],bsumlin)
```



- Even with 3 users > 2 antennas, linear loses much.
  - ~ 20 dB (from "link budget)
- **Linear curve variation** is because $\overline{N}$ is finite and the simultaneous water filling is not necessarily best solution under linear restriction
- When would "linear receiver be best?"

Vector-coding cannot be used, on MAC in general.

The linear max-sum prob is not convex, See OSB in Sec 5.6

```
Rxx(:,:,1)
   1.4504   0       0
       0    0       0
       0    0   1.3050
Rxx(:,:,2) =
   1.4512   0       0
       0    0       0
       0    0   1.3120
Rxx(:,:,3) =
   1.4528   0       0
       0    0       0
       0    0   1.3274
.........
Rxx(:,:,9) =
   1.4419        0       0
       0    0.0670       0
       0         0  1.3303
Rxx(:,:,11) =
   1.3170        0       0
       0    1.0228       0
       0         0  0.5748
Rxx(:,:,15) =

   1.3889        0       0
       0    1.4116       0
       0         0  0.1513
Rxx(:,:,26) =
   0.1384        0       0
       0    1.4184       0
       0         0  1.3192
```

```
.........
Rxx(:,:,27) =
       0         0       0
       0    1.4939       0
       0         0  1.3767
Rxx(:,:,28) =
       0         0       0
       0    1.4885       0
       0         0  1.3761
Rxx(:,:,31) =

       0         0       0
       0    1.4229       0
       0         0  1.3689
Rxx(:,:,32) =
       0         0       0
       0    1.3394       0
       0         0  1.3606
Rxx(:,:,39) =
       0         0       0
       0    1.4939       0
       0         0  1.3767
Rxx(:,:,40) =
   0.1384        0       0
       0    1.4184       0
       0         0  1.3192
Rxx(:,:,51) =
   1.3889        0       0
       0    1.4116       0
       0         0  0.1513
Rxx(:,:,52) =
   1.3871        0       0
       0    1.3929       0
       0         0  0.1700
```

```
Rxx(:,:,53) =
   1.3678        0       0
       0    1.3329       0
       0         0   0.243
Rxx(:,:,57) =

   1.4419        0       0
       0    0.0670       0
       0         0  1.3303
...........
Rxx(:,:,58) =
   1.4577        0       0
       0         0       0
       0         0  1.3736
Rxx(:,:,59) =
   1.4573   0       0
       0    0       0
       0    0   1.3693
Rxx(:,:,60) =
   1.4567        0       0
       0         0       0
       0         0  1.3633
Rxx(:,:,61) =
   1.4557        0       0
       0         0       0
       0         0  1.3548
Rxx(:,:,62) =
   1.4544        0       0
       0         0       0
       0         0  1.3428
Rxx(:,:,63) =
   1.4528        0       0
       0         0       0
       0         0  1.3274
```

$\wp_{H,n} < U$??

With $\bar{N} > 1$, there can be some tones that use all 3 dimensions. **Sep Theorem** applies only over a single user's tones.

These share the physical layer
with the same order on all tones.

Freq sharing cannot happen when $\bar{N} = 1$ & $\wp_H < U$.

Also L_y > 1,
so an FDM solution is not assured.

Macmax use

```
[RxxEsum, bsumEsum , bsum_linEsum] = macmax(3*64/67, h, [1 1 1], 64 , 1);

bsumEsum = 1011.3 > 1011.2                    (just slightly)
bsum_linEsum = 571.7289 < 578.85              (no guarantee that linear version is best)


>> sum(real(Rxx),3) =
 61.1343        0        0
       0  61.1343        0
       0        0  61.1343
>> 64^2/67 =   61.1343 checks on each dimension
>> trace(sum(real(Rxx),3)) =  183.4030 (clearly 3x single dimensional energy)
```

Any order can be used for same rate sum (in this case maximum).

But, the same order needs to be used on ALL tones.

Secondary user COMPONENTS are 0.

