STANFORD

Lecture 10

Broadcast Channels Continued

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JOHN M. CIOFFI

Hitachi Professor Emeritus of Engineering

Instructor EE379B – Spring 2024

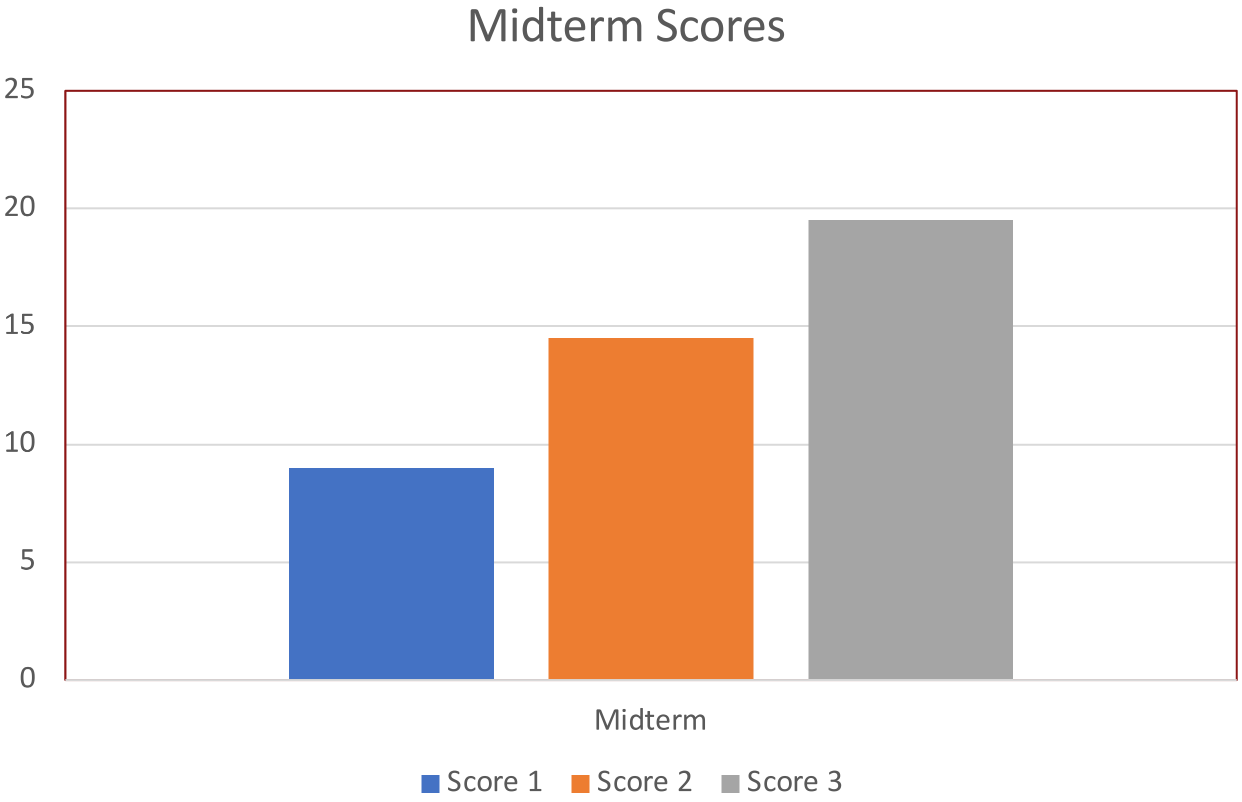
Announcements & Agenda

- Announcements
  - Problem Set #5 due Wednesday May 15
  - Midterms grades (and PS4) at canvas
    - Feedback/assessment from exam

- Problem Set 5 = PS5 (due **May 15**)
  1. 2.28 modulo precoding function
  2. 2.29 scalar BC region
  3. 2.30 vector BC design
  4. 2.31 2-user IC region
  5. 2.32 bonded relay channel

- Agenda
  - Scalar Gaussian BC
  - Vector Gaussian BC Design
  - Worst-case-Noise BC Design
  - Vector WCN-BC Design
  - Maximum BC rate sum

Midterm



[Midterm Link]

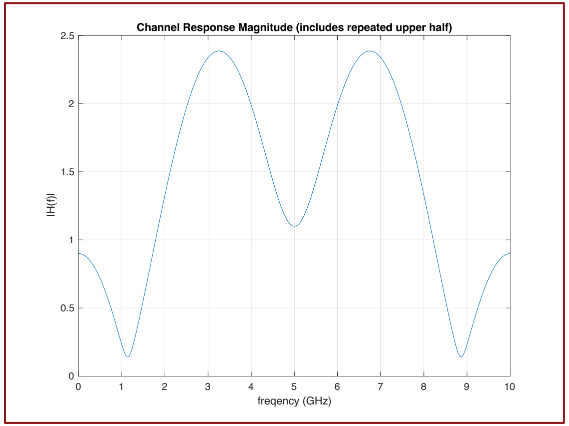
[Solutions Link]

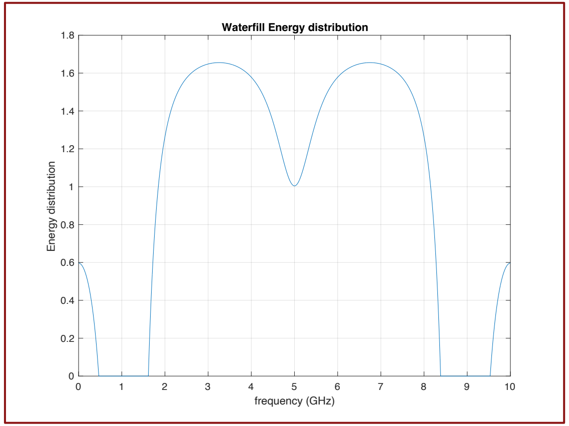
Problem 1

- $H(D) = \mathbf{1 - .9 \cdot D + .8 \cdot D^3}$; h = [1.0000   -0.9000   0   0.8000];
  - `>> f=10*[0:1023]/1024;  plot(f,abs(fft(h,1024)))`
  - `>> sig=1 ; Exbar=1`
  - `[gn, en_bar,bn_bar,Nstar,b_bar,SNRdmt]=DMTra(h,sig,Exbar,8192,0);`
  - **b_bar =   0.844 ; SNRdmt =   3.46 dB**
  - `f=10*[0:8191]/8192;  plot(f,abs(fft(h,8192)))`
  - `>> plot(f, en_bar)`

- The **high transfer at Nyquist** (5 GHz) means likely passage >5GHz
  - 2 bands roughly DC to 500MHz, and then about 1.7 GHz to 5GHz.
    - **mask**= en_bar > 0;
    - >> [a, indexa] = **min(mask);** indexa % =   **385**
    - >> [b, indexb] = **max(mask (386:4096));** indexb % =   **940**
    - >> babar=sum(bn_bar(1:384))/(8195/2) % = 0.0180 ; 10*log10(2^(2*babar)-1)% = **-15.9647 dB**
    - >> bbbarb=sum(bn_bar(940:4096))/(8195/2)% = 0.83; 10*log10(2^(2*bbbarb)-1) % = **3.31 dB**
    - >> btot=sum(bn_bar(1:4096))/4099 % =   **0.844** ; 10*log10(2^(2*btot)-1) % =   **3.46 dB**

- DFE, yes but with uncoded, there is **error propagation** likely and so there would be loss (or precoder loss).
  - The DMT solution here is MAP and has no such loss.



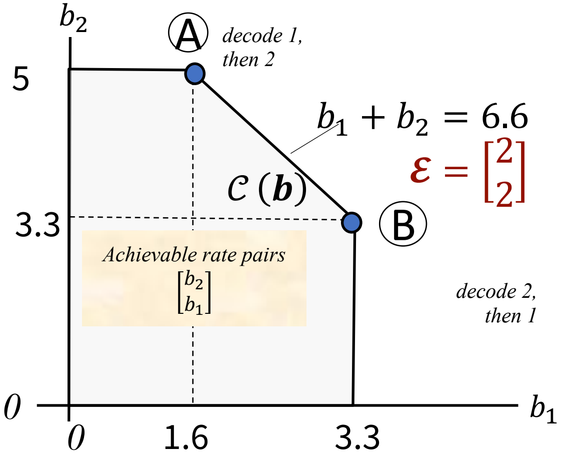


Problem 2

- $\mathrm{H} = [7\ 5\ \ -6\ ;\ 3\ 8\ 5];\mathrm{Rnn} = [1\ 0\ ;\ 0\ 2];\ \mathrm{Htilde} = \mathrm{sqrt}\big(\mathrm{inv}(\mathrm{Rnn})\big) * \mathrm{H}$

$$\begin{matrix} \mathbf{7.0000} & \mathbf{5.0000} & \mathbf{-6.0000} \\ \mathbf{2.1213} & \mathbf{5.6569} & \mathbf{3.5355} \end{matrix}$$

