STANFORD

Lecture 13

Optimized GDFE Inputs

May 16, 2024

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Announcements & Agenda

- Announcements
  - May want to revisit L8 (MAC)
  - PS6 due May 22
  - Section 5.4
- Agenda
  - Special GDFE Forms
  - Tonal GDFE
  - Input Optimization
  - Circulant DFE (CDFE) with optimum (or other) designed inputs
  - ZF/MMSE Convergence Conditions

PS5 Feedback

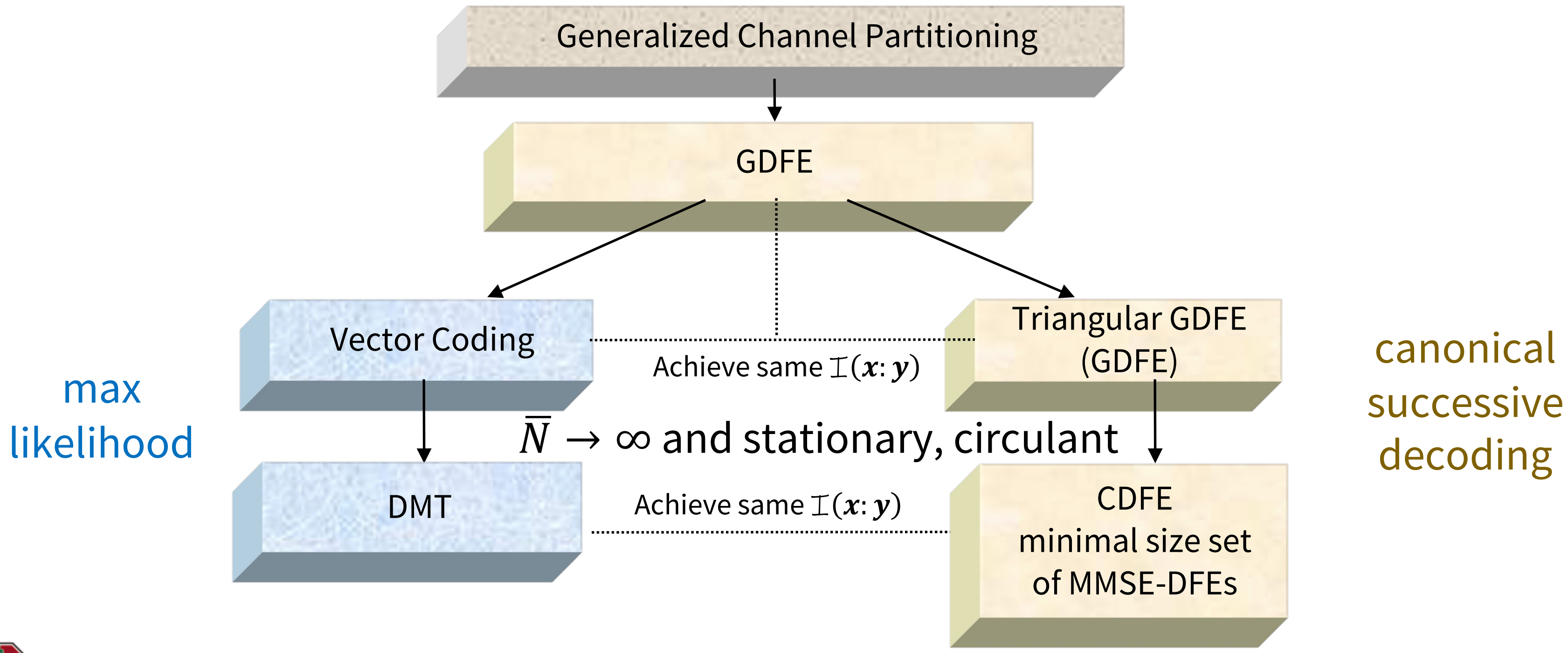
1. 14-15 hrs.
2. Decoding order takes time. And where to use Cholesky.

Special GDFE Forms

Section 5.2

Canonical Reminder, any Square Root

- All GDFEs have **canonical** performance $SNR = 2^{2\bar{\mathbb{I}}(\boldsymbol{x}:\boldsymbol{y})} - 1$, specific to the input they use.
  - Same good codes for AWGN ($\Gamma \rightarrow 0$ dB) work on GDFE-generated dimensions to drive $P_e \rightarrow 0$ if $\tilde{b} \leq \tilde{C}$.



Triangular GDFEs

- Any GDFE, especially when $G \neq I$, is triangular.

- So (canonical) GDFE's may fit multiuser better than VC.

$$R_{\boldsymbol{uu}} = \Phi \cdot \Phi^* = G_\phi \cdot S_x \cdot G_\phi^* \text{ where } G_\phi \text{ is monic upper triangular (Cholesky[16])}$$

$$|R_{\boldsymbol{uu}}| = |S_x|$$

- Discrete modulator is $A = \left(P_{A,C}\right) \cdot G_\phi \cdot S_\phi^{1/2}$ where $\left(P_{A,C}\right)$ removes singularity to get to $\boldsymbol{u} = G_\phi \cdot S_\phi^{1/2} \cdot \boldsymbol{v}$.

[16]use lohc.m command

Circulant DFE

- The CDFE uses a cyclic prefix and models each energized band in time domain.

- The consequent square-circulant channel matrix H has full rank, so $\mathcal{N} = \emptyset$,
  - as long as H is not a matrix of all constant equal values.

- With circulant $\bar{N}\times\bar{N}$ input $R_{\boldsymbol{uu}} = \Phi \cdot \Phi^* = R_{\boldsymbol{xx}}$, the CDFE receiver ignores the cyclic prefix.
  - The time-domain convolution appears periodic for any symbol.
  - The factorization $R_{\boldsymbol{uu}}$=$\Phi \cdot \Phi^*$corresponds to “causal” filter from input $\boldsymbol{v}$.
    - Special case $R_{\boldsymbol{xx}} = \mathrm{I}$, then simply direct input to channel, $\boldsymbol{x} = \boldsymbol{u} = \boldsymbol{v}$ with cyclic prefix.
  - The CDFE Imitates EE379’s MMSE-DFE as $\bar{N} \rightarrow \infty$; and indeed, the CDFE is better for $\bar{N} < \infty$.

- However, good input design (almost always) introduces singularity (think water-filling).
  - This requires some care to create single-carrier OFDM bands.
  - And, it’s not as simple as just transmit data $\boldsymbol{x} = \boldsymbol{v}$ into the channel – interpolation of some type needed.

- Also known as “Single-carrier OFDM” in standards (CDFE name and publication predates SC-OFDM by more than 10 years).

CDFE Example

```
>> H=toeplitz([.9 zeros(1,7)]',[.9 1 zeros(1,6)]);
>> H(8,1)=1;
>> H=1/sqrt(.181)*H;
>> [snrGDFEu, GU, WU, S0, MSWMFU, b, bbar,LE] = computeGDFE(H, eye(8), 2, 9);
>> snrGDFEu =   7.1666 dB
>> GU =
   1.0000  0.4972       0        0        0        0        0   0.4972
        0  1.0000  0.6414        0        0        0        0  -0.2899
        0       0  1.0000   0.6930        0        0        0   0.1780
        0       0       0   1.0000   0.7128        0        0  -0.1113
        0       0       0        0   1.0000   0.7206        0   0.0702
        0       0       0        0        0   1.0000   0.7237  -0.0444
        0       0       0        0        0        0   1.0000   0.7531
        0       0       0        0        0        0        0   1.0000
>> MSWMFU =
   0.2115        0        0        0        0        0        0   0.2351
   0.1798   0.2729        0        0        0        0        0  -0.1371
  -0.1104   0.1601   0.2948        0        0        0        0   0.0841
   0.0691  -0.1002   0.1525   0.3033        0        0        0  -0.0526
  -0.0435   0.0631  -0.0961   0.1495   0.3066        0        0   0.0332
   0.0275  -0.0399   0.0608  -0.0945   0.1483   0.3079        0  -0.0210
  -0.0174   0.0253  -0.0385   0.0598  -0.0939   0.1478   0.3084   0.0133
  -0.0933   0.0418   0.0010  -0.0439   0.0961  -0.1687   0.2772   0.1647
>> b' =
   1.7297   1.5648   1.5156   1.4978   1.4909   1.4882   1.4871   1.0792
>> bbar =   1.3170
LE =   4.4409 % dB (linear is 2.73 dB worse).
```

- For very long symbol, → 8.4 dB
- G_u→ .725 single feedback coefficient = (7.85/6.85)x.633
- W→ constant-row feedforward filter
- CDFE's limit is Chapter 3's MMSE-DFE for infinite symbol length
- Same as DMT (which it should be):

```
>> [V,D]=eig(H);
>> [snrGDFEu, GU, WU, S0, MSWMFU, b, bbar,LE] = computeGDFE(H, V, cb, 9)
snrGDFEu =   7.1666
>> GU-eye(8) =   1.0e-13 *  (It's an identity for DMT!)
>> MSWMFU(:,1:3) = (note that it is complex – why? Channel is real? V is complex)
-1.5042 + 0.0000i   1.5042 + 0.0000i  -1.5042 + 0.0000i
  0.1018 + 0.1782i   0.0540 - 0.1980i  -0.1782 + 0.1018i
  0.1018 - 0.1782i   0.0540 + 0.1980i  -0.1782 - 0.1018i
 -0.0748 + 0.0831i   0.0831 + 0.0748i   0.0748 - 0.0831i
 -0.0748 - 0.0831i   0.0831 - 0.0748i   0.0748 + 0.0831i
  0.0792 + 0.0000i   0.0792 - 0.0000i   0.0792 + 0.0000i
  0.0798 + 0.0311i   0.0784 - 0.0345i   0.0311 - 0.0798i
  0.0798 - 0.0311i   0.0784 + 0.0345i   0.0311 + 0.0798i
(the receiver FFT is included when we use computeGDFE)

>> b' =
   0.0388   0.9942   0.9942   1.7297   1.7297   2.1943   2.0862   2.0862
Note it's different, but the sum is the same as CDFE
>> bbar =   1.3170
>> LE % =   7.1666 (linear is the same).
```