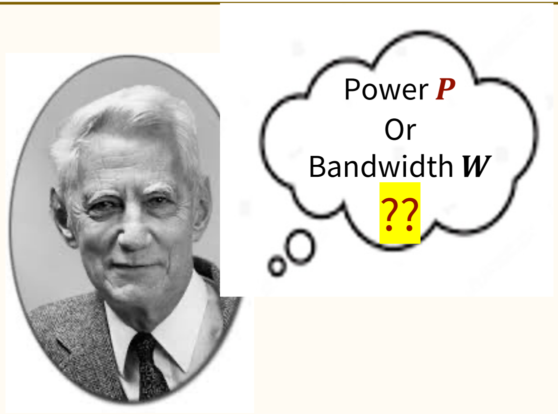
(There is no need for time-sharing two vertices for a SWF nor macmax point).

- The RxxEsum are very similar to those from SWF.m.
- While many tones individually zero one user (consistent with secondary-component concept).
  - It is not the same user for all such tones.
  - Some tones energize all 3 users.
- $\sum_n \wp_{H,n} > U$, significantly so. This means there is effectively dimension-sharing occurring over the 64 tones, at least for the rate-sum max.

Recall Shannon AWGN Measure

$$\mathcal{C} = \underbrace{W}_{Bandwidth} \cdot \log_2\left(1+\frac{P/W}{\mathcal{N}_0}\right)$$

- CAPACITY ALWAYS INCREASES with BANDWIDTH (W)!!
  - *in a vacuum with no other effects – that is, an ideal Additive White Gaussian Noise (AWGN) Channel.*



ALWAYS TAKE
Bandwidth
(if you can)!

Spatial/Spectral Shannon Capacity?

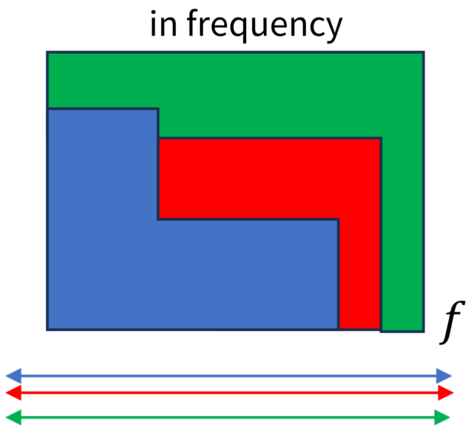
$$\mathcal{C} = \underbrace{W \cdot L}_{Bandwidth} \cdot \log_2\left(1 + \frac{P/(W \cdot L)}{\mathcal{N}_0}\right)$$

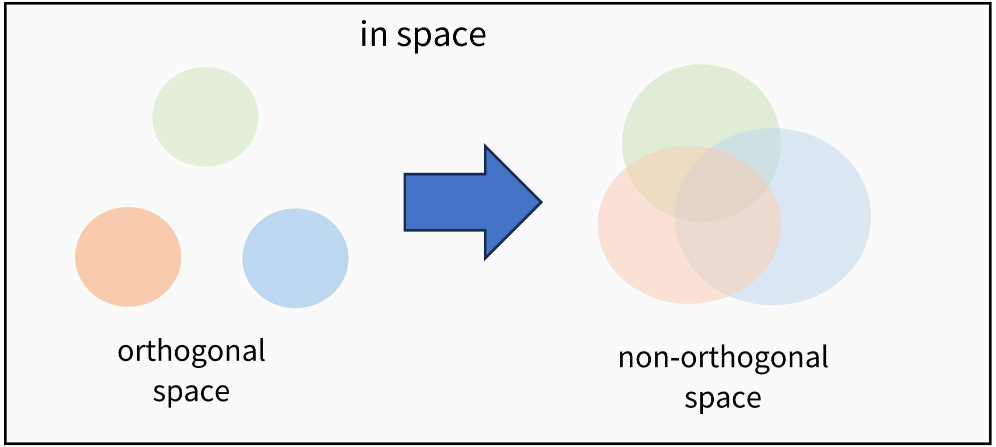
- CAPACITY ALWAYS INCREASES with **DIMENSIONS** ($W \cdot L$)!!
  - *L is spatial rank (bounded by number of antennas – spatial degrees of freedom or DoF)*
- So then why not "spread spectrum" everywhere?
  - *Multipath slightly alters the capacity above (but for wireless still increasing W helps).*
  - *Crosstalk also alters this formula (with multiple users sharing the dimensions).*
  - *Regulators limit available bandwidth for specific users or uses.*
  - *Spatial degrees of freedom have limits.*
  - *Electronics won't support infinitely wide transmission efficiently.*

Factor increase of N – do you want it in Power or Dimensions?