- User 2 is 2x2 so can do a 2x2 VC if it is only user present
  - >> [F,L,M]=svd(Htilde(1:2,1:2));
  - >> [bn en Nstar]= waterfill_gn([L(1,1)^2 L(2,2)^2],1,0,2)
  - bn =   3.3841   1.5653
  - en =   1.0560   0.9440
  - >> sum(bn) =   **4.9494**
- Max rate sum thus uses Rxx=diag([en(1) en(2) 2]);
  - >> bsum=0.5*log2(det(Htilde*Rxx*Htilde'+eye(2))) % =   **6.5713 (User 2)**
  - >> bsum-sum(bn) % =   **1.6219 (User 1)**
  - >> b1max=0.5*log2(det(2*Htilde(:,3)*Htilde(:,3)'+eye(2))) % =   **3.3074 (User 1)**
  - >> bsum-b1max =   **3.2640 (user 2)**



```
e). >> [b, GU, WU, S0, MSWMFU] = mu_mac(Htilde*[M zeros(2,1) ; 0 0 1], sqrt(Rxx), [2 1] , 2)
b =   4.9494   1.6671
GU =   1.0000   0.0000   0.4190
            0   1.0000   3.1717
            0        0   1.0000
WU =  0.0093        0        0
     -0.0000   0.1289        0
     -0.0457  -0.3092   0.1101
diag(S0) = 109.0061  8.7576  10.0851
MSWMFU =
  -0.0805  -0.0528
  -0.1969   0.3002
  -0.0646   0.0904
```

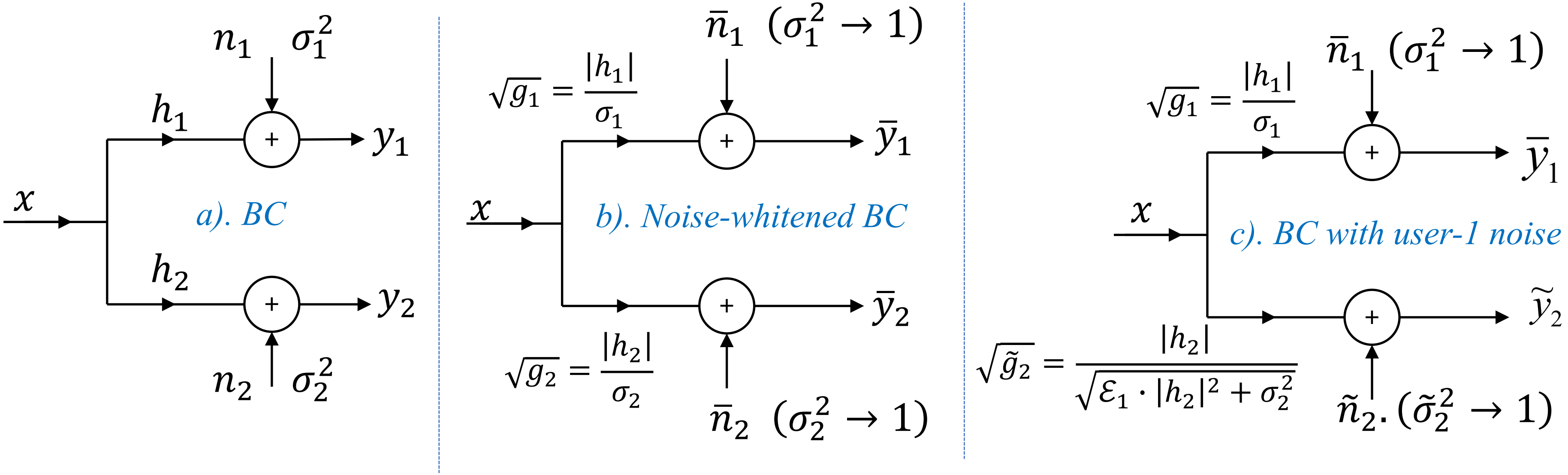
Xmit is Rxx = sqrt(diag(en(1),en(2),sqrt(2))]*[M [0 ;0] ; 0 0 1]

```
f). N=1; cb=2;
[Rxx, bsum , bsum_lin]=macmax(4, Htilde, [2 1],N,cb)
Rxx =
   1.3860   1.9056        0
   1.9056   2.6201        0
        0        0   3.9939
bsum =  8.0018
bsum_lin =  7.8726
```

Scalar Gaussian BC

PS 5.2 - 2.29 scalar BC region

3 *SCALAR*-BC "scalings"



n_1 σ_1^2
h_1
x
y_1
a). BC
h_2
y_2
n_2 σ_2^2
$\bar{n}_1$ $(\sigma_1^2 \to 1)$
$\sqrt{g_1} = \frac{|h_1|}{\sigma_1}$
$\bar{y}_1$
b). Noise-whitened BC
$\bar{y}_2$
$\sqrt{g_2} = \frac{|h_2|}{\sigma_2}$
$\bar{n}_2$ $(\sigma_2^2 \to 1)$
$\bar{n}_1$ $(\sigma_1^2 \to 1)$
$\sqrt{g_1} = \frac{|h_1|}{\sigma_1}$
$\bar{y}_1$
c). BC with user-1 noise
$\tilde{y}_2$
$\sqrt{\tilde{g}_2} = \frac{|h_2|}{\sqrt{\mathcal{E}_1 \cdot |h_2|^2 + \sigma_2^2}}$
$\tilde{n}_2.$ $(\tilde{\sigma}_2^2 \to 1)$

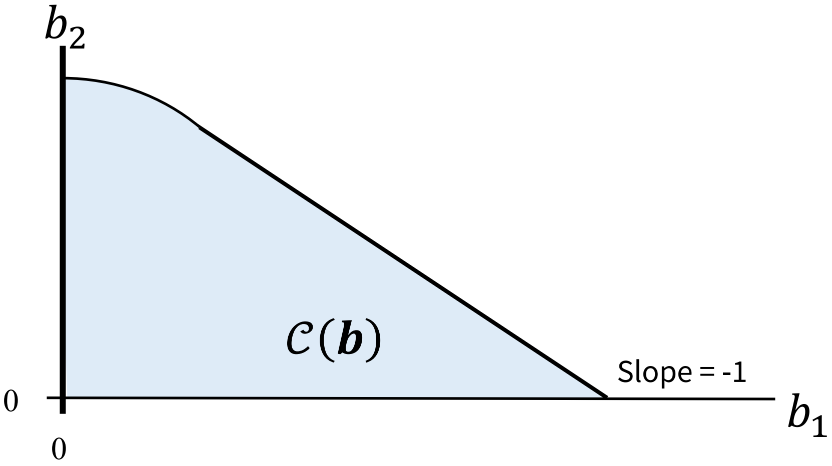
- They're all equivalent, but the 3 scalings are different.

Rate region

$$g_1 > g_2$$

$$\bar{b}_1 \leq \mathbb{I}(x_1 : y_1 / x_2) = \tfrac{1}{2} \cdot \log_2(1 + \alpha \cdot \bar{\mathcal{E}}_x \cdot g_1)$$

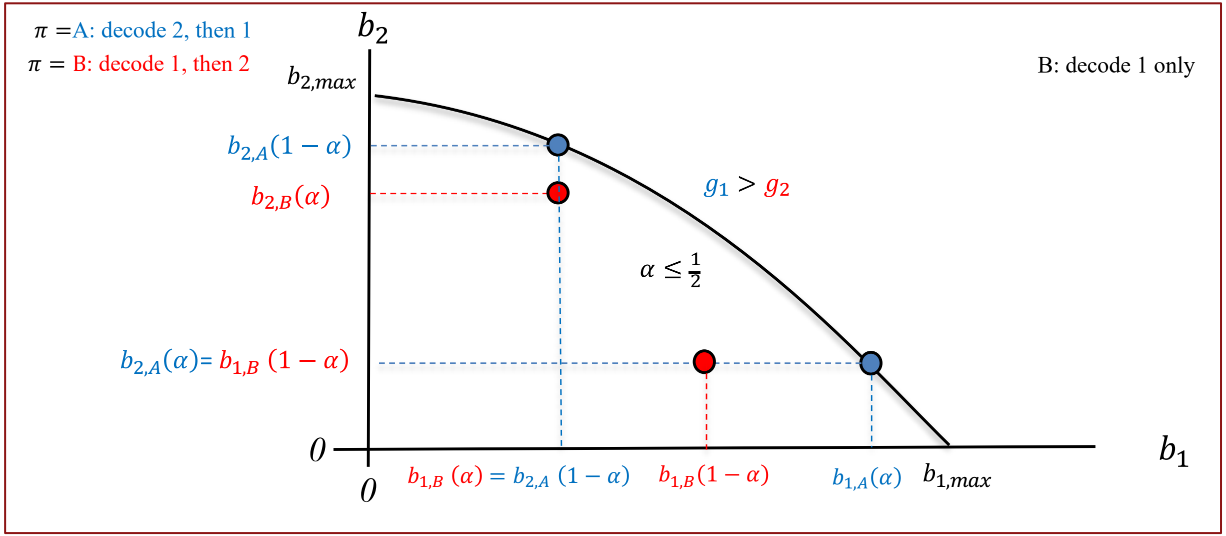
$$\bar{b}_2 \leq \mathbb{I}(x_2 : y_2) = \tfrac{1}{2} \cdot \log_2\left(1 + \frac{(1-\alpha) \cdot \bar{\mathcal{E}}_x \cdot g_2}{1 + \alpha \cdot \bar{\mathcal{E}}_x \cdot g_2}\right)$$



- $\mathcal{C}(\boldsymbol{b})$'s calculation runs through all energy splits (this is single parameter α in 2-user BC).