Triangular (with guard period) and no energy in $\mathcal{N}$

```
>> H=(1/sqrt(.181))*toeplitz([.9 zeros(1,7)]',[.9 1 zeros(1,7)])
H=(1/sqrt(.181))*toeplitz([.9 zeros(1,7)]',[.9 1 zeros(1,7)]);
>> [Ct, Ot, Ruutt] = fixmod(H, eye(9), eye(9));
>> [A, OA, Ruupp] = fixin(Ruutt, Ct);

A =
  0.7764   0.2012  -0.1811   0.1630  -0.1467   0.1320  -0.1188   0.1069
  0.2012   0.8189   0.1630  -0.1467   0.1320  -0.1188   0.1069  -0.0962
 -0.1811   0.1630   0.8533   0.1320  -0.1188   0.1069  -0.0962   0.0866
  0.1630  -0.1467   0.1320   0.8812   0.1069  -0.0962   0.0866  -0.0779
 -0.1467   0.1320  -0.1188   0.1069   0.9038   0.0866  -0.0779   0.0702
  0.1320  -0.1188   0.1069  -0.0962   0.0866   0.9221   0.0702  -0.0631
 -0.1188   0.1069  -0.0962   0.0866  -0.0779   0.0702   0.9369   0.0568
  0.1069  -0.0962   0.0866  -0.0779   0.0702  -0.0631   0.0568   0.9489
 -0.0962   0.0866  -0.0779   0.0702  -0.0631   0.0568  -0.0511   0.0460

>> Gxbar=lohc(Ruupp);
>> Gx=Gxbar*inv(diag(diag(Gxbar))) ;
>> Xmit=A*Gxbar;

  0.8812  -0.0000   0.0000   0.0000  -0.0000   0.0000  -0.0000   0.0000
  0.2283   0.8757   0.0000   0.0000  -0.0000  -0.0000  -0.0000   0.0000
 -0.2055   0.2397   0.8681  -0.0000   0.0000   0.0000   0.0000   0.0000
  0.1850  -0.2157   0.2554   0.8574  -0.0000   0.0000   0.0000   0.0000
 -0.1665   0.1942  -0.2299   0.2779   0.8416   0.0000   0.0000  -0.0000
  0.1498  -0.1747   0.2069  -0.2501   0.3120   0.8163   0.0000   0.0000
 -0.1348   0.1573  -0.1862   0.2251  -0.2808   0.3678   0.7710  -0.0000
  0.1213  -0.1415   0.1676  -0.2026   0.2527  -0.3310   0.4733   0.6690
 -0.1092   0.1274  -0.1508   0.1823  -0.2274   0.2979  -0.4260   0.7433
```

```
>> [snrGDFEu, GU, WU, S0, MSWMFU, b, bbar, LE] = computeGDFE(H,
Xmit,2,9);

>> snrGDFEu =    7.4896 dB

>> GU =
  1.0000   0.8573   0.0000   0.0000  -0.0000  -0.0000  -0.0000  -0.0000
       0   1.0000   0.7628   0.0000  -0.0000   0.0000   0.0000   0.0000
       0        0   1.0000   0.7174  -0.0000   0.0000   0.0000   0.0000
       0        0        0   1.0000   0.6899   0.0000   0.0000  -0.0000
       0        0        0        0   1.0000   0.6624   0.0000   0.0000
       0        0        0        0        0   1.0000   0.6189   0.0000
       0        0        0        0        0        0   1.0000   0.5233
       0        0        0        0        0        0        0   1.0000

>> MSWMFU =
  0.4165   0.0000   0.0000  -0.0000  -0.0000  -0.0000  -0.0000  -0.0000
  0.0471   0.3738   0.0000  -0.0000   0.0000   0.0000   0.0000   0.0000
 -0.0294   0.0650   0.3560  -0.0000  -0.0000  -0.0000   0.0000   0.0000
  0.0178  -0.0394   0.0693   0.3487   0.0000   0.0000  -0.0000  -0.0000
 -0.0104   0.0231  -0.0407   0.0669   0.3452  -0.0000   0.0000   0.0000
  0.0058  -0.0129   0.0227  -0.0374   0.0599   0.3415   0.0000   0.0000
 -0.0029   0.0065  -0.0114   0.0188  -0.0302   0.0479   0.3328   0.0000
  0.0011  -0.0024   0.0042  -0.0069   0.0111  -0.0176   0.0279   0.3023

> b' =
  1.3789   1.4498   1.4859   1.5067   1.5249   1.5512   1.6042   1.7592
>> bbar =  1.3623
>> LE =   5.8689 (linear 1.6 dB worse)
```

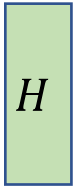
Outperforms CDFE
Same as VC w.r.t. DMT

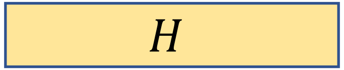
- Better!
- No clear convergence though

Try running computeGDFE to implement Vector Coding – what Xmit? (Hint: This is easy with this white input.)

Massive MIMO = diagonal dominance

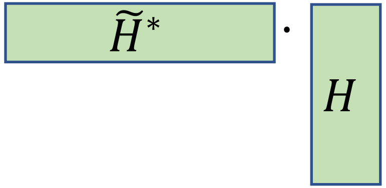
- The channel matrix H is very :
  - tall (many more receiver dimensions than ϱ_H) $-$ uplink.
  - fat (many more transmitter dimensions than ϱ_H) – downlink.





- IF **tall** channel matrix $\tilde{H}$ has entries $\tilde{h}_{il}$ that are largely uncorrelated, except for same index, then

$$\mathbb{E}\left[\tilde{h}_{li}^{*} \cdot \tilde{h}_{lk}\right] = |h|^2 \cdot L_y \cdot \delta_{ik}.$$



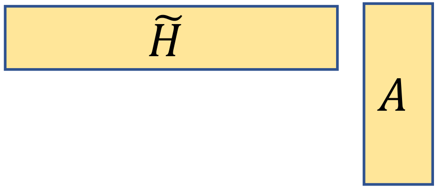
- THEN also $R_f = \tilde{H}^* \cdot \tilde{H}$ off-diag's contain uncorrelated values.
  - Law of large numbers – off diagonals ($\cdot\ 1/L_y$) go to zero, while diagonal grows relatively.

- Then R_b is also almost diagonal, like R_f.
- $G = I$ → canonical with no feedback!
- Linear is sufficient and returns to ML.

Massive MIMO Downlink continued

- Fat case (downlink) means small number of receivers.
  - Best transmit A matrix (special square root in BC case, but even with single-user) attempts to match its long columns to the long rows of **fat** $\tilde{H}$, think water-filling (formal GDFE xmit optimization comes soon).
- IF **fat** channel matrix $\tilde{H}$ has entries $\tilde{h}_{il}$ that are largely uncorrelated, except for same index, then

$$\mathbb{E}\left[\tilde{h}_{il} \cdot a_{kl}\right] = \mathcal{E} \cdot L_y \cdot \delta_{ik} .$$



- THEN also $R_b = A^* \cdot \tilde{H}^* \cdot \tilde{H} \cdot A + I$ is diagonally dominant,
  - law of large numbers again.
- Here $R_{vv} = I$ by design, then and again there is no feedback.
- $G = I$ → canonical with no feedback!

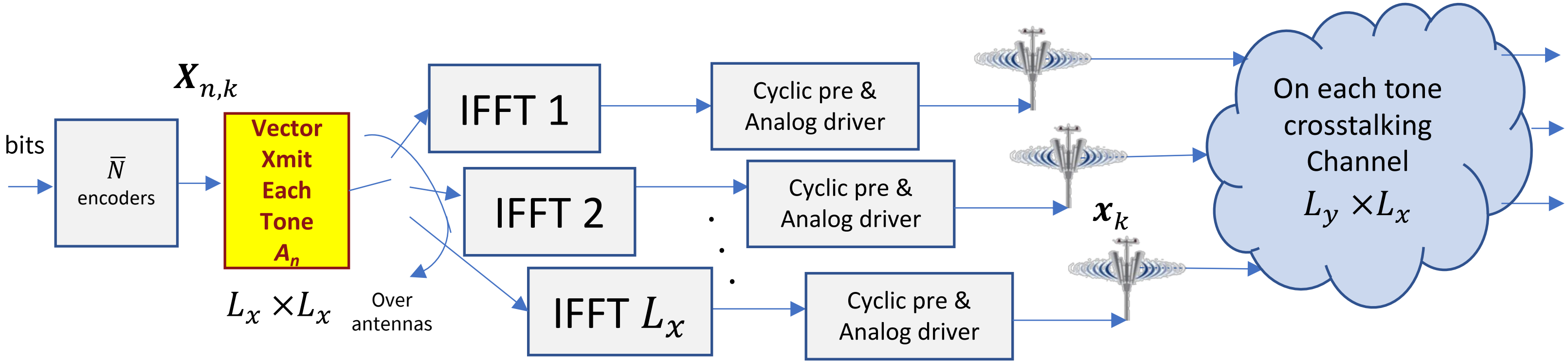
Again: no precoder (linear is sufficient).

Wireless: use many antennas at base

Wireline: vectored DSLs (the Gbps DSLs)
often happens even when large square (not fat nor tall)

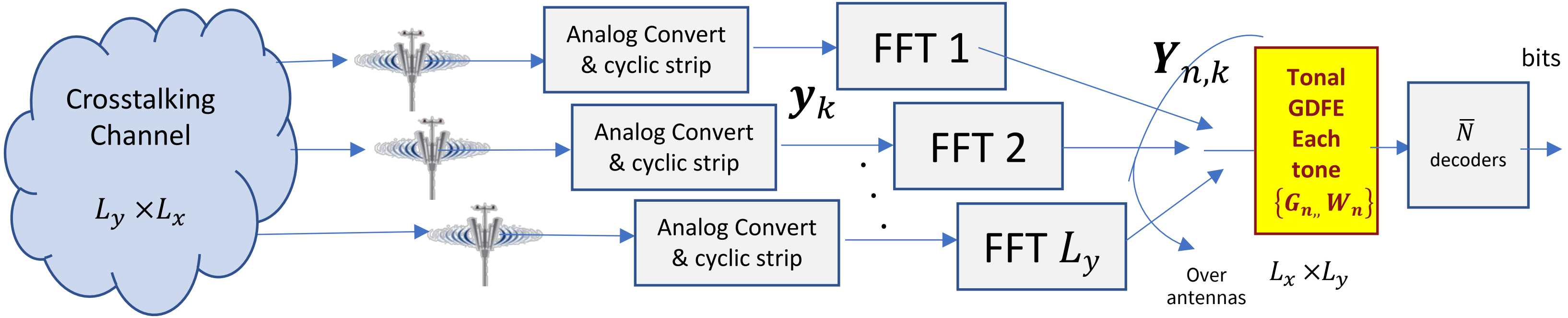
Tonal GDFE

Tonal GDFE Transmitter (see L2:31)



- Synchronize them all (like Vector DMT in Chapter 4) – the transmit filter changes (not necessarily M from SVD)

Tonal GDFE Receiver (see L2:32)



- Common symbol boundary for all antennas also at receiver.