Fundamentally – ALWAYS TAKE DIMENSIONS!!

Canonical Design: Users may occupy same dimensions!





- Users can all “spread” and get the positive bandwidth-increase effect (in sum rate or otherwise).
- This spreading has many names:
  - Superposition coding,
  - Generalized Decision Feedback Equalization (GDFE) or successive decoding,
  - NOMA (NON-orthogonal multiple access),
  - Rate-splitting,
  - Time/dimension sharing,
  - Spread spectrum, &
  - …

>10x ENERGY SAVINGS (same rates), sustainable

Wireless – uplink Cellular or Wi-Fi

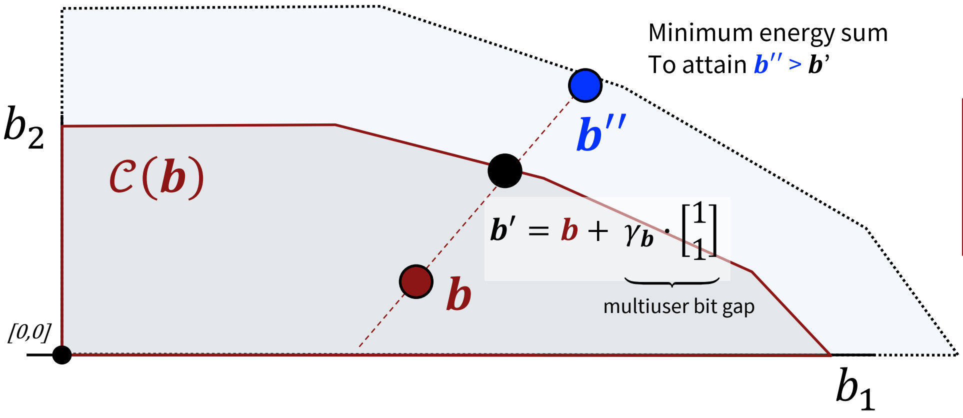
- The C-OFDM systems are as in Lecture 6 (Wi-Fi and cellular).
- A single MAC receiver uses common FFTs (one for each, max #, of spatial stream/dimension with MIMO).
- They share common frequencies.
- Usually, no feedback sections are used ... Yet; so linear setting from computeGDFE provides comparison.

OK – All good, but what is $R_{xx}(u)$ when we don't just maximize a rate sum??

Designs with weighted sums

Section 5.4.3

Capacity region revisited for design margin



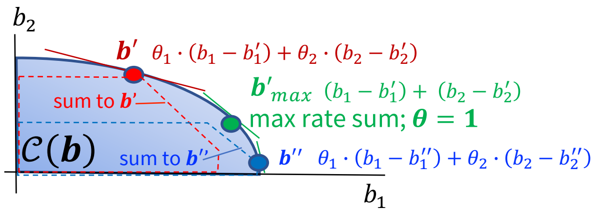
Design for zero margin at $\boldsymbol{b}'$, and reduce rate to $\boldsymbol{b} = \boldsymbol{b}' - \gamma_{\boldsymbol{b}} \cdot \begin{bmatrix}1\\1\end{bmatrix}$

- $\mathcal{C}(\boldsymbol{b})$ contains all possible weighted **rate** sums $\sum_{u=1}^{U} \theta_u \cdot b_u$ that meet **energy-vector** constraint $\mathcal{E} \preccurlyeq \mathcal{E}_x$.
- The max-b-sum point is "highest" (tangent to plane $\mathbf{1}^t \cdot \boldsymbol{b}$) with $\boldsymbol{b}$ in $\mathcal{C}(\boldsymbol{b})$, **but we want another $\boldsymbol{b}$**!
- If $\mathcal{E} \preccurlyeq \mathcal{E}_x$, **then $\boldsymbol{b}$ is admissible.**
- If not, $\boldsymbol{b}''$, a design might still target the minimum energy-sum that achieves $\boldsymbol{b}''$ **or** $\boldsymbol{b}'' - \gamma_{\boldsymbol{b}} \cdot \begin{bmatrix}1\\1\end{bmatrix}$.

Design for points on traced boundary (margin at end)

- **Maximize weighted rate** sum b, given $\boldsymbol{\mathcal{E}}$:
  - Rate is not limited.
  - rate weights satisfy $\boldsymbol{\theta} \succcurlyeq \mathbf{0}$.