- Can also reverse order and take convex hull (not necessary though, see next slide).

Single best order for scalar BC, $g_1 > g_2$

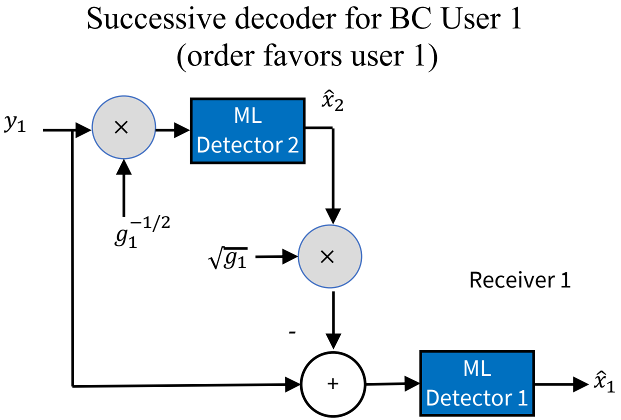

π =A: decode 2, then 1
π = B: decode 1, then 2
b_2
$b_{2,max}$
B: decode 1 only
$b_{2,A}(1-\alpha)$
$b_{2,B}(\alpha)$
$g_1 > g_2$
$\alpha \leq \frac{1}{2}$
$b_{2,A}(\alpha)= b_{1,B}\ (1-\alpha)$
0
b_1
0
$b_{1,B}\ (\alpha) = b_{2,A}\ (1-\alpha)$
$b_{1,B}(1-\alpha)$
$b_{1,A}(\alpha)$
$b_{1,max}$

$$\boldsymbol{b}_A = \begin{bmatrix} \frac{1}{2}\cdot\log_2(1+\alpha\cdot\bar{\mathcal{E}}_x\cdot g_1) \\ \frac{1}{2}\cdot\log_2\left(1+\frac{(1-\alpha)\cdot\bar{\mathcal{E}}_x\cdot g_2}{1+\alpha\cdot\bar{\mathcal{E}}_x\cdot g_2}\right) \end{bmatrix}$$

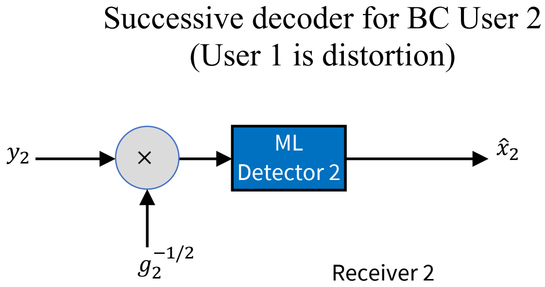
$$\boldsymbol{b}_B = \begin{bmatrix} \frac{1}{2}\cdot\log_2\left(1+\frac{\alpha\cdot\bar{\mathcal{E}}_x\cdot g_1}{1+(1-\alpha)\cdot\bar{\mathcal{E}}_x\cdot g_1}\right) \\ \frac{1}{2}\cdot\log_2(1+(1-\alpha)\cdot\bar{\mathcal{E}}_x\cdot g_2) \end{bmatrix}$$

- Best order? $g_1 > g_2$ **both users'** data rates are on boundary if 2 is decoded first with 1 as noise.
  - If we try to reverse order at RCVR2, decoding user 1 first, this then limits user 1 at RCVR 1 (even if 1 is last decoded at RCVR 1 because user 1 must be decodable at RCVR 2 also.)
  - See equations in text.
- Inductively, $g_1 > \cdots > g_U$ is the single best order (no search needed on scalar Gaussian BC!).

BC Successive Decoders



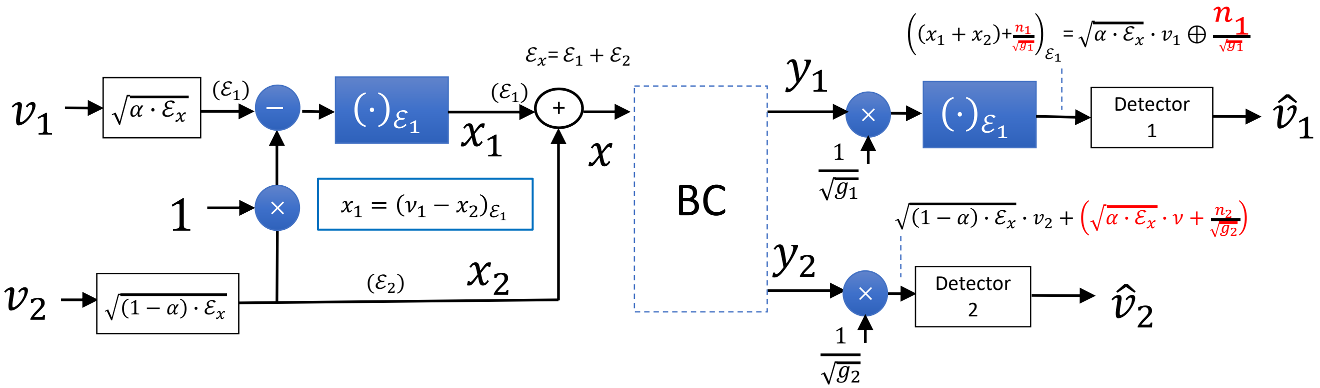
Successive decoder for BC User 1
(order favors user 1)



Successive decoder for BC User 2
(User 1 is distortion)

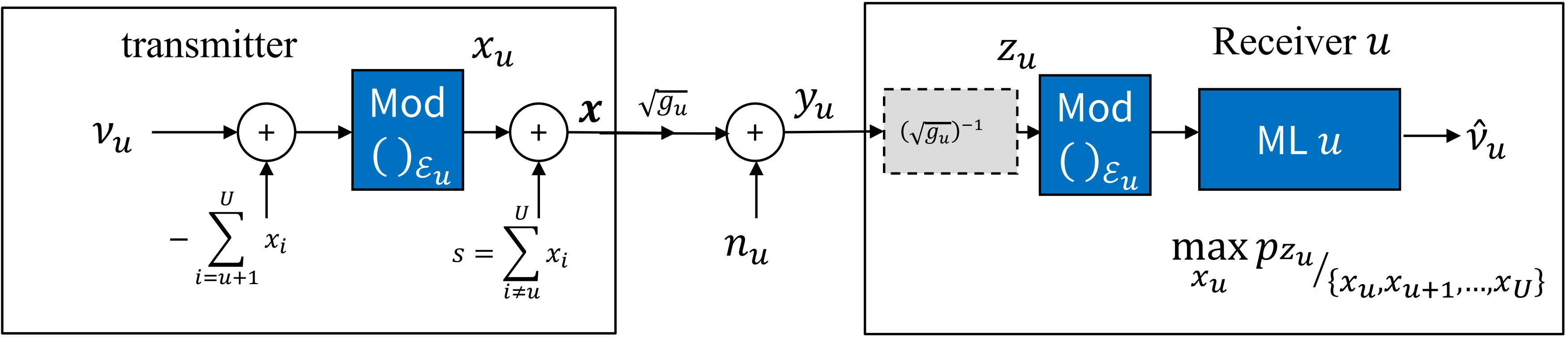
- U ML-U detectors; or really $\sum_{u=1}^{U} u = \frac{U}{2} \cdot (U+1)$ total detectors.
- A precoder simplifies to U uses of the same modulo at transmitter (+ 1 modulo at each receiver).

Scalar Precoder


$\left((x_1+x_2)+\frac{n_1}{\sqrt{g_1}}\right)_{\mathcal{E}_1} = \sqrt{\alpha\cdot\mathcal{E}_x}\cdot v_1 \oplus \frac{n_1}{\sqrt{g_1}}$
$\mathcal{E}_x = \mathcal{E}_1 + \mathcal{E}_2$
v_1
$\sqrt{\alpha\cdot\mathcal{E}_x}$
$(\mathcal{E}_1)$
$-$
$(\cdot)_{\mathcal{E}_1}$
$(\mathcal{E}_1)$
$+$
x_1
x
BC
y_1
$\times$
$(\cdot)_{\mathcal{E}_1}$
Detector 1
$\hat{v}_1$
$\frac{1}{\sqrt{g_1}}$
1
$\times$
$x_1 = (v_1 - x_2)_{\mathcal{E}_1}$
$\sqrt{(1-\alpha)\cdot\mathcal{E}_x}\cdot v_2 + \left(\sqrt{\alpha\cdot\mathcal{E}_x}\cdot \nu + \frac{n_2}{\sqrt{g_2}}\right)$
v_2
$\sqrt{(1-\alpha)\cdot\mathcal{E}_x}$
$(\mathcal{E}_2)$
x_2
y_2
$\times$
Detector 2
$\hat{v}_2$
$\frac{1}{\sqrt{g_2}}$

Gaussian distortion

- The side information becomes x_2 and $\mathcal{E}_x = \mathcal{E}_1 + \mathcal{E}_2$; receiver 1's modulo removes x_2.
- Precoder applies inductively (recursively) applied from U … 1.