Tonal GDFE Example

```
% tensor, time on index 3=
>> h= cat(3, h0 , h1);
>> N=8;
>> H=(10)*fft(h, N, 3)
H(:,:,1) =
 20.0000 + 0.0000i  -9.0000 + 0.0000i
  6.0000 + 0.0000i   1.0000 + 0.0000i
H(:,:,2) =
 17.0711 - 7.0711i  -7.8284 + 2.8284i
  6.8787 + 2.1213i   3.6360 + 6.3640i
H(:,:,3) =
 10.0000 -10.0000i  -5.0000 + 4.0000i
  9.0000 + 3.0000i  10.0000 + 9.0000i
H(:,:,4) =
  2.9289 - 7.0711i  -2.1716 + 2.8284i
 11.1213 + 2.1213i  16.3640 + 6.3640i
H(:,:,5) =
  0.0000 + 0.0000i  -1.0000 + 0.0000i
 12.0000 + 0.0000i  19.0000 + 0.0000i
H(:,:,6) =
  2.9289 + 7.0711i  -2.1716 - 2.8284i
 11.1213 - 2.1213i  16.3640 - 6.3640i
H(:,:,7) =
 10.0000 +10.0000i  -5.0000 - 4.0000i
  9.0000 - 3.0000i  10.0000 - 9.0000i
H(:,:,8) =
 17.0711 + 7.0711i  -7.8284 - 2.8284i
  6.8787 - 2.1213i   3.6360 - 6.3640i
```

$$H(D) = \begin{bmatrix} 1+D & -.5 - .4\cdot D \\ .9 - .3\cdot D & 1 - .9\cdot D \end{bmatrix}$$

$$\boldsymbol{h}_0 = \begin{bmatrix} 1 & -.5 \\ .9 & 1 \end{bmatrix} \quad \boldsymbol{h}_1 = \begin{bmatrix} 1 & -.4 \\ -.3 & -.9 \end{bmatrix}$$

$$R_{\boldsymbol{nn}} = .01 \cdot I$$

```
% 2x2 discrete mod for each tone (also tensor)
>> A=zeros(2,2,8); for n=1:N
A(:,:,n) = sqrt(8/9)*eye(2);
end

 cb=1;
Lx = 2*(9/8);  %use this Lx and it is # of real dimensions
GU=zeros(2,2,8);
WU=zeros(2,2,8);
S0=zeros(2,2,8);
MSWMFU=zeros(2,2,8);

b=zeros(2,1,8);
bbar=zeros(1,8);

for n=1:N
[snrGDFEu(1,n), GU(:,:,n), WU(:,:,n), S0(:,:,n), MSWMFU(:,:,n),
b(:,:,n), bbar(n)] = ...
computeGDFE(H(:,:,n), A(:,:,n), cb, Lx);
end
```

Loop the design for Each tone

```
>> snrGDFEu % = (in dB)
 16.2546  19.6868  20.8260  19.4629  11.9618  19.4629  20.8260  19.6868
>> GU
GU(:,:,1) =
  1.0000 + 0.0000i  -0.3991 + 0.0000i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,2) =
  1.0000 + 0.0000i  -0.2928 + 0.0737i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,3) =
  1.0000 + 0.0000i   0.0931 + 0.1414i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,4) =
  1.0000 + 0.0000i   0.9056 + 0.1552i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,5) =
  1.0000 + 0.0000i   1.5833 + 0.0000i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,6) =
  1.0000 + 0.0000i   0.9056 - 0.1552i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,7) =
  1.0000 + 0.0000i   0.0931 - 0.1414i
  0.0000 + 0.0000i   1.0000 + 0.0000i
GU(:,:,8) =
  1.0000 + 0.0000i  -0.2928 - 0.0737i
  0.0000 + 0.0000i   1.0000 + 0.0000i
```

8 FB Sections

Continued

```
>> MSWMFU

MSWMFU(:,:,1) =
   0.0487 + 0.0000i   0.0146 + 0.0000i
  -0.0865 + 0.0000i   0.2821 + 0.0000i
MSWMFU(:,:,2) =
   0.0460 + 0.0191i   0.0186 - 0.0057i
  -0.0409 + 0.0060i   0.0705 - 0.0787i
MSWMFU(:,:,3) =
   0.0366 + 0.0366i   0.0329 - 0.0110i
  -0.0364 - 0.0175i   0.0476 - 0.0370i
MSWMFU(:,:,4) =
   0.0166 + 0.0402i   0.0632 - 0.0120i
  -0.0381 - 0.0564i   0.0431 - 0.0177i
MSWMFU(:,:,5) =
   0.0000 + 0.0000i   0.0884 + 0.0000i
  -0.2792 + 0.0000i   0.0411 + 0.0000i
MSWMFU(:,:,6) =
   0.0166 - 0.0402i   0.0632 + 0.0120i
  -0.0381 + 0.0564i   0.0431 + 0.0177i
MSWMFU(:,:,7) =
   0.0366 - 0.0366i   0.0329 + 0.0110i
  -0.0364 + 0.0175i   0.0476 + 0.0370i
MSWMFU(:,:,8) =
   0.0460 - 0.0191i   0.0186 + 0.0057i
  -0.0409 - 0.0060i   0.0705 + 0.0787i
```

8 FF Sections

- Bit distribution
  - [space, freq]
- Lx=2.25; (2*9/8) handled cyclic-prefix dimension loss already
  - 2x because cmplx (cb=1)
  - bbar is bits/ real-dim
  - Only divide by 2*8 then for equivalent overall SNR

```
%Overall SNR from bbar
>> reshape(b,[2,8]) =
   8.6020   8.4535   8.0156   7.3838   7.0112   7.3838   8.0156   8.4535
   3.6233   6.2959   7.5772   7.1999   2.1297   7.1999   7.5772   6.2959

>> bbar =  % This is where computeGDFE used the Lx=2.25
 5.4334   6.5553   6.9301   6.4817   4.0627   6.4817   6.9301   6.5553
>> 8.6+3.6/2.25 =  5.4

>> sum(b,'all')  =  111.2179

>> sum(bbar)/16 % = 3.0894 - only 16 because 2.25 already in
>> SNRgeo = 10*log10(2^(2*ans)-1)  = 18.5396 dB
```

Can now design for multiple antennas, ISI, crosstalk – and its canonical! If we know the Rxx → A.

Time to/from Freq

- **MIMO Channel:** $\boldsymbol{H}(D) = \boldsymbol{h}_0 + \boldsymbol{h}_1 \cdot D + \boldsymbol{h}_2 \cdot D^2 + \cdots + \boldsymbol{h}_\nu \cdot D^\nu$.

- Matlab's conversion to Frequency-Domain with FFT is NOT unitary (it increases energy by FFT size).

```
h= cat(3, h0 , h1, ..., hnu);
H=fft(h, N, 3)
```

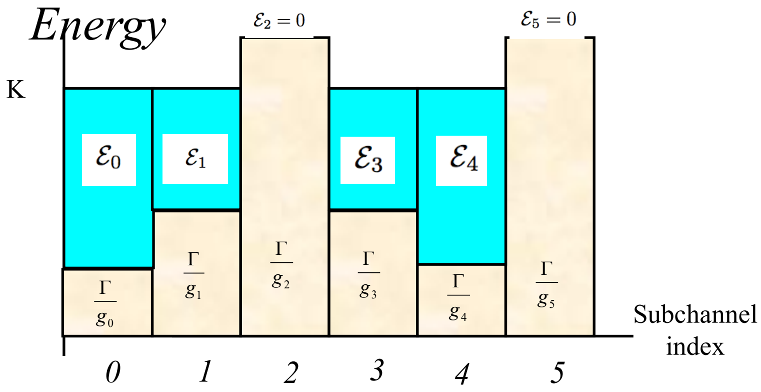
- **Why not $^1/_{\sqrt{\bar{N}}}$?** With **fixed sampling period** T':
  - $T = \bar{N} \cdot T'$; which means the energy per symbol grows by $\bar{N}$ for both input and noise.
  - $\mathcal{E}_{\boldsymbol{x}} = \bar{N} \cdot \tilde{\mathcal{E}}_x = N \cdot \bar{\mathcal{E}}_x$.
  - So make sure, depending on program/analysis, that the energy **per symbol** is $\bar{N}$ times larger.

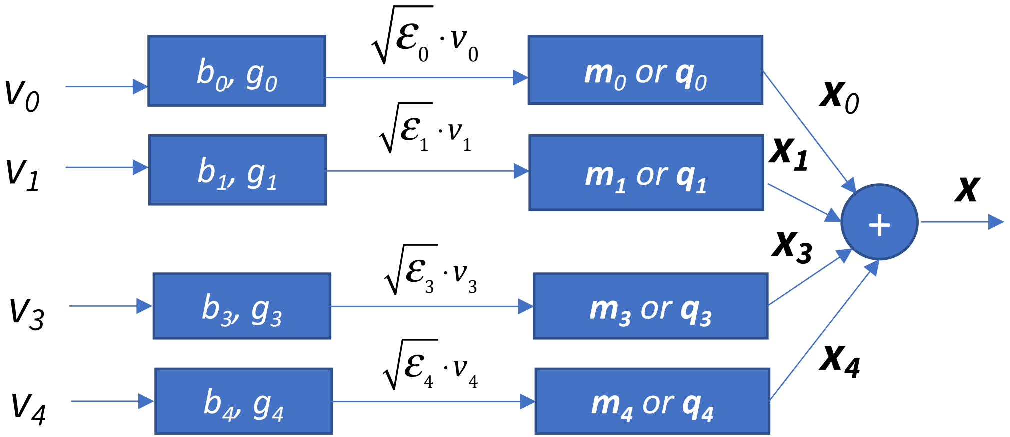
- But also: noise-whitening to $R_{\boldsymbol{nn}} = I \cdot \delta_k$ (time domain) so 1 unit/sample effectively increases the noise variance by $\bar{N}$ factor, so by amplitude $\sqrt{\bar{N}}$. This means the whitened-channel $\boldsymbol{h}_k$ needs to match this increase of $\sqrt{\bar{N}}$.
  - So dividing $\boldsymbol{h}_k$ by the **single-sample** (two-sided) square-root PSD level σ effectively increases the noise.
  - Easiest accommodation uses matlab's FFT directly with no scaling.