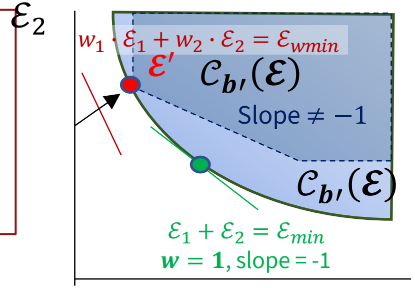
$$\max_{\{R_{\boldsymbol{xx}}(u)\}} \sum_{u=1}^{U} \theta_u \cdot b_u$$
$$ST: \quad \boldsymbol{\mathcal{E}_x} \preceq [\mathcal{E}_{1,max}\ \mathcal{E}_{2,max}\ ...\mathcal{E}_{U,max}]^* = \boldsymbol{\mathcal{E}}^*_{max} \succeq \mathbf{0}$$
$$\boldsymbol{b} \succeq \mathbf{0} \quad .$$



Any boundary point is a vertex in pentagon, which is
Also a simultaneous waterfill **(only)** for its $\boldsymbol{\theta}$ **at one vertex.**

- **Minimize weighted energy** sum $\mathcal{E}_{\boldsymbol{x}}$, given $\boldsymbol{b}$:
  - Energy is not limited.
  - Energy weights satisfy $\boldsymbol{w} \succcurlyeq \mathbf{0}$.

$$\min_{\{R_{\boldsymbol{xx}}(u)\}} \sum_{u=1}^{U} w_u \cdot \underbrace{\text{trace}\left\{R_{\boldsymbol{xx}}(u)\right\}}_{\mathcal{E}_u}$$
$$ST: \quad \boldsymbol{b} \succeq [b_{1,min}\ b_{2,min}\ ...b_{U,min}]^* = \boldsymbol{b}^*_{min} \succeq \mathbf{0}$$
$$\boldsymbol{\mathcal{E}} \succeq \mathbf{0} \quad .$$

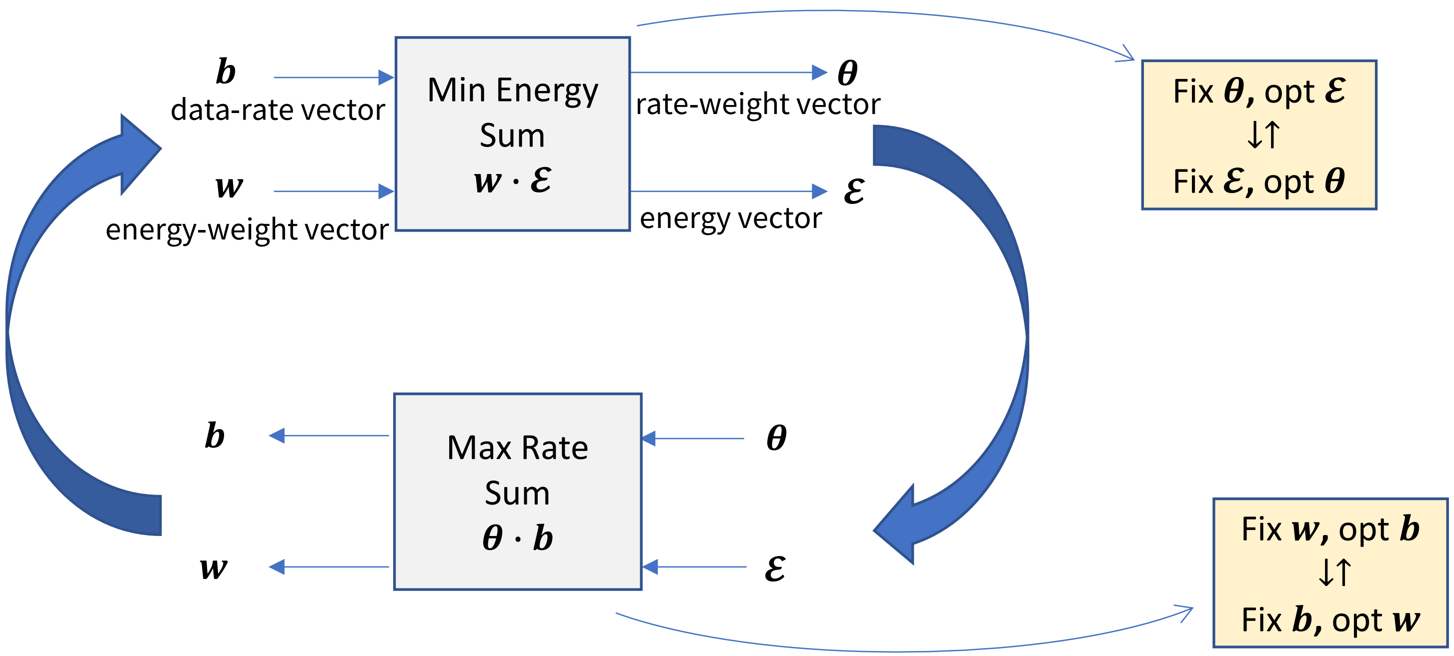


- These are "dual" problems

$$L_{minE}(R_{\boldsymbol{xx}}, \boldsymbol{b}, \boldsymbol{w}, \boldsymbol{\theta}) = \max_{\boldsymbol{\theta}} \min_{R_{\boldsymbol{xx}}} \underbrace{\sum_{u=1}^{U} [w_u \cdot trace\left\{R_{\boldsymbol{xx}}(u)\right\} + \theta_u \cdot b_u}_{\text{common term}} - \theta_u \cdot b_{min,u}]$$

$$L_{maxR}(R_{\boldsymbol{xx}}, \boldsymbol{b}, \boldsymbol{w}, \boldsymbol{\theta}) = \min_{\boldsymbol{w}} \max_{R_{\boldsymbol{xx}}} \underbrace{\sum_{u=1}^{U} [w_u \cdot trace\left\{R_{\boldsymbol{xx}}(u)\right\} + \theta_u \cdot b_u}_{\text{common term}} - w_u \cdot \mathcal{E}_{max,u}]$$

Basic Solution Cycles



- Each of these “boxes” (subnetworks) can be intense calculation, but convex and convergent.
- The overall recursive cycling also converges if $\boldsymbol{b} \in \mathcal{C}(\boldsymbol{b})$ for $\boldsymbol{\mathcal{E}}$.

Use Convex, or ML/AI methods …

Tonal Lagrangian