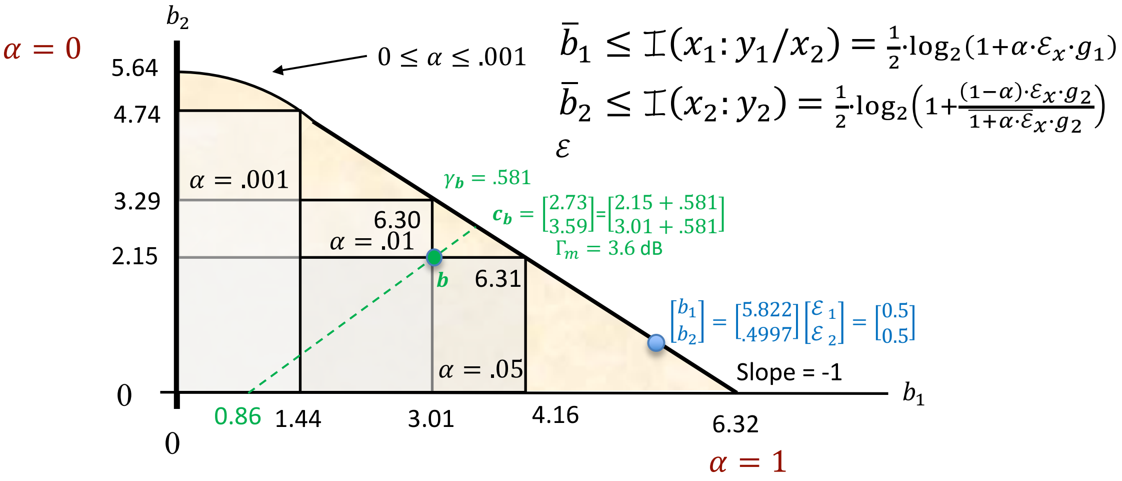


Example

- $h_1 = 0.8\,; h_2 = 0.5; \sigma_1^2 = \sigma_2^2 = .0001$

$$\mathbb{I}(\boldsymbol{x} : \boldsymbol{y}) = \frac{1}{2}\cdot\log_2\left(\frac{|R_{\boldsymbol{yy}}|}{|R_{\boldsymbol{nn}}|}\right) = \frac{1}{2}\cdot\log_2\left(\frac{(.6401)\cdot(.2501)-.4^2}{.01^2}\right) = 6.56$$

α	$\bar{b}_1$	$\bar{b}_2$	$\bar{b} = \bar{b}_1 + \bar{b}_2$
1.0	6.32	0	6.32
.75	6.12	.20	6.32
.50	5.82	.50	6.32
.25	5.32	1.0	6.32
.10	4.66	1.66	6.32
.05	4.16	2.15	6.31
.01	3.01	3.29	6.30
.001	1,44	4.74	6.18
0	0	5.64	5.64



$$\bar{b}_1 \le \mathbb{I}(x_1 : y_1/x_2) = \frac{1}{2}\cdot\log_2(1+\alpha\cdot\mathcal{E}_x\cdot g_1)$$

$$\bar{b}_2 \le \mathbb{I}(x_2 : y_2) = \frac{1}{2}\cdot\log_2\left(1+\frac{(1-\alpha)\cdot\mathcal{E}_x\cdot g_2}{1+\alpha\cdot\mathcal{E}_x\cdot g_2}\right)$$

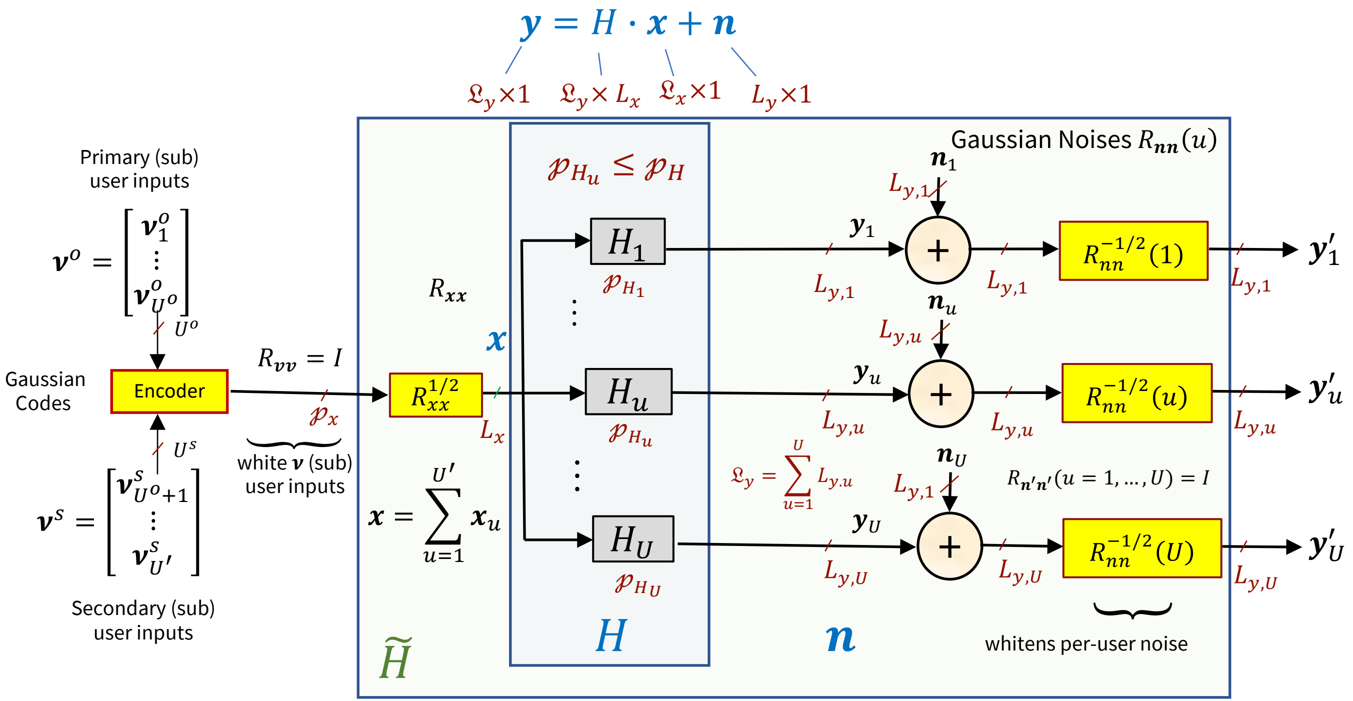
$\mathcal{E}$

- User 1 has highest sum rate when User 2 has zero energy.
  - User 1 is a primary user/component.
  - User 2 is a secondary user/component.

Vector MMSE BC Design

Known $R_{xx}(u)$ Section $2.8.3.1$

Vector Gaussian BC


$\boldsymbol{y} = H \cdot \boldsymbol{x} + \boldsymbol{n}$
$\mathfrak{L}_y \times 1$
$\mathfrak{L}_y \times L_x$
$\mathfrak{L}_x \times 1$
$L_y \times 1$
Primary (sub) user inputs
$\boldsymbol{v}^o = \begin{bmatrix} \boldsymbol{v}_1^o \\ \vdots \\ \boldsymbol{v}_{U^o}^o \end{bmatrix}$
U^o
Gaussian Codes
Encoder
U^s
$\boldsymbol{v}^s = \begin{bmatrix} \boldsymbol{v}_{U^o+1}^s \\ \vdots \\ \boldsymbol{v}_{U'}^s \end{bmatrix}$
Secondary (sub) user inputs
$R_{\boldsymbol{vv}} = I$
$\mathscr{p}_x$
white $\boldsymbol{v}$ (sub) user inputs
$R_{\boldsymbol{xx}}$
$R_{\boldsymbol{xx}}^{1/2}$
$\boldsymbol{x}$
L_x
$\boldsymbol{x} = \sum_{u=1}^{U'} \boldsymbol{x}_u$
$\mathscr{p}_{H_u} \leq \mathscr{p}_H$
H_1
$\mathscr{p}_{H_1}$
H_u
$\mathscr{p}_{H_u}$
H_U
$\mathscr{p}_{H_U}$
H
$\widetilde{H}$
Gaussian Noises $R_{\boldsymbol{nn}}(u)$
$\boldsymbol{n}_1$
$L_{y,1}$
$\boldsymbol{y}_1$
$L_{y,1}$
$+$
$L_{y,1}$
$R_{\boldsymbol{nn}}^{-1/2}(1)$
$L_{y,1}$
$\boldsymbol{y}_1'$
$\boldsymbol{n}_u$
$L_{y,u}$
$\boldsymbol{y}_u$
$L_{y,u}$
$+$
$L_{y,u}$
$R_{\boldsymbol{nn}}^{-1/2}(u)$
$L_{y,u}$
$\boldsymbol{y}_u'$
$\mathfrak{L}_y = \sum_{u=1}^{U} L_{y,u}$
$\boldsymbol{n}_U$
$L_{y,1}$
$\boldsymbol{y}_U$
$L_{y,U}$
$+$
$L_{y,U}$
$R_{\boldsymbol{nn}}^{-1/2}(U)$
$L_{y,U}$
$\boldsymbol{y}_U'$
$R_{\boldsymbol{n'n'}}(u = 1, \ldots, U) = I$
$\boldsymbol{n}$
whitens per-user noise