Input Optimization

Section 5.3

Water-filling (or LC or ...) occurs first



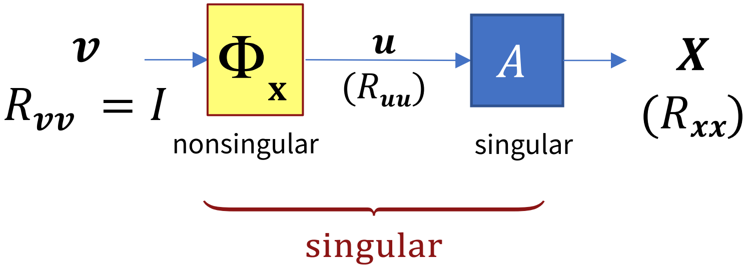


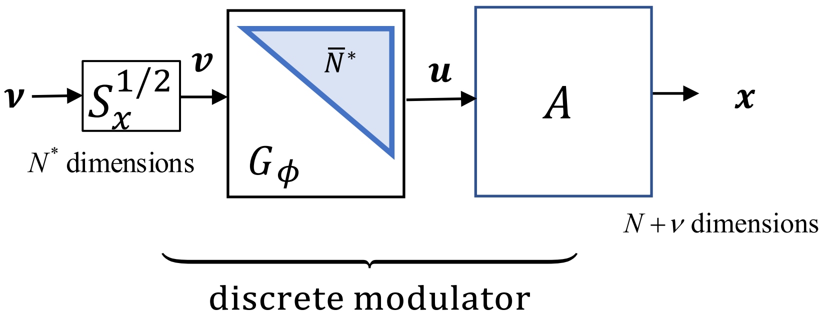
- The input optimization can be done with VC or DMT.
- Optimization finds an R_{xx} that subsequently can be factored (don't really need bit dist'n, Sep Thm).
  - Design constructs $R_{xx}^{1/2}$ as modulator, which can use sq rt:
  - M (vector coding) and/or Q (IDFT, DMT) matrices

$$R_{xx} = M \cdot diag\{\mathcal{E}_x\} \cdot M^*$$

- The energies need not be water-filling, just some optimization or design.
- For MIMO with ISI, water-fill is over spectrum and space.

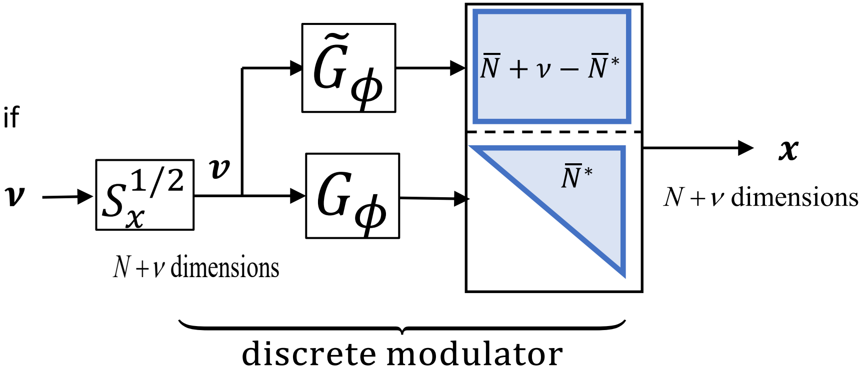
Optimum Transmit Structures





- Design uses an optimum or good $R_{\boldsymbol{xx}}$ from VC or DMT.

- Factorizations:
  - Cholesky on $R_{\boldsymbol{uu}}$ if A already known (e.g., via singularity elimination) if
    - $R_{\boldsymbol{uu}}$ is nonsingular, or
    - other square roots also allowed (including eigen decomposition).
  - Generalized Cholesky (if A not yet known) is:



- *1/T** and f_c* implemented digitally (corresponds to MMSE-DFE) – next section.

- The singular part is removed by receiver's matched-filter-matrix to form R_f.

Example – 8 dimensions waterfill DMT; 7 dimensions GDFE

- From Section 4.6:
  - [gn,en_bar,bn_bar,Nstar,bbar,SNRdmt]=DMTra([.9 1],.181,1,8,0)
  - en_bar = 1.2415 1.2329 1.1916 0.9547 0 0.9547 1.1916 1.2329

Optimize INPUT

```
>> rXX=diag([en_bar(8:-1:1)]) =
 1.2329       0       0       0       0       0       0       0
      0  1.1916       0       0       0       0       0       0
      0       0  0.9547       0       0       0       0       0
      0       0       0       0       0       0       0       0
      0       0       0       0  0.9547       0       0       0
      0       0       0       0       0  1.1916       0       0
      0       0       0       0       0       0  1.2329       0
      0       0       0       0       0       0       0  1.2415
>> rXXbar=diag([en_bar(8:-1:6) en_bar(4:-1:1)]) =
 1.2329       0       0       0       0       0       0
      0  1.1916       0       0       0       0       0
      0       0  0.9547       0       0       0       0
      0       0       0  0.9547       0       0       0
      0       0       0       0  1.1916       0       0
      0       0       0       0       0  1.2329       0
      0       0       0       0       0       0  1.2415
>> J=hankel([zeros(1,7),1]);
>> Q=(1/sqrt(8))*J*fft(J);
>> J7=hankel([zeros(1,6),1]);
>> Qtilde=(1/sqrt(7))*J7*fft(J7);
>> ruu=real(Qtilde'*rXXbar*Qtilde); =
>>norm(imag(Qtilde'*rXXbar*Qtilde))=1e-16
(proves taking real part only for appearance)
>> Gubar=lohc(ruu);
```

Cholesky needs Nonsingular; Here 7x7

Interpolate INPUT with two IFFT's

```
>> Jg = [
   1   0   0   0   0   0   0
   0   1   0   0   0   0   0
   0   0   1   0   0   0   0
   0   0   0   0   0   0   0
   0   0   0   1   0   0   0
   0   0   0   0   1   0   0
   0   0   0   0   0   1   0
   0   0   0   0   0   0   1 ];
>>A=real(Q'*Jg*Qtilde*Gubar);
>> C=[.9
zeros(6,1)
1];
>> R=[.9 1 0 0 0 0 0 0];
>> H=toeplitz(C,R);
>> Ht=(1/sqrt(.181))*H;
```

8x7

FFT, then IFFT

Compute GDFE

```
>> [snrGDFEu, GU, WU, S0, MSWMFU, b, bbar] = computeGDFE(Ht, A, 2, 9)

snrGDFEu =   7.6247 dB
>> GU =
   1.0000   0.4654  -0.0309  -0.0024   0.0245  -0.0574   0.4464
        0   1.0000   0.5340  -0.0310  -0.0052   0.0327  -0.2178
        0        0   1.0000   0.5510  -0.0307  -0.0091   0.1120
        0        0        0   1.0000   0.5554  -0.0289  -0.0499
        0        0        0        0   1.0000   0.5549  -0.0102
        0        0        0        0        0   1.0000   0.5555
        0        0        0        0        0        0   1.0000

>> MSWMFU =
   0.2144   0.0140  -0.0036   0.0019  -0.0022   0.0042  -0.0120   0.1767
   0.0788   0.2646   0.0434  -0.0124   0.0080  -0.0090   0.0157  -0.0972
  -0.0572   0.0191   0.2737   0.0812  -0.0222   0.0152  -0.0179   0.0609
   0.0413  -0.0281  -0.0237   0.2623   0.1233  -0.0296   0.0215  -0.0421
  -0.0318   0.0250  -0.0033  -0.0541   0.2364   0.1663  -0.0321   0.0320
   0.0279  -0.0225   0.0093   0.0179  -0.0731   0.1999   0.2058  -0.0254
  -0.1112   0.0652  -0.0290  -0.0061   0.0501  -0.1141   0.2104   0.1753

>> b' =   1.8981   1.8022   1.7816   1.7762   1.7752   1.7762   1.6233

>> bbar =   1.3814
```

SNR Improves (xmit opt)

Circulant DFE

3GPP calls this "Single-Carrier OFDM"

Repeated for each of $L_y \cdot L_x$ spatial paths

Start with OFDM-like signal

- IFFT from tone inputs $\boldsymbol{X}$ is $\boldsymbol{x} = Q^* \cdot \boldsymbol{X}$

$$\boldsymbol{X} = \begin{bmatrix} \widetilde{\boldsymbol{X}}_M \\ \vdots \\ \widetilde{\boldsymbol{X}}_2 \\ 0 \\ \widetilde{\boldsymbol{X}}_1 \\ 0 \end{bmatrix} = J_g \cdot \begin{bmatrix} \widetilde{\boldsymbol{X}}_M \\ \vdots \\ \widetilde{\boldsymbol{X}}_1 \end{bmatrix}$$

$$R_{\boldsymbol{xx}} = Q^* \cdot J_g \cdot R_{\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{X}}} \cdot J_g \cdot Q$$

complex baseband

$$J_g = \begin{bmatrix} 0_{\bar{N}_{z,M+1} \times \bar{N}_M} & 0_{\bar{N}_{z,M+1} \times \bar{N}_{M-1}} & \cdots & 0_{\bar{N}_{z,M+1} \times \bar{N}_1} \\ I_{\bar{N}_M \times \bar{N}_M} & 0_{\bar{N}_M \times \bar{N}_{M-1}} & \cdots & 0_{\bar{N}_M \times \bar{N}_1} \\ 0_{\bar{N}_{z,M} \times \bar{N}_M} & 0_{\bar{N}_{z,M} \times \bar{N}_{M-1}} & \cdots & 0_{\bar{N}_{z,M} \times \bar{N}_1} \\ 0_{\bar{N}_{z,M-1} \times \bar{N}_M} & I_{\bar{N}_{M-1} \times \bar{N}_{M-1}} & \cdots & 0_{\bar{N}_{z,M-1} \times \bar{N}_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\bar{N}_{z,1} \times \bar{N}_M} & 0_{\bar{N}_{z,1} \times \bar{N}_{M-1}} & \cdots & I_{\bar{N}_1 \times \bar{N}_1} \\ 0_{\bar{N}_{z,1} \times \bar{N}_M} & 0_{\bar{N}_{z,1} \times \bar{N}_{M-1}} & \cdots & 0_{\bar{N}_{z,1} \times \bar{N}_1} \end{bmatrix}$$