- Minimize (over $R_{\boldsymbol{xx}}(u)$) weighted sum at any given (think temporary) $\boldsymbol{\theta}$ where $\boldsymbol{b}$ and $\boldsymbol{w}$ are the specified values:

$$L(R_{\boldsymbol{XX}}, \boldsymbol{b}, \boldsymbol{w}, \boldsymbol{\theta}) = \sum_{n=0}^{\bar{N}-1} \left\{ \underbrace{\sum_{u=1}^{U} \left[w_u \cdot \text{trace}\left\{ R_{\boldsymbol{XX}}(u,n) \right\} - \sum_{u=1}^{U} \theta_u \cdot b_{u,n} \right]}_{L_n(R_{\boldsymbol{XX}}(n), \boldsymbol{b}_n, \boldsymbol{w}, \boldsymbol{\theta})} \right\} + \theta_u \cdot b_u \quad ,$$

With fixed $\boldsymbol{\theta} \succcurlyeq \boldsymbol{0}$ each tone can be individually minimized

- which produces then for tone n:

$$L_{min}(\boldsymbol{\theta}, n) \stackrel{\Delta}{=} \min_{\{R_{\boldsymbol{XX}}(u,n)\}, b_{u,n}} L_n(R_{\boldsymbol{XX}}(n), \boldsymbol{b}_n, \boldsymbol{w}, \boldsymbol{\theta}) \quad .$$

- Then, max over $\boldsymbol{\theta}$

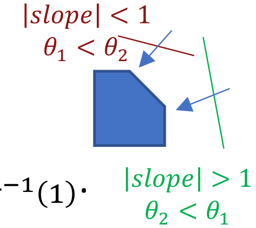
$$L^* = \max_{\boldsymbol{\theta}} \sum_{n=0}^{\bar{N}-1} L_{min}(\boldsymbol{\theta}, n) \stackrel{\Delta}{=} \max_{\boldsymbol{\theta}} L_{min}(\boldsymbol{\theta}) \quad ,$$

- and satisfy tonal GDFE (achievable region) constraint

$$\boldsymbol{b}_n \in \left\{ \boldsymbol{b}_n \mid 0 \le \sum_{\boldsymbol{u} \subseteq \boldsymbol{U}} b_{u,n} \le \log_2 \left| \left(\sum_{u=1}^{U} \tilde{H}_{u,n} \cdot R_{\boldsymbol{XX}}(u,n) \cdot \tilde{H}^*_{u,n} \right) + I \right| \right\} = \mathcal{A}_n \left(\left\{ R_{\boldsymbol{XX}}(n) \right\}, \bar{H}_n \right).$$

(5.2

The tonal achievable-region constraint



- Maximum $\sum_{u=1}^{U} \theta_u \cdot b_{u,n}$ occurs at $\mathcal{A}_n(R_{\boldsymbol{XX}}(u,n), \boldsymbol{b}_n)$ vertex (think slope -1 line and pentagon),
  - Given $R_{\boldsymbol{XX}}(u,n) \rightarrow R_{\boldsymbol{XX}}(u)$; equivalently max $\sum_{u=1}^{U} \theta_u \cdot b_u$ occurs at $\mathcal{A}\,(R_{\boldsymbol{XX}}(u), \boldsymbol{b}\,)$ vertex.