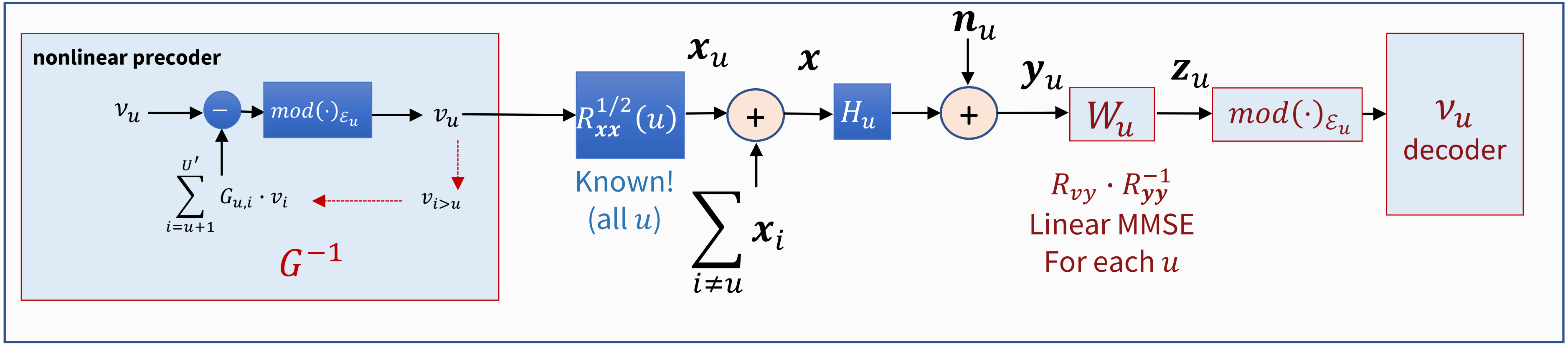
- The users' independent message subsymbol vectors sum to a single BC input $\boldsymbol{x}$.

$$\# \text{ of subusers} = U' \leq \sum_{u=1}^{U} \min\left(\mathscr{p}_x, \mathscr{p}_{H_u}\right) \leq \mathfrak{L}_y \cdot L_x$$
$$\leq U \cdot L_x \text{ (our designs)}$$

Modulator $A = R_{xx}^{1/2}$ need not be square because it includes the sum.

MMSE – BC and Mutual Information – user u

- $\mathcal{I}(\boldsymbol{x}_u : \boldsymbol{y}_u / \boldsymbol{x}_{u+1,\ldots,U}) = \frac{1}{2} \cdot \log_2 \frac{|R_{\boldsymbol{xx}}(\boldsymbol{u})|}{|R_{\boldsymbol{ee}}(\boldsymbol{u})|}$ corresponds to a MMSE problem (like MAC, except $\boldsymbol{y}_u$).



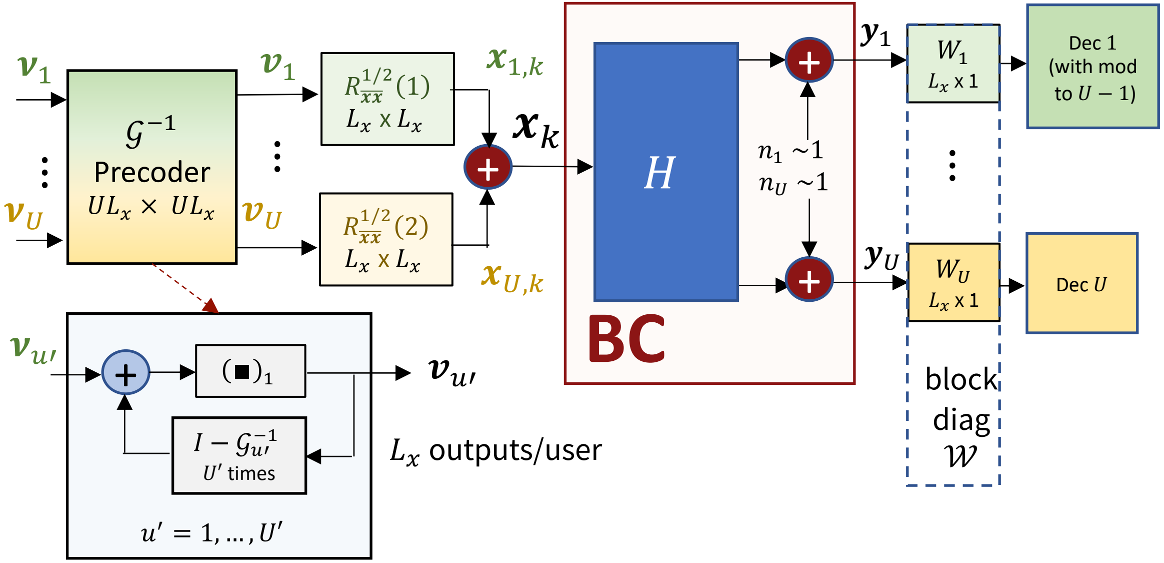
nonlinear precoder
v_u
−
$mod(\cdot)_{\mathcal{E}_u}$
v_u
$\sum_{i=u+1}^{U'} G_{u,i} \cdot v_i$
$v_{i>u}$
G^{-1}
$R_{\boldsymbol{xx}}^{1/2}(u)$
Known!
(all u)
$\boldsymbol{x}_u$
+
$\sum_{i \neq u} \boldsymbol{x}_i$
$\boldsymbol{x}$
H_u
$\boldsymbol{n}_u$
+
$\boldsymbol{y}_u$
W_u
$R_{vy} \cdot R_{\boldsymbol{yy}}^{-1}$
Linear MMSE
For each u
$\boldsymbol{z}_u$
$mod(\cdot)_{\mathcal{E}_u}$
v_u
decoder

- There is successive-decoding ("GDFE") canonical performance (up to U' components).
  - BC implements the G^{-1} with a lossless precoder at the transmitter.

- This structure reliably achieves highest rate for given input $R_{\boldsymbol{xx}}(u)$, and order $\boldsymbol{\pi}_u$.
  - We'll see why shortly.

- The catch? Designer must know $\{R_{\boldsymbol{xx}}(\boldsymbol{u})\}$ and order beforehand.

Structure for all user components $u \in U'$



- This structure needs a little more interpretation when channel rank < number of energized users.

The program mu_bc.m

```
function [Bu, GU, S0, MSWMFunb , B] = mu_bc(H, AU, Lyu , cb)
 ------------------------------------------------------------------------
  Inputs: Hu, AU , Usize, cb
  Outputs: Bu, Gunb, Wunb, S0, MSWMFunb

  H: noise-whitened BC matrix [H1 ; ... ; HU] (with actual noise, not wcn)
     sum-Ly x Lx x N
  AU: Block-row square-root discrete modulators, [A1 ... AU]
      Lx  x (U * Lx) x N
  Lyu: # of (output, Lyu) dimensions for each user U ... 1 in 1 x U row vector
  cb: = 1 if complex baseband or 2 if real baseband channel

  GU: unbiased precoder matrices: (Lx U) x (Lx U) x N
      For each of U users, this is Lx x Lx matrix on each tone
  S0: sub-channel dimensional channel SNRs: (Lx U) x (Lx U) x N
  MSWMFunb: users' unbiased diagonal mean-squared whitened matched matrices
     For each of U cells and Ntones, this is an Lx x Lyu  matrix
  Bu - users bits/symbol 1 x U
     the user should recompute SNR if there is a cyclic prefix
  B - the user bit distributions (U x N) in cell array
```

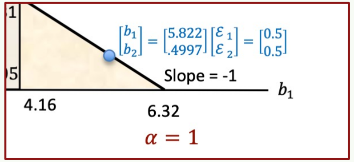
Set $N = 1$ (for now)

```
>> H =
   80
   50
>> Lyu=[1 1];
>> [Bu, Gunb, S0, MSWMFunb] = mu_bc(H, [1/sqrt(2) 1/sqrt(2)], Lyu, 2)
Bu =
   5.8222    0.4997
>> Gunb{:,:} =
   1.0000    1.0000
---
       0         1
S0 =
  2 x 1 cell array
   {[3.2010e+03]}    User 1 SNR
   {[    1.9992]}    User 2 SNR
MSWMFunb =
  2 x 1 cell array
   {[0.0177]}        multiplies y1
   {[0.0283]}        multiplies y2
>> sum(Bu) =    6.3219
>> [Bu, GU, S0, MSWMFunb , B] = mu_bc(H, [1 0], 1 , 2);
>> B =  1 × 1 cell array    {[6.3220]}
```

Equal energy on both users

G adds user 2 to user 1 – we already knew this

Add nothing to user 2



Receivers are each simple scaling

- L10:10's blue rate vector has same values.
- This channel's rate sum is already close to maximum, which occurs at $b = 6.3220$.