real baseband

$$J_g = \begin{bmatrix} J_g^- & 0_{\frac{N}{2} \times \left(N_1^- + N_M^- + \sum_{m=2}^{M-1} N_m\right)} \\ 0_{\frac{N}{2} \times \left(N_1^+ + N_M^+ + \sum_{m=2}^{M-1} N_m\right)} & J_g^+ \end{bmatrix},$$

where J_g^- and J_g^+ are respectively defined by

$$J_g^- = \begin{bmatrix} 0_{N_{z,1}^- \times N_1^-} & 0_{N_{z,1}^- \times N_2} & \cdots & 0_{N_{z,1}^- \times N_M^-} \\ I_{N_1^- \times N_1^-} & 0_{N_1^- \times N_2} & \cdots & 0_{N_1^- \times N_M^-} \\ 0_{N_{z,2} \times N_1^-} & 0_{N_{z,2}^- \times N_2} & \cdots & 0_{N_{z,2} \times N_M^-} \\ 0_{N_2 \times N_1^-} & I_{N_2 \times N_2} & \cdots & 0_{N_2 \times N_M^-} \\ \vdots & \vdots & \iddots & \vdots \\ 0_{N_M^- \times N_1^-} & 0_{N_M^- \times N_2} & \cdots & I_{N_M^- \times N_M^-} \\ 0_{N_{z,M}^- \times N_1^-} & 0_{N_{z,1}^- \times N_2} & \cdots & 0_{N_{z,M}^- \times N_M^-} \end{bmatrix}$$

and

$$J_g^+ = \begin{bmatrix} 0_{N_{z,M}^+ \times N_M^+} & 0_{N_{z,M}^+ \times N_{M-1}} & \cdots & 0_{N_{z,M}^+ \times N_1^+} \\ I_{N_M^+ \times N_M^+} & 0_{N_M^+ \times N_{M-1}} & \cdots & 0_{N_M^+ \times N_1^+} \\ 0_{N_{z,M-1} \times N_M^+} & 0_{N_{z,M-1} \times N_{M-1}} & \cdots & 0_{N_{z,M-1} \times N_1^+} \\ 0_{N_{M-1} \times N_M^+} & I_{N_{M-1} \times N_{M-1}} & \cdots & 0_{N_{M-1} \times N_1^+} \\ \vdots & \vdots & \iddots & \vdots \\ 0_{N_1^+ \times N_M^+} & 0_{N_1^+ \times N_{M-1}} & \cdots & I_{N_1^+ \times N_1^+} \\ 0_{N_{z,1}^+ \times N_M^+} & 0_{N_{z,1}^+ \times N_{M-1}} & \cdots & 0_{N_{z,1}^+ \times N_1^+} \end{bmatrix}.$$

- Dimensions $\quad \bar{N}^* = \sum_{i=1}^{M} \bar{N}_i$

- Nonsingular and $\boldsymbol{x}$ is circulant. $\quad |R_{\widetilde{\boldsymbol{x}}\widetilde{\boldsymbol{x}}}| > 0$

New autocorrelation is nonsingular

- Form the individual bands from time-domain signals :

$$\tilde{\boldsymbol{X}} = \begin{bmatrix} \tilde{\boldsymbol{X}}_M \\ \vdots \\ \tilde{\boldsymbol{X}}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} Q_{\bar{N}_M} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{\bar{N}_1} \end{bmatrix}}_{\tilde{Q}} \cdot \begin{bmatrix} \boldsymbol{u}_M \\ \vdots \\ \boldsymbol{u}_1 \end{bmatrix}.$$

- Real baseband case imposes conjugate symmetry - see text.

- Each band has an $R_{\boldsymbol{uu}}(i)$; $i = 1, \ldots, M$

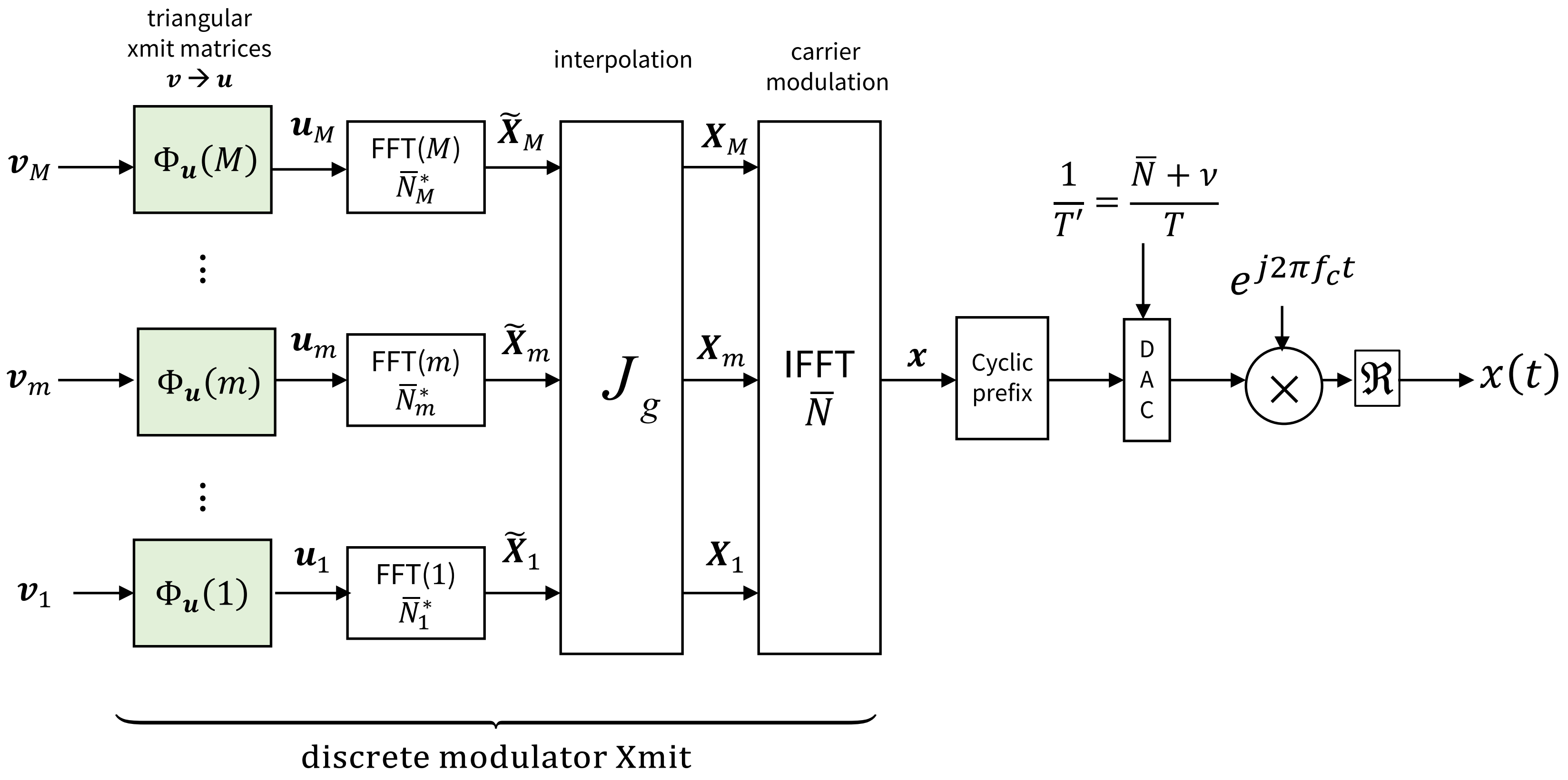
$$R_{\boldsymbol{uu}} = \begin{bmatrix} R_{\boldsymbol{uu}}(M) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R_{\boldsymbol{uu}}(1) \end{bmatrix} = \begin{bmatrix} \Phi(M)\cdot\Phi^*(M) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Phi(1)\cdot\Phi^*(1) \end{bmatrix}$$

$$R_{\boldsymbol{uu}}(i) = \Phi(i)\cdot\Phi^*(i) = G_x(i)\cdot S_x(i)\cdot G_x^*(i)$$

- So far, the input is

$$\boldsymbol{x} = Q^* \cdot J_g \cdot \tilde{Q} \cdot \boldsymbol{u} \qquad = Q^* \cdot J_g \cdot \tilde{Q} \cdot \Phi \cdot \boldsymbol{v}$$

The CDFE Transmitter(s)