- That max-weighted-sum vertex has specific θ_u, which must satisfy $\theta_{\pi^{-1}(U)} \geq \theta_{\pi^{-1}(U-1)} \geq \ldots \geq \theta_{\pi^{-1}(1)}$.
  - Proof see Theorem 5.4.1, and also discussion in Sec 5.4.4.1. (Same order all tones)
  - Alternative to testing all orders – optimum order is inferred from the (converged) real vector θ.
- The user data rates in $\mathcal{A}_n(R_{\boldsymbol{XX}}(u,n), \boldsymbol{b}_n)$ must satisfy the (sum of) **tonal-GDFE constraint(s):**

$$b_{u,n} = \log_2\left\{\frac{|R_{yy}(u,n)|}{|R_{yy}(u-1,n)|}\right\} = \log_2\left|\sum_{i=1}^{u} \tilde{H}_{\pi^{-1}(i),n} \cdot R_{\boldsymbol{XX}}(\pi^{-1}(i),n) \cdot \tilde{H}^*_{\pi^{-1}(i),n} + I\right| - \log_2\left|\sum_{i=1}^{u-1} \tilde{H}_{\pi^{-1}(i),n} \cdot R_{\boldsymbol{XX}}(\pi^{-1}(i),n) \cdot \tilde{H}^*_{\pi^{-1}(i),n} + I\right|$$

- For given $\boldsymbol{\theta}$, min weighted rate sum over $R_{\boldsymbol{XX}}(u,n)$ maximizes convex sum

$$\sum_{u=1}^{U} \theta_u \cdot b_{u,n} = \sum_{u=1}^{U}\left\{\underbrace{\left[\theta_{\pi^{-1}(u)} - \theta_{\pi^{-1}(u-1)}\right]}_{\delta_{\pi^{-1}(u)} \geq 0} \cdot \log_2\left|\sum_{i=u}^{U} \tilde{H}_{\pi^{-1}(i),n} \cdot R_{\boldsymbol{XX}}(\pi^{-1}(i),n) \cdot \tilde{H}^*_{\pi^{-1}(i),n} + I\right|\right\} .$$

Equal Theta

- Successive equal theta values
  - can happen and will in congested multiuser systems (more users active than channel rank).
  - This happens when there are secondary user components active.
- The corresponding rate-sum difference term(s) is (are) zero.
- Only the sum rate of the corresponding users can be varied $b_{\pi^{-1}(u)} + b_{\pi^{-1}(u)+1}$ is optimized.
- The corresponding vertices for swapping the order (more generally varying when 3 or more) need to be “vertex-shared” in a proportion that causes the desired data rate to be achieved.
- Complicated program in matlab to do this, although concept is straightforward.