More Examples

```
 H=[50 30
10 20];
>> A =
   0.5000       0   0.5000       0
        0   0.5000       0   0.5000
[Bu, Gunb, S0, MSWMFunb] = mu_bc(H, A, [1 1], 2);
Bu =
   4.8665   0.4971
>> Gunb{:,:} =
   1.0000   0.6000   1.0000   0.6000
        0   1.0000   1.6667   1.0000
  -----------------
        0        0   1.0000   2.0000
        0        0   0        1.0000
>> MSWMFunb{:,:} =
   0.0400
   0.0667
------
   0.2000
   0.1000
```

$U' = U^{L_x} =$ 4 subuser components

user 1’s own xtalk and user 2 also

user 2’s own xtalk

Each receiver estimates 2 input dimensions for its user, each a subuser.

```
>> >> S0{:,:} =
  626.0000        0
        0   1.3594
-------------------
   1.1984        0
        0   1.6623

>> sum(Bu) =   5.3636
```

User 1’s dimensional SNR’s

User 2’s dimensional SNR’s

- mu_bc.m solves two MMSE problems here (for receiver 1 and receiver 2).
- It also aggregates them into right places in single matrix (cell array) of feedback/precoder, receiver filters.
- The receiver filters’ rows apply to only their specific user/component (subuser) through MSWMFunb.

Worst-Case-Noise BC Design

Sections 2.8.3.2 and 2.8.3.5

$L_{y,u} = 1$ Case: finding the primary components

- Find each user's normalized channel $\widetilde{H}_u \triangleq R_{\boldsymbol{nn}}^{-1/2}(u) \cdot H_u$.

- Later → a general $(L_{y,u} > 1)$ way that uses worst-case noise; however, $L_{y,u} = 1$ is simpler to describe.

$$y_u = \tilde{h}_{u,1} \cdot x_1 + \cdots + \tilde{h}_{u,L_x} \cdot x_{L_x}$$

- Find largest BC element:
  - This becomes user i_1 and is first in order π.

$$h_{max} = \max_{i,j} \left|\tilde{h}_{i,j}\right| \; i \in I_{BC} \wedge j \in J_{BC}$$

- Find next largest channel gain with user 1 as noise:

$$\tilde{h}_{max} = \max_{i,j} \left|\tilde{h}_{i,j}\right|^2 / (\left|\tilde{h}_{i,i_1}\right|^2 + 1) \; \forall \, i \in \{I_{BC} \setminus i_1\} \wedge j \in \{J_{BC} \setminus \{i_1\}\}$$

  - That is user i_2 and is second in order π.

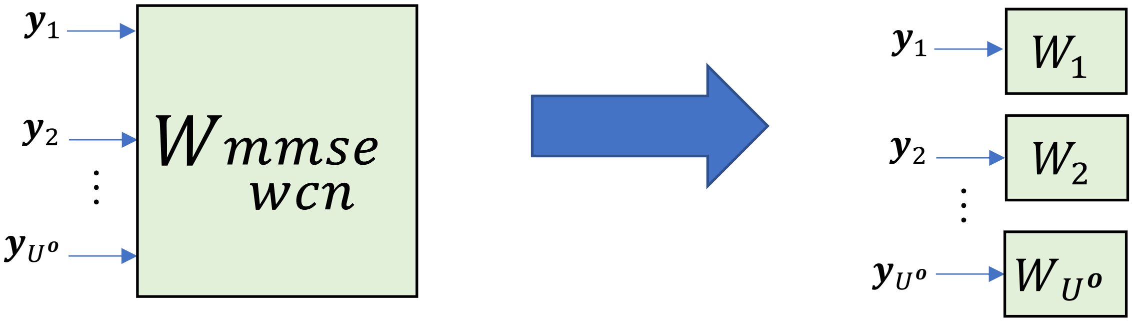
- This continues recursively $U^o = \wp^H$ times.
- Any energy on users $\{\min(L_x, \wp_H) + 1, \ldots, U\}$ reduces rate sum and is from secondary components.

Worst-Case Noise (2.8.3.3)

- "Worst-case" noise has covariance $R_{\boldsymbol{nn}}$ that minimizes $\mathcal{I}(\boldsymbol{x};\boldsymbol{y})$ for a fixed $R_{\boldsymbol{xx}}$.
  - Only the receivers' local noises $R_{\boldsymbol{nn}}(u)$ are fixed, but the correlation between different user/receivers' noise may vary.

- Thm: $R_{wcn}^{-1} - [H \cdot R_{\boldsymbol{xx}} \cdot H^*]^{-1} + R_{psd} = \mathcal{S}_{wcn}$ where $\mathcal{S}_{wcn}$ is $\mathfrak{L}_y \times \mathfrak{L}_y$ block (sub-block sizes $L_{y,u}$) diagonal.
  - Further: $\varrho_{R_{wcn}} = \varrho_{\mathcal{S}_{wcn}}$ = # of primary components = U^0 .
  - Further: The secondary components correspond to S_{wcn}'s zeroed diagonal elements (equivalently nonzero elements correspond to primary components).
  - R_{psd} is a Lagrange constraint for the positive semidefinite nature of the worst-case noise, and $\mathcal{S}_{wcn}$ is the Lagrange constraint for the block diagonal noises. R_{psd} is absent unless R_{wcn} is singular – many treatments ignore the singular case.
- Proof: See notes (also Appendix C on matrix calculus).

- When noise has R_{wcn} , the MMSE estimate $\hat{\boldsymbol{v}} = W \cdot \boldsymbol{y}$ has W that **IS DIAGONAL (block).**

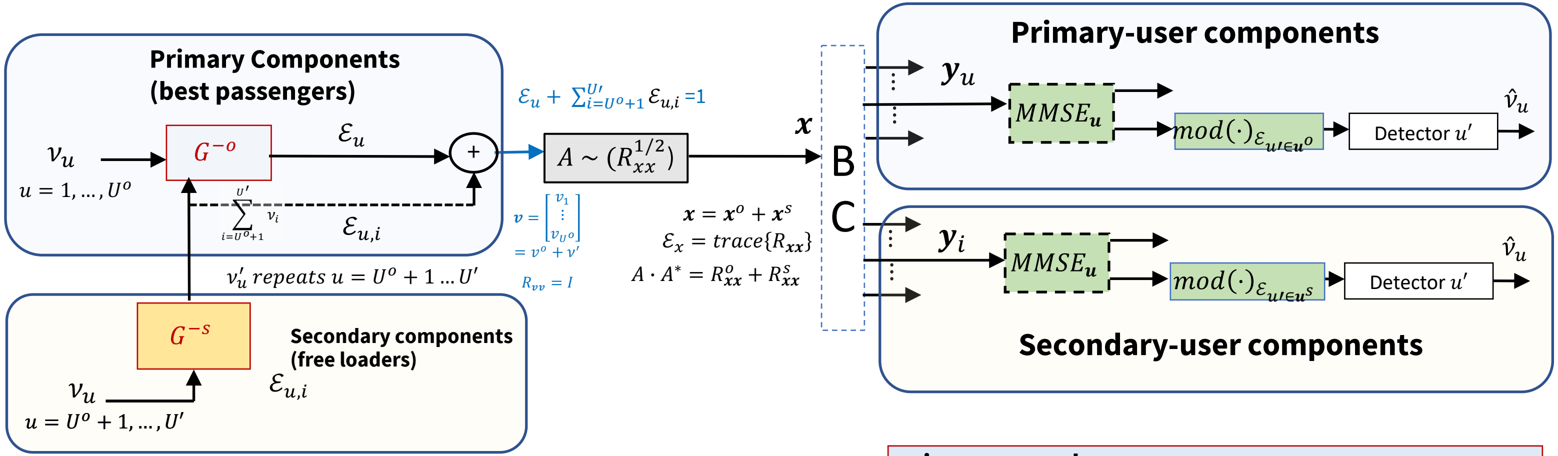


So, WCN corresponds to best BC receiver(s)
$b = \mathcal{I}_{wcn}(\boldsymbol{x};\boldsymbol{y})$

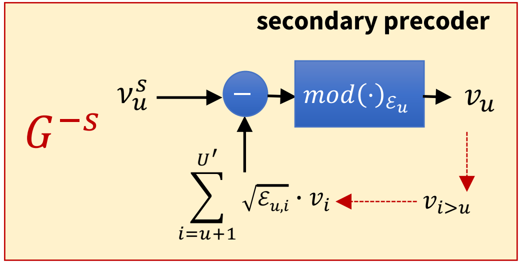
- Energized secondary components simply "free load" on the primary-components' dimensions.

Remove Singularity and now Precode

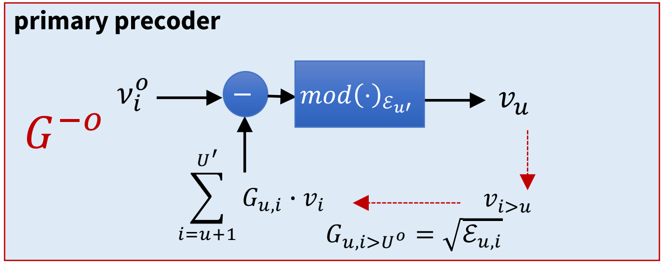
- Now with known primary (sub-) users, original channel's SVD (no noise-whiten) is $H \cdot R_{xx}^{1/2} \triangleq [F \quad \boldsymbol{f}\,] \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M^* \\ \boldsymbol{m}^* \end{bmatrix}$.
- Input transforms to $\boldsymbol{x} \to M \cdot \boldsymbol{v}$ (M becomes part of square root, & relabel); $H \cdot R_{xx}^{1/2} \to F\Lambda$.