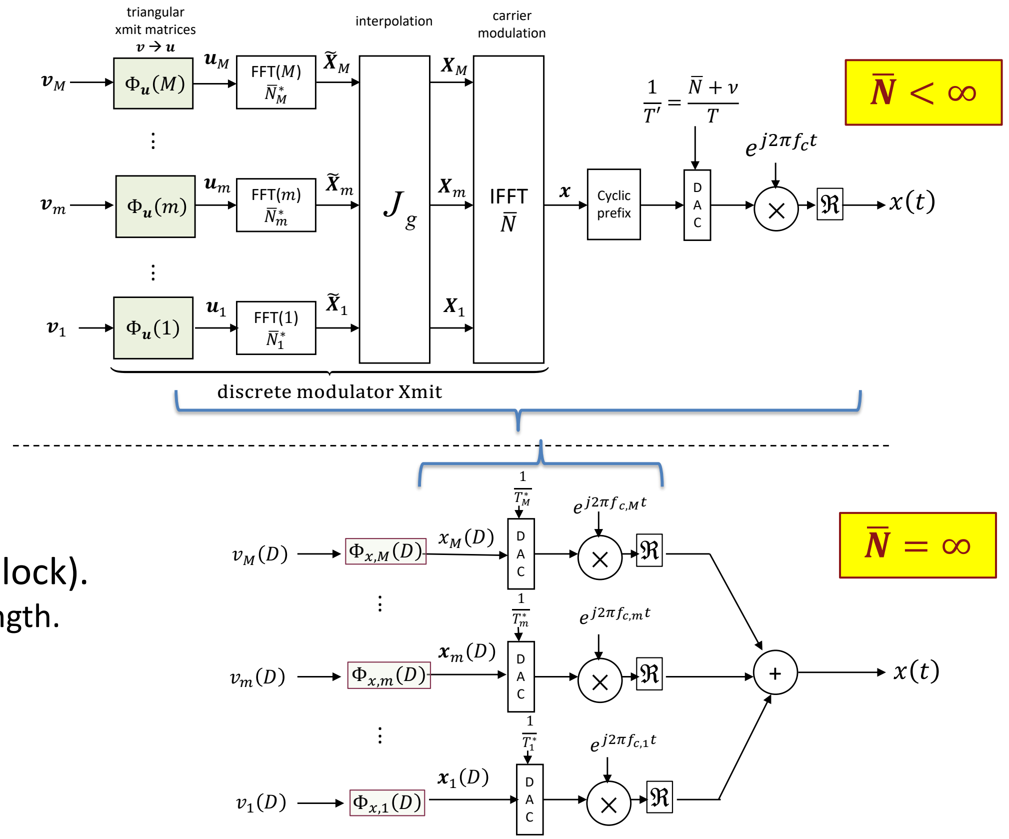


Cyclic Convergence (Toeplitz Dist'n) – EACH Band

- Applies to temporal dimensionality (time-frequency) when (block) stationary, so $R_{\boldsymbol{xx}}$ is (block) Toeplitz.
- Start with linear prediction (Cholesky Factors define this):
  - $\boldsymbol{v}_N = \boldsymbol{x}_N - \boldsymbol{\phi}_1^* \cdot \boldsymbol{x}_{N-1} - \cdots - \boldsymbol{\phi}_{N-1}^* \cdot \boldsymbol{x}_0$,
- which is found through orthogonality principle:
  - $\mathbb{E}\left[\boldsymbol{v}_N \cdot \boldsymbol{x}_{N-i}^*\right] = 0 \;\forall\; i = 1, \ldots, \bar{N} - 1$.
- The filter has limiting value as the stationary predictor in D-Transform notation:
  - $\lim_{N \to \infty} [I \quad \boldsymbol{\phi}_1^* \quad \cdots \quad \boldsymbol{\phi}_{N-1}^*] = \boldsymbol{\Phi}^*(D)$.
- The energy per sample (or one block of block Toeplitz autocorrelation) is
  - $\lim_{N \to \infty} R_{\boldsymbol{vv}}(i) = \mathbb{E}[\boldsymbol{v}_i \cdot \boldsymbol{v}_i^*] = S_{\boldsymbol{v}} = \mathcal{E}_{\boldsymbol{x}} = trace\left\{R_{\boldsymbol{x}_i \boldsymbol{x}_i}\right\} \forall\; i \geq 0$.

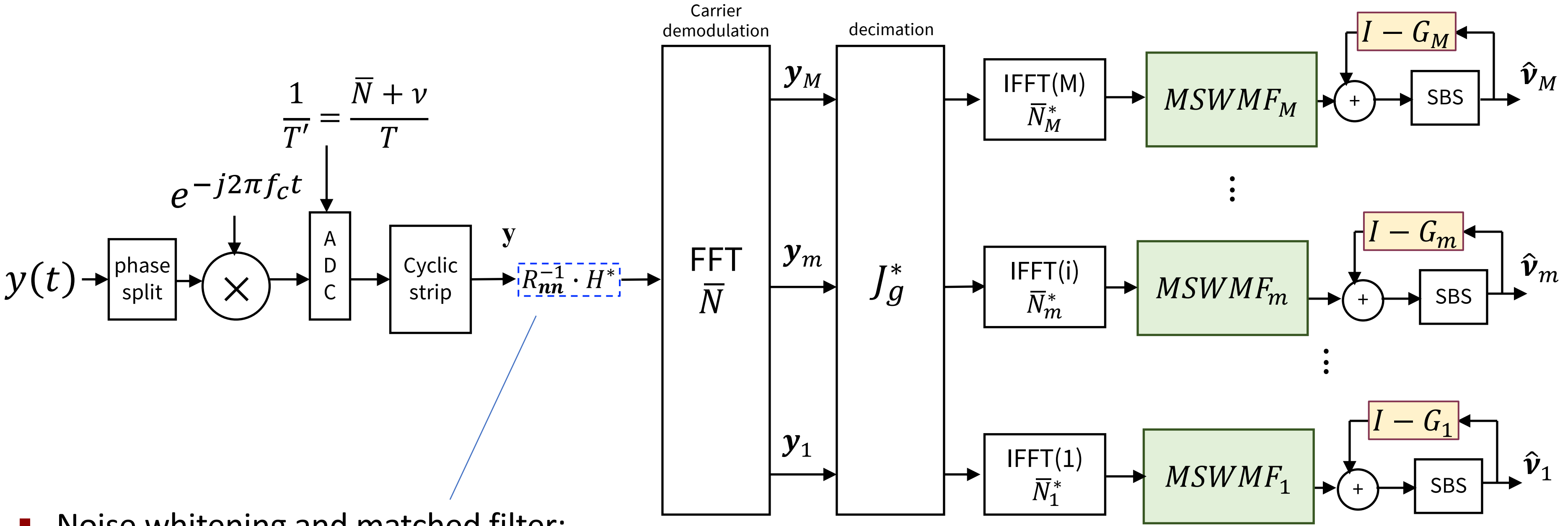
Transmitters as $\bar{N} \to \infty$

- Upper diagram is digital signal processing.
  - 3GPP's Uplink uses this.
  - Localized mapping has one active $G_{\boldsymbol{u}}(m)$.
    - Think of this as $[I \quad \boldsymbol{\phi}_1^* \quad \cdots \quad \boldsymbol{\phi}_{N-1}^*]$ from previous page.
  - The distributed mapping has >1 $G_{\boldsymbol{u}}(m)$ active.



- Lower diagram is continuous time (& infinite block).
  - Cyclic prefix is unnecessary with infinite block length.
  - 3GPP SC-OFDM uses cp with finite length.
  - Must synchronize between devices (think MAC).
  - The $G_x(D) = \boldsymbol{\Phi}^*(D)$ from previous page.

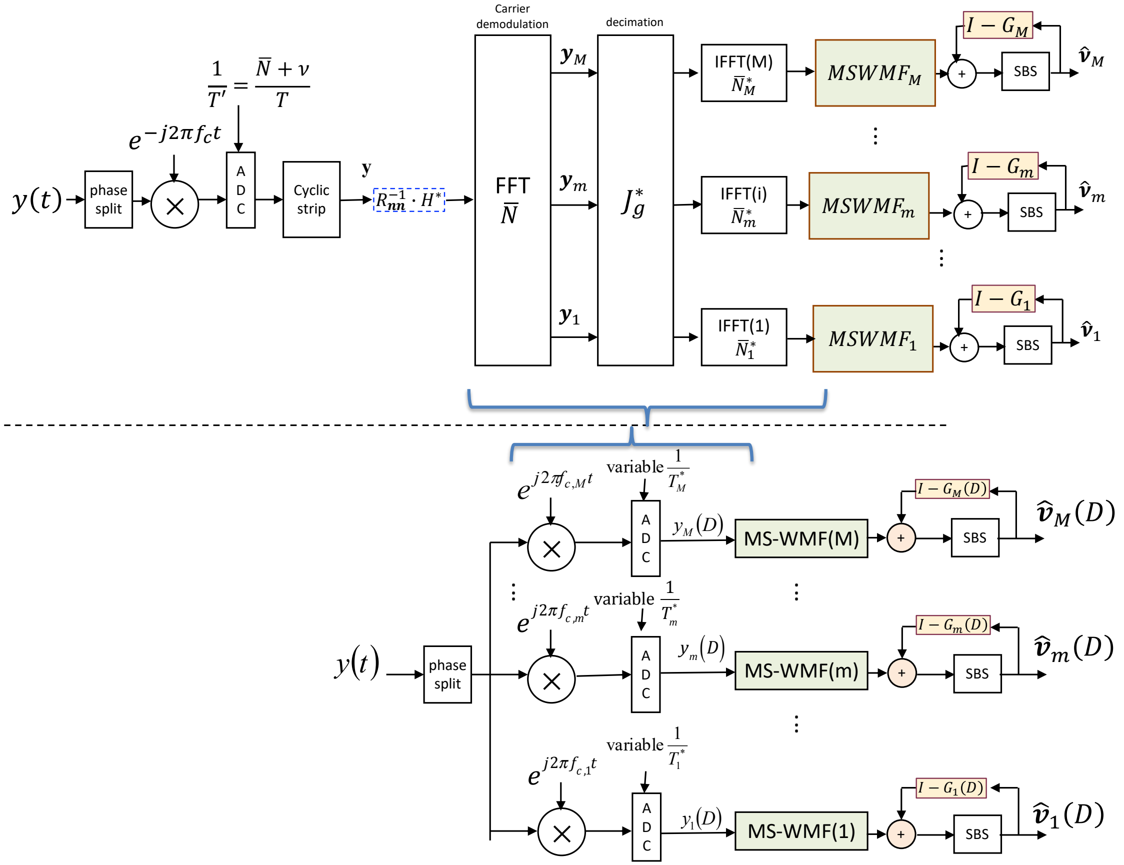
The CDFE Receiver(s)



- Noise whitening and matched filter:
  - can commute if large $\bar{N}$ for each band, which is
  - close enough to cyclic to absorb into (green) FF matrix filter.

Compare Receivers

- Digital has a clear multi-band structure.
- Remember GDFE is conditional linear prediction problem.
- So $[I \quad \boldsymbol{\phi}_1^* \quad \cdots \quad \boldsymbol{\phi}_{\bar{N}-1}^*] \rightarrow G_i(D)$.



- Commutation of MS-WMF has occurred.
  - Matched filter is stationary (because H is).
  - $G_i(D)$ is stationary in each band.