Coming Attraction: The Stanford minPMAC program(s)

STANFORD

End Lecture 13

Two iterated steps

- $R_{XX}(u,n)$ step: With the given (current) $\boldsymbol{\theta}$, $\boldsymbol{w}$, $\{b_{u,n}\}$, minimize the (neg) weighted rate sum over $R_{XX}(u,n)$
  - Each tone separately and sum

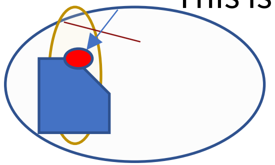
$$L_{min}(\boldsymbol{\theta}, n) = \underbrace{\sum_{u=1}^{U} \left[w_u \cdot \text{trace}\left\{ R_{\boldsymbol{XX}}(u,n) \right\} - \sum_{u=1}^{U} \theta_u \cdot b_{u,n} \right]}_{L_n(R_{\boldsymbol{XX}}(n), \boldsymbol{b}_n, \boldsymbol{w}, \boldsymbol{\theta})}$$

e.g. $$L_{k+1} = L_k - \mu \cdot \left(\nabla^2 L_k \right)^{-1} \cdot \nabla L_k$$ Weighted steepest descent ("Newton")

- Order step: With the given (current) $R_{XX}(u,n)$, $\boldsymbol{w}$, $\{b_{u,n}\}$, maximize the Lagrangian over $\boldsymbol{\theta}$

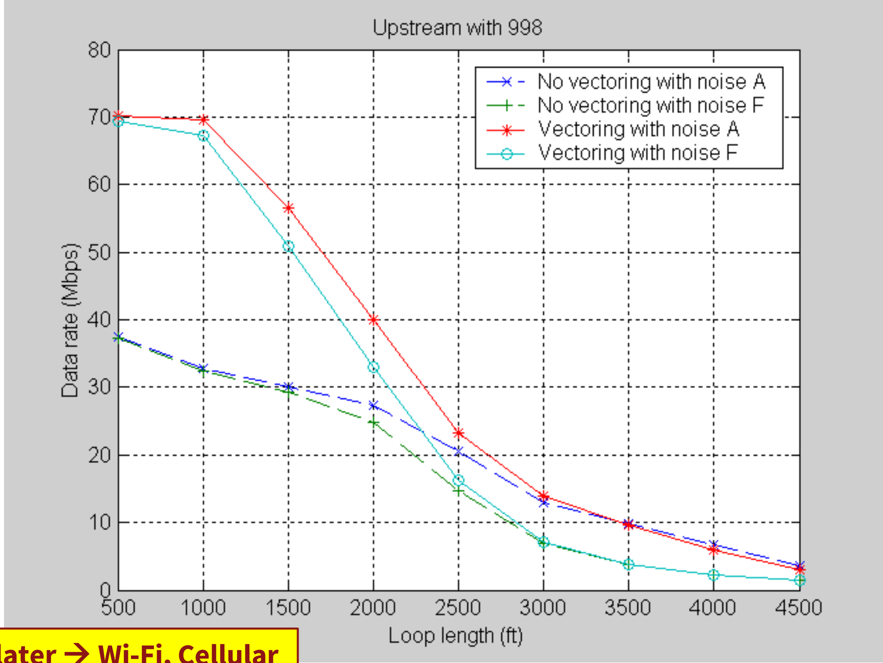
$$L(\boldsymbol{\theta}) = \sum_{n=1}^{\bar{N}} L_{min}(\boldsymbol{\theta}, n)$$

Initialize (first time only) with FM SWF for given $\boldsymbol{b}$
This is the "find the vertex set" – elliptic algorithm, see text



Gain is larger when crosstalk is larger

- Binders of copper wires (think ethernet or your/neighbors' cable of telephone wires) crosstalk.
  - Highly variable with twisting (even measuring point can lead to 20dB or more variation if moved an inch or two).
  - Probabilistic models (like wireless' distributions) are also used.
  - Average xtalk is larger on SHORTER wires because xtalk coupling increases with frequency.
    - Shorter wires use higher frequencies that are less attenuated
- Example is vectored VDSL (upstream MAC).
  - Each user has its own "link" that terminates (upstream) on a common receiver – by default all primary users (no time-sharing needed).
    - "perfect massive MIMO" all (used) tones (plot is for 25 links)
    - Can see up to *U=384* links vectored (predates "massive MIMO" in invention and use by 10 years).
  - The GDFE cancels the crosstalk.
  - It exhibits diagonal dominance too. Why?
    - So typicaly no feedback section is used.
  - Actually, some "mgfast" (ITU G.9711) to 5 Gbps multiuser and 10 Gbps "fastback" (ITU G.9702), 2 pairs – 3 channels, single user can use some GDFE's at lowest frequencies.



This was the first MultiUser MIMO (MU-MIMO), Stanford 2001 patent, later → Wi-Fi, Cellular