Primary Components (best passengers)
v_u
$u = 1, \dots, U^o$
G^{-o}
$\mathcal{E}_u$
$+$
$\sum_{i=U^o+1}^{U'} v_i$
$\mathcal{E}_{u,i}$
$\mathcal{E}_u + \sum_{i=U^o+1}^{U'} \mathcal{E}_{u,i} = 1$
$A \sim (R_{xx}^{1/2})$
$\boldsymbol{x}$
$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_{U^o} \end{bmatrix} = v^o + v'$
$R_{vv} = I$
v'_u repeats $u = U^o + 1 \dots U'$
$\boldsymbol{x} = \boldsymbol{x}^o + \boldsymbol{x}^s$
$\mathcal{E}_x = trace\{R_{xx}\}$
$A \cdot A^* = R_{xx}^o + R_{xx}^s$
B
C
G^{-s}
Secondary components (free loaders)
$\mathcal{E}_{u,i}$
v_u
$u = U^o + 1, \dots, U'$
Primary-user components
$\boldsymbol{y}_u$
$MMSE_{\boldsymbol{u}}$
$mod(\cdot)_{\mathcal{E}_{u' \in u^o}}$
Detector u'
$\hat{v}_u$
$\boldsymbol{y}_i$
$MMSE_{\boldsymbol{u}}$
$mod(\cdot)_{\mathcal{E}_{u' \in u^s}}$
Detector u'
$\hat{v}_u$
Secondary-user components



secondary precoder
G^{-s}
v_u^s
$-$
$mod(\cdot)_{\mathcal{E}_u}$
v_u
$\sum_{i=u+1}^{U'} \sqrt{\mathcal{E}_{u,i}} \cdot v_i$
$v_{i>u}$



primary precoder
G^{-o}
v_i^o
$-$
$mod(\cdot)_{\mathcal{E}_{u'}}$
v_u
$\sum_{i=u+1}^{U'} G_{u,i} \cdot v_i$
$v_{i>u}$
$G_{u,i>U^o} = \sqrt{\mathcal{E}_{u,i}}$

Mohseni’s nonsingular-WCN Program (no CVX)

```
function [Rwcn, bsum] = wcnoise(Rxx, H, Ly, dual_gap, nerr)

 inputs
 H is U*Ly by Lx, where
     Ly is the (constant) number of antennas/receiver,
     Lx is the number of transmit antennas, and
     U is the number of users. H can be a complex matrix
 Rxx is the Lx by Lx input (nonsingular) autocorrelation matrix
     which can be complex (and Hermitian!)
 dual_gap is the duality gap, defaulting to 1e-6 in wcnoise
 nerr is Newton's method acceptable error, defaulting to 1e-4 in
     wcnoise
```

```
outputs
Rwcn is the U*Ly by U*Ly worst-case-noise autocorrelation matrix.
bsum is the rate-sum/real-dimension.

I = 0.5 * log(det(H*Rxx*H'+Rwcn)/det(Rwcn)), Rwcn has Ly x Ly diagonal blocks
  that are each equal to an identity matrix
```

- Example

```
>> H=[80
50] =
   80
   50            (noise-whitened/normalized channel)
>> [Rwcn,b]=wcnoise(1,H,1,1e-6,1e-4)
Rwcn =
   1.0000    0.6250
   0.6250    1.0000
b =    6.3220.
```

```
>> Htilde=inv(Rwcn)*H =
  80.0000
   0.0000
>>Swcn=inv(Rwcn)-inv(H*H' + Rwcn). =
   0.9998    0.0000
   0.0000    0.0000
>> Ryy=H*H'+Rwcn =   1.0e+03 *
   6.4010    4.0006
   4.0006    2.5010
>> SNRp1=det(Ryy)/det(Rwcn)  = 6.4010e+03
>> 0.5*log2(SNRp1)  =    6.3220 (checks!)
```

- $S_{wcn}(u,u)$ = sensitivity to $R_{\boldsymbol{nn}}(u)$ change, if = 0 ➔ secondary user.
- Note WCN program permits easy sketch of the capacity region.
  - Hint: you know noise-free user 1, and you also know the sum, so user 2’s rate is the difference (PS5.3).

Singular 3x3 BC, nonsingular WCN

```
>> H =

  80   60   40
  60   45   30
  20   20   20

>> Rxx  =


   3    0    0
   0    4    0
   0    0    2

>> [Rwcn, b]=wcnoise(Rxx, H, 1, 1e-5 , 1e-4);

 >>Rwcn
  1.0000   0.7500   0.0016
  0.7500   1.0000   0.0012
  0.0016   0.0012   1.0000
>> b
  11.3777
>> Swcn=inv(Rwcn)-inv(H*Rxx*H'+Rwcn) =

  0.9995   0.0000   0.0000
  0.0000  -0.0000   0.0000
  0.0000   0.0000   0.9948
```

- Works for any Rxx **IF** WCN produced is nonsingular

```
>> sr=[5 0 0
7 8 9
10 11 12];
>> Rxx=sr*sr' =
   25   35   50
   35  194  266
   50  266  365
>> [Rwcn, b]=wcnoise(Rxx, H, 1, 1e-5 , 1e-4);
>> Rwcn
   1.0000   0.7500   0.0002
   0.7500   1.0000   0.0001
   0.0002   0.0001   1.0000
>> b = 16.4029
>> Swcn=inv(Rwcn)-inv(H*Rxx*H'+Rwcn)

Swcn =

   0.9999   0.0000   0.0000
   0.0000  -0.0000  -0.0000
   0.0000  -0.0000   0.9996
```

- Note position of zeros – user 2 is a single secondary component.
- wcnoise.m is sufficient if WCN is nonsingular.
- Nonsingular R_{xx} forces ID of secondary components (WCN nonsingular).

Liao’s Generalized Worst-Case Noise (uses CVX)

```
function [Rnn, sumRatebar, S1, S2, S3, S4] = cvx_wcnoise(Rxx, H, Lyu)

cvx_wcnoise This function computes the worst-case noise for any given input
autocorrelation Rxx and channel matrix.
  Arguments:
    - Rxx: input autocorrelation, size(Lx, Lx)
    - H: channel response, size (Ly, Lx)
    - Lyu: number of antennas at each user, scalar/vector of length U
  Outputs:
    - Rnn: worst-case noise autocorrelation, with white local noise
    - sumRatebar: maximum sum rate/real-dimension
    - S1 is the lagrange multiplier for the real part of Rnn diagonal
        elements.  Zero values indicate secondary users.
    - S2 is the imaginary part
    - S3 is for the positive semidefinite constraint on Rwcn
    - S4 is for a larger Schur compliment used in the optimization
```

also allows variable number of antennas/user – not so in wcnoise.m

S1 and S2 together are S_{wcn} ; S2 prevents quantization-error accumulation on imaginary part

S3 plus S4 together in upper $U\times U$ positions equal S_{wcn}

- This program accommodates singular WCN (which is common in well-designed systems).
- The S3 plus S4 are the R_{psd} value, which must add to the S_{wcn} (block) diagonal = S1, see text 2.8.3.3.
- Secondary users are only identified when R_{xx} is nonsingular or if singular, the best or “water-filling for WCN” case,
    - recalling that secondary indentification occurs only for maximum rate sum.

Singular WCN

- Can occur if input is lower rank (optimization):

```
>> H =
   0.4054 - 0.1990i   0.3641 + 0.6869i   3.6004 + 0.5569i   0.5318 + 0.0080i
   1.8406 + 1.3469i  -1.3014 - 1.1630i   2.7217 + 1.0820i   0.0947 - 1.0710i
  -2.3367 + 1.1594i  -0.3949 + 0.7899i  -1.4024 + 0.8380i   0.8085 + 0.3019i
   1.1210 + 1.4423i   0.2611 + 1.6376i   2.9534 - 0.3945i   0.2962 - 0.8347i

>> Rxx =
   0.1382 + 0.0000i   0.0077 - 0.0664i  -0.0701 + 0.0449i  -0.0594 - 0.0207i
   0.0077 + 0.0664i   0.2005 + 0.0000i  -0.0155 - 0.0152i   0.0281 - 0.0130i
  -0.0701 - 0.0449i  -0.0155 + 0.0152i   0.0522 + 0.0000i   0.0262 + 0.0288i
  -0.0594 + 0.0207i   0.0281 + 0.0130i   0.0262 - 0.0288i   0.0331 + 0.0000i
>> rank(H) =     3
>> rank(Rxx) =     2
>>[V, D]=eig(Rxx);
diag(D) = -0.0001  0.0001    0.1418     0.2821
>> Rxx=V(:,3)*V(:,3)'*D(3,3)+ V(:,4)*V(:,4)'*D(4,4);
rank(iRxx) = 2
[F,L,M]=svd(H);
diag(L) =  6.7116    3.1056    2.0988    0.0000
H=F(:,1)*M(:,1)'*L(1,1)+ F(:,2)*M(:,2)'*L(2,2)+F(:,3)*M(:,3)'*L(3,3);
rank(H) =    3
```