Revisit optimum 1+.9D^{-1} from L13:12

```
>> [gn,en_bar,bn_bar,Nstar,b_bar]=DMTra([.9 1],.181,1,8,0)
gn = 19.9448   17.0320   10.0000    2.9680    0.0552    2.9680   10.0000   17.0320
en_bar = 1.2415    1.2329    1.1916    0.9547         0    0.9547    1.1916    1.2329
bn_bar = 2.3436    2.2297    1.8456    0.9693         0    0.9693    1.8456    2.2297
Nstar = 7
b_bar = 1.3814
>> 10*log10(2^(2*b_bar) -1) = 7.6247 dB
>> rXX=diag([en_bar(8:-1:1)]) =
   1.2329        0        0        0        0        0        0        0
        0   1.1916        0        0        0        0        0        0
        0        0   0.9547        0        0        0        0        0
        0        0        0        0        0        0        0        0
        0        0        0        0   0.9547        0        0        0
        0        0        0        0        0   1.1916        0        0
        0        0        0        0        0        0   1.2329        0
        0        0        0        0        0        0        0   1.2415
>> rXXbar=diag([en_bar(8:-1:6) en_bar(4:-1:1)]) =
   1.2329        0        0        0        0        0        0
        0   1.1916        0        0        0        0        0
        0        0   0.9547        0        0        0        0
        0        0        0   0.9547        0        0        0
        0        0        0        0   1.1916        0        0
        0        0        0        0        0   1.2329        0
        0        0        0        0        0        0   1.2415
>> J=hankel([zeros(1,7),1]);
>> Q=(1/sqrt(8))*J*fft(J);
>> J7=hankel([zeros(1,6),1]);
>> Qtilde=(1/sqrt(7))*J7*fft(J7);
>> ruu=real(Qtilde'*rXXbar*Qtilde); %(avoid finite-prec error on imag part)
>> Phibar=lohc(ruu) ;
```

```
>> Jg = [
   1   0   0   0   0   0   0
   0   1   0   0   0   0   0
   0   0   1   0   0   0   0
   0   0   0   0   0   0   0
   0   0   0   1   0   0   0
   0   0   0   0   1   0   0
   0   0   0   0   0   1   0
   0   0   0   0   0   0   1 ];
```

```
>> A=real(Q'*Jg*Qtilde*Phibar) =
    0.9690   -0.0430    0.0265   -0.0400    0.0643   -0.1047    0.2044
    0.3040    0.9208   -0.1373    0.0784   -0.0748    0.0937   -0.1427
   -0.1969    0.4609    0.8251   -0.1903    0.1093   -0.0932    0.1060
    0.1576   -0.2170    0.6189    0.6929   -0.2052    0.1182   -0.0938
   -0.1314    0.1408   -0.2126    0.7640    0.5365   -0.1870    0.1060
    0.1064   -0.0937    0.1084   -0.1770    0.8833    0.3691   -0.1427
   -0.0729    0.0515   -0.0489    0.0606   -0.1068    0.9658    0.2044
   -0.0000   -0.0000   -0.0000   -0.0000   -0.0000   -0.0000    1.0000
>> C=[.9 ; zeros(6,1) ; 1];
>> R=[.9 1 0 0 0 0 0 0];
>> Ht=(1/sqrt(.181))*toeplitz(C,R);
>> [snrGDFEu, GU, WU, S0, MSWMFU,b,bbar] = computeGDFE(Ht, A, 2, 9)
(note use of Nx/Lx = 9 because nu = 1, but also A reduced to 7 dimensions)
snrGDFEu =  7.6247 dB
GU =
    1.0000    0.4654   -0.0309   -0.0024    0.0245   -0.0574    0.4464
         0    1.0000    0.5340   -0.0310   -0.0052    0.0327   -0.2178
         0         0    1.0000    0.5510   -0.0307   -0.0091    0.1120
         0         0         0    1.0000    0.5554   -0.0289   -0.0499
         0         0         0         0    1.0000    0.5549   -0.0102
         0         0         0         0         0    1.0000    0.5555
         0         0         0         0         0         0    1.0000
MSWMFU =
    0.2144    0.0140   -0.0036    0.0019   -0.0022    0.0042   -0.0120    0.1767
    0.0788    0.2646    0.0434   -0.0124    0.0080   -0.0090    0.0157   -0.0972
   -0.0572    0.0191    0.2737    0.0812   -0.0222    0.0152   -0.0179    0.0609
    0.0413   -0.0281   -0.0237    0.2623    0.1233   -0.0296    0.0215   -0.0421
   -0.0318    0.0250   -0.0033   -0.0541    0.2364    0.1663   -0.0321    0.0320
    0.0279   -0.0225    0.0093    0.0179   -0.0731    0.1999    0.2058   -0.0254
   -0.1112    0.0652   -0.0290   -0.0061    0.0501   -0.1141    0.2104    0.1753
>> b %=
1.8981   1.8022   1.7816   1.7762   1.7752   1.7762   1.6233
>> bbar % =   1.3814
```

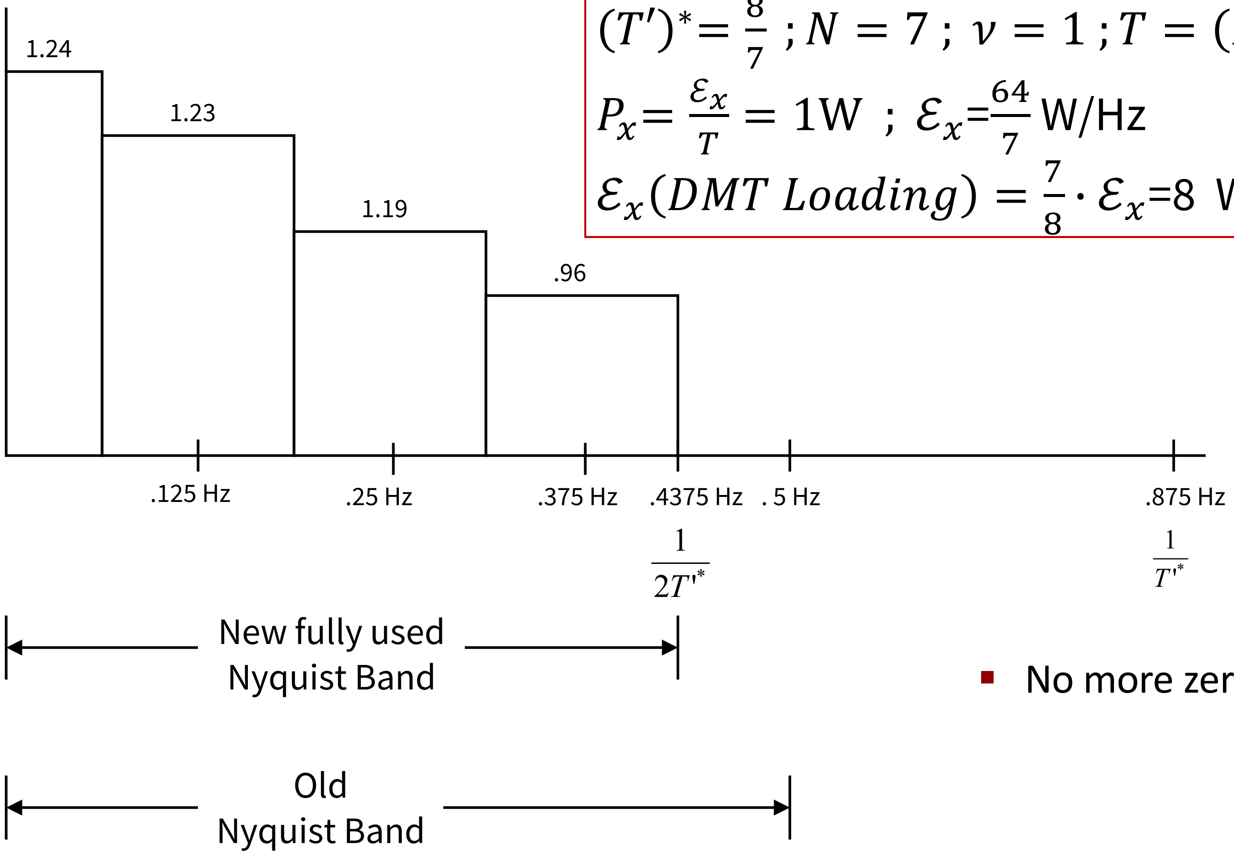
7 Dimensions

Better Perf (input opt)

Converges On GU

(and WU Not shown)

Resampled $1+.9D^{-1}$



$(T')^* = \frac{8}{7}$; $N = 7$; $\nu = 1$; $T = (N+\nu)\cdot(T')^* = \frac{64}{7}$;

$P_x = \frac{\mathcal{E}_x}{T} = 1\text{W}$; $\mathcal{E}_x = \frac{64}{7}$ W/Hz

$\mathcal{E}_x(DMT\ Loading) = \frac{7}{8}\cdot\mathcal{E}_x = 8$ W/Hz

- No more zeroed bands

Resampled Design Commands

```
h=[.9 1];
h7=resample(h,7,8);
norm(h)^2 % =    1.8100
norm(h7)^2 %  =    1.5517
7*1.81/8 % check and close =    1.5837
% form matrix
H7 = toeplitz([h7(1); zeros(5,1) ; h7(2)],[h7(1:2) zeros(1,5)]);
H7 = sqrt(1/((7/8)*.181))*H7;
g7=svd(H7).*svd(H7) %=
>> g7' %= 19.5764   15.8947   15.8947    7.6218  7.6218    0.9874    0.9874
[bn, en, Nstar]=waterfill_gn(g7', (8/7*9/8) , 0 ,2); %approx.
en % =
   1.5867    1.5749    1.5749    1.5066    1.5066    0.6251    0.6251
%(note all nonzero after decimation)
% form input
rXX = diag([en]);
J7=hankel([zeros(1,6),1]');
Q7=(1/sqrt(7))*J7*fft(J7);
rxx=real(Q7'*rXX*Q7);
Phibar=lohc(rxx);
 A=Phibar;
```

```
[snrGDFEu, GU, WU, S0, MSWMFU, b, bbar] = computeGDFE(H7, A)

snrGDFEu = 9.0212 dB (higher, but at lower symbol rate)
GU %=
 1.0000    0.2943   -0.1997   -0.0846   -0.0641   -0.1280    0.3119
      0    1.0000    0.4663   -0.2183   -0.0872   -0.0440   -0.4196
      0         0    1.0000    0.5299   -0.2212   -0.0765    0.1812
      0         0         0    1.0000    0.5589   -0.2226   -0.2173
      0         0         0         0    1.0000    0.5809   -0.1146
      0         0         0         0         0    1.0000    0.5381
      0         0         0         0         0         0    1.0000\
>> MSWMFU %=
 0.2156         0         0         0         0         0    0.2026
 0.1290    0.2782         0         0         0         0   -0.1244
-0.0887    0.1002    0.3025         0         0         0    0.0791
 0.0572   -0.0720    0.0879    0.3143         0         0   -0.0547
-0.0411    0.0470   -0.0648    0.0822    0.3189         0    0.0354
 0.0249   -0.0332    0.0415   -0.0605    0.0789    0.3196   -0.0278
-0.1765    0.1385   -0.1234    0.1227   -0.1425    0.1749    0.2314
>> b' % =
 1.8174    1.6652    1.6142    1.5900    1.5815    1.5840    1.2324
>> bbar = 1.5835
>> R=bbar*(7/8) = 1.3856    (bits/sec) ~ 1.3814 bits/sec  (same as interp)
This is not exact
```

Also
7 Dimensions

Converges
On GU

(and WU
Not shown)

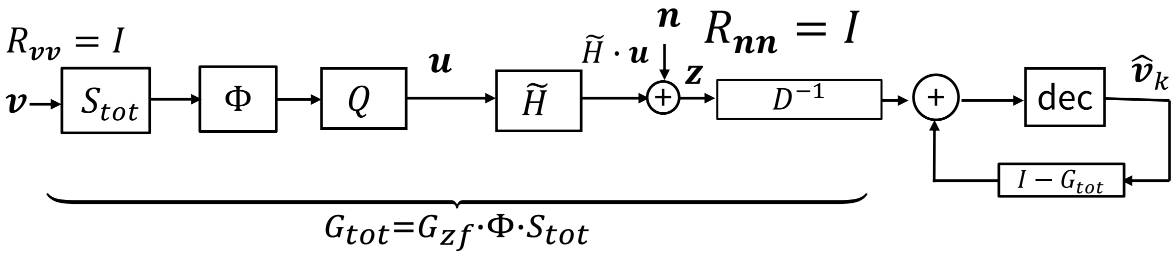
- See also the two-band example in Section 5.3
  - Tedious but could be helpful in following details for a multiband CDFE (e.g. – uplink carrier aggregation with multiple resource blocks in Cellular)

ZF/MMSE convergence conditions

Section 5.3.5

Use Zero Forcing GDFE with Water-fill?

- **To Show:** MMSE GDFE's triangular shaping G equal the triangular factor of $\tilde{H} = D \cdot G_{zf} \cdot Q^*$?
  - **IF the input is water-fill and nonsingular (so resampled):**
- $D^{-1} \cdot \tilde{H} = F \cdot \Lambda \cdot M^*$ with energies $diag\{\mathcal{E}\} = K - \Lambda^{-2}$ → $R_{\boldsymbol{uu}} = M \cdot (K - \Lambda^{-2}) \cdot M^* = Q \cdot \Phi \cdot \Phi^* \cdot Q^*$
  - To form this, Cholesky–Factor $Q^* \cdot R_{\boldsymbol{uu}} \cdot Q = \Phi \cdot \Phi^*$.
  - Define monic triangular: $G_{tot} = G_{zf} \cdot \Phi \cdot S_{tot}$ and note corresponding MSWF and ZF-GDFE with D^{-1} is



Zero-forcing with WF

always	$\tilde{H} = F \cdot \Lambda \cdot M^* = D \cdot G_{zf} \cdot Q^*$
nonsingular	Cholesky: $Q^* \cdot R_{\boldsymbol{uu}} \cdot Q = \Phi \cdot \Phi^*$

This zero-forcing actually has G_{tot} , not G_{zf}, as feedback; however designers often use flat input $S_{tot} = \bar{\mathcal{E}}_x \cdot I$, where $K \approx \bar{\mathcal{E}}_x$ which sets G_{tot}= G_{zf} ($M = Q$, so $\Phi = I$).

Equivalently, the input v as N→ ∞ has S_{tot}→ constant, so then exactly true.

The two water-fill receivers

Rearrange Water-fill

$R_{\boldsymbol{uu}}$=$M \cdot [K \cdot I - \Lambda^{-2}] \cdot M^* = Q \cdot \Phi \cdot \Phi^* \cdot Q^*$

Noting

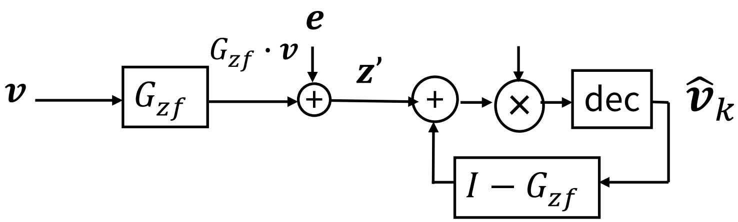
$M \cdot \Lambda^{-2} \cdot M^* = \tilde{H}^{-1} \cdot D^2 \cdot \tilde{H}^{-*}$=$R_f^{-1}$

$$K \cdot I = Q \cdot \Phi \cdot \Phi^* \cdot Q^* + Q \cdot G_{zf}^{-1} \cdot G_{zf}^{-*} \cdot Q^*$$

$$K \cdot G_{zf} \cdot G_{zf}^* = \underbrace{G_{tot} \cdot S_{tot}^{-2} \cdot G_{tot}^*}_{R_f} + I$$

This is R_b^{-1} !

Equivalent MMSE with WF (same $G = G_{zf}'$):
Performance still not same, MMSE slightly better



$G = G_{zf}$ so water-filling leads to MMSE having same feedback as the water-fill zero-forcing, at least the flat-energy approximation to wf

However, G_{tot} , is really the ZF cascade when non flat or for finite symbol length.

- The K calculation needs to be positive (or increase K so slightly positive and then scale down the resulting energies); the water-fill input was not full rank so there is a loss; can be small in wireless.
- However, MMSE still has (slightly) higher SNR so use of G_{zf} with waterfill as feedback, not G_{tot}, is highest SNR.
  - Note carefully on previous slide that the ZF-GDFE feedforward filter is not the same as MMSE-GDFE, even if feedback is same.

Worst-Case Noise equates ZF and MMSE

- Easy proof: WCN diagonalizes the primary-user receivers from BC in Chapter 2.

  - Step 1: Separates noise-whitened-noise-matched channel's triangular part
    - $\tilde{H} \triangleq R_{wcn}^{+} \cdot H = R_{zf} \cdot Q_{zf}^{*} = \begin{bmatrix} 0 & R_1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} q \\ Q_1^{*} \end{bmatrix}$

  R_1 is triangular part
  Q_1^{*} is corresponding column set

  - Step 2: cascade that triangular part with Cholesky of rotated input:
    - $R_{\tilde{x}\tilde{x}} = Q_1^{*} \cdot R_{xx} \cdot Q_1 = \Phi \cdot \Phi^{*}$ where Φ is triangular Cholesky factor.
    - $A = Q_1 \cdot \Phi$.

  - Step3: find the channel gains/SNR and feedback section:
    - The cascade of receiver triangular inverse is $D_A \cdot G_{zf} = R_1 \cdot \Phi$.

Example of WCN's RCVR Diagonalization

```
>> H =
   0.9000   1.0000        0        0
        0   0.9000   1.0000        0
        0        0   0.9000   1.0000
H=(1/sqrt(.181))*H;
>> Rxx=eye(4);
>>  [Rwcn,b]=wcnoise(Rxx,H,1);
>> b =   4.8024
>> Htilde=inv(Rwcn)*H;
>> [R,Q]=rq(Htilde);
>> R =
        0  -2.7323  -1.4039   0.0071
        0        0  -2.7077  -1.2346
        0        0        0  -3.0719
Q1=Q(:,2:4);
Rxxrot=Q1'*Rxx*Q1;
Rup=R(:,2:4) ;
Phibar=lohc(Rxxrot);
DA=diag(diag(Rup*Phibar));
>> G=inv(DA)*Rup*Phibar
   1.0000   0.5138  -0.0026
        0   1.0000   0.4559
        0        0   1.0000
>> A=Q1*Phibar;
>> sri=inv(sqrtm(Rwcn));
```

Rwcn is nonsingular here

Finding ZF modulator G

```
>> [snrGDFEu, GU, WU, S0, MSWMFU,b,bbar] = computeGDFE(sri*H, A, 2, 4)
snrGDFEu =   1.1340 dB

GU =
   1.0000   0.5826  -0.0030
        0   1.0000   0.5160
        0        0   1.0000
WU =
   0.1339        0        0
  -0.0676   0.1317        0
   0.0244  -0.0470   0.1031
S0 =
   8.4657        0        0
        0   8.5956        0
        0        0  10.7006
MSWMFU =
  -0.3657  -0.0128   0.0054
  -0.0125  -0.3560  -0.0125
   0.0047  -0.0111  -0.3164
>> MSWMFU*sri' =
  -0.3660  -0.0000   0.0000
  -0.0000  -0.3565   0.0000
   0.0000  -0.0000  -0.3167
>> b' =   1.5408   1.5518   1.7098
>> bbar =   1.2006
>> sum(b) =   4.8024 (checks)
```

Diagonal !
(so ZF = MMSE GDFE)

Only need
RQ fact and lohc

- See Example 5.3.6 with non-white Rxx

Some Final Comments

- The GDFE is canonical – capacity rate is reliably achievable with $\Gamma = 0$ (or capacity less shaping loss).
- GDFE can have error propagation (limited to $\bar{N}$) if $\Gamma > 0$ dB.
  - Unless it is VC (~DMT) , which is ML decoder uniquely amoung all GDFEs.
  - Other GDFE's becoming increasingly less favorable performance relative to VC/DMT as gap grows.
- The DMT form benefits from FFT algorithms so also more cost effective than the others.
- By Separation Theorem, Coded-OFDM can capture the DMT benefits also without error propagation.
  - But will lose more rapidly lose performance relatively if input is not water filling.
- The MMSE-DFE is limiting (stationary) case of the CDFE and can be canonical.
  - Set of MMSE-DFE's for each of which PWC holds, which
  - has unlimited error propagation (use precoder) and also degrades more rapidly for nonzero-gap codes

Eventual Global Conclusion: Use DMT (wireline) or C-OFDM (wireless) on almost all difficult single-user transmission systems.

End Lecture 13