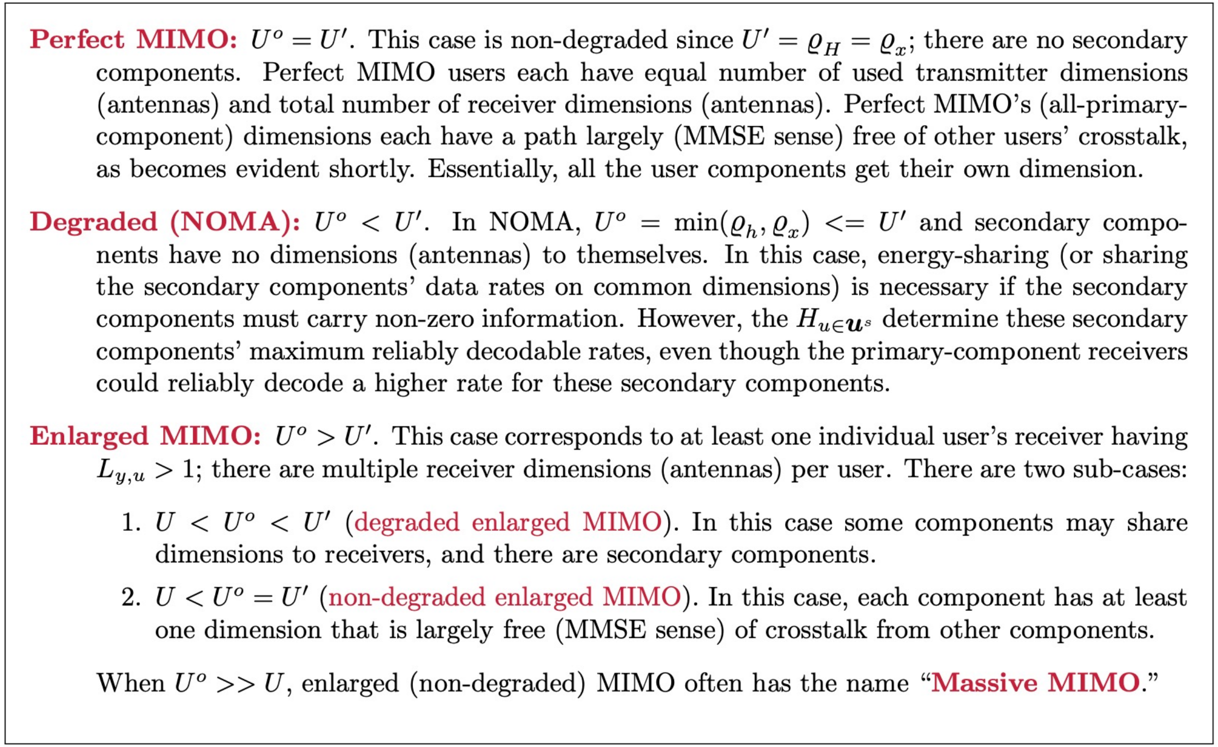
Only Necessary for copy-paste.

```
[>> [Rwcn, b , S1, S2, S3, S4] = cvx_wcnoise(Rxx, H, ones(1,4))
Rwcn =
   1.0000 + 0.0000i  -0.5084 + 0.0458i   0.1251 + 0.5916i  -0.0042 + 0.3065i
  -0.5084 - 0.0458i   1.0000 + 0.0000i  -0.4394 + 0.2473i  -0.6341 + 0.0932i
   0.1251 - 0.5916i  -0.4394 - 0.2473i   1.0000 + 0.0000i   0.7215 + 0.3923i
  -0.0042 - 0.3065i  -0.6341 - 0.0932i   0.7215 - 0.3923i   1.0000 + 0.0000i
>> rank(Rwcn) =    3
b =    0.8241
S1 =  4x1 cell array
    {[    0.2288]}
    {[    0.3149]}
    {[    0.3073]}
    {[5.9295e-09]}
S2 =  4x1 cell array
    {[0]}
    {[0]}
    {[0]}
    {[0]}
S3+S4(1:4,1:4) =
    0.2288         0         0         0
         0    0.3149         0         0
         0         0    0.3073         0
         0         0         0    0.0000
>> pinv(Rwcn)*H =   1.0e+09*
   1.0647 + 0.7898i  -0.1492 + 0.2605i   2.1940 + 0.5313i   0.1632 - 0.5248i
   0.4982 + 1.1166i  -0.2378 + 0.1418i   1.5228 + 1.4199i   0.3687 - 0.3479i
  -0.6578 + 1.1464i  -0.2754 - 0.1173i  -0.2699 + 2.2345i   0.5387 + 0.1003i
  -0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0001 - 0.0000i  -0.0000 + 0.0000i
```

- 0 sensitivity with singular WCN may **not** identify secondary user,
  - which appears to be last user here.

- It is secondary, **IF** also maximum rate sum (see L11).

3 BC Cases



Perfect MIMO: $U^o = U'$. This case is non-degraded since $U' = \varrho_H = \varrho_x$; there are no secondary components. Perfect MIMO users each have equal number of used transmitter dimensions (antennas) and total number of receiver dimensions (antennas). Perfect MIMO's (all-primary-component) dimensions each have a path largely (MMSE sense) free of other users' crosstalk, as becomes evident shortly. Essentially, all the user components get their own dimension.

Degraded (NOMA): $U^o < U'$. In NOMA, $U^o = \min(\varrho_h, \varrho_x) <= U'$ and secondary components have no dimensions (antennas) to themselves. In this case, energy-sharing (or sharing the secondary components' data rates on common dimensions) is necessary if the secondary components must carry non-zero information. However, the $H_{u\in\boldsymbol{u}^s}$ determine these secondary components' maximum reliably decodable rates, even though the primary-component receivers could reliably decode a higher rate for these secondary components.

Enlarged MIMO: $U^o > U'$. This case corresponds to at least one individual user's receiver having $L_{y,u} > 1$; there are multiple receiver dimensions (antennas) per user. There are two sub-cases:

1. $U < U^o < U'$ (degraded enlarged MIMO). In this case some components may share dimensions to receivers, and there are secondary components.
2. $U < U^o = U'$ (non-degraded enlarged MIMO). In this case, each component has at least one dimension that is largely free (MMSE sense) of crosstalk from other components.

When $U^o >> U$, enlarged (non-degraded) MIMO often has the name "**Massive MIMO**."

- Wi-Fi, 4G/5G, Vector DSL (sometimes called MU-MIMO) are all "perfect MIMO."
- Time-sharing (TDMA) really just increases channel rank until perfect MIMO, not optimum, & extra delay.
- NOMA is more general; IoT, Metaverse, FWA when congested are NOMA.

Vector WCN-BC Design

PS5.3 - 2.30 Vector BC Design

WCN Design focuses on primary users

- Any secondary-user components “free load” on the dimensions best used by primary-user components.
- Delete the secondary-user components’ rows from $\tilde{H}$.
- The precoder-coefficients’ design, which pre-inverts channel, depends on those primary components.
  - Any energized secondary components “dimension-share” those primary dimensions (and reduce overall rate sum).
- This design method can provide BC insights.
- Chapter 5 will find a way for any desired $\boldsymbol{b}' \in \mathcal{C}(\boldsymbol{b})$ to derive the $\{R_{\boldsymbol{xx}}(u)\}$, but the choice of $\boldsymbol{b}'$ (scheduling) may want to know about primary/secondary components.
- **Every user component has its own single-user good code (so use your 379A favorite code).**

Only for primary users (→ *NONSINGULAR* WCN)

- Use R_{wcn} directly:
  - General $\mathcal{S}_{wcn}$ is block diagonal.
  - Find A.

$$R_{wcn}^{-1} - \left[H \cdot A \cdot A^* \cdot H^* + R_{wcn}\right]^{-1} = \mathcal{S}_{wcn}$$

- Indeed, that is the backward MMSE channel in there!
  - Q_{wcn} is also block diagonal.

$$\begin{aligned} \mathcal{S}_{wcn} &= R_{wcn}^{-1} - \left[R_{wcn}^{-1} - R_{wcn}^{-1} \cdot H \cdot A \left(I + A^* \cdot H^* \cdot R_{wcn}^{-1} \cdot H \cdot A\right)^{-1} A^* \cdot H^* \cdot R_{wcn}^{-1}\right] \\ Q_{wcn}^* \cdot \mathcal{S}_{wcn}' \cdot Q_{wcn} &= R_{wcn}^{-1} \cdot H \cdot A \boxed{\underbrace{\left(I + A^* \cdot H^* \cdot R_{wcn}^{-1} \cdot H \cdot A\right)^{-1}}_{R_b \triangleq G^{-1} \cdot S_0^{-1} \cdot G^{-*}}} A^* \cdot H^* \cdot R_{wcn}^{-1} \qquad (2.432) \\ \underbrace{\mathcal{S}_{wcn}'}_{\text{diagonal}} &= \underbrace{Q_{wcn} \cdot R_{wcn}^{-1} \cdot H \cdot A}_{\text{triangular inverse}} \cdot R_b \cdot \underbrace{A^* \cdot H^* \cdot R_{wcn}^{-1} \cdot Q_{wcn}^*}_{\text{triangular inverse}} \;, \qquad (2.433) \end{aligned}$$

- QR factorization (primaries' channel)
  - extract $A = R_{\boldsymbol{xx}}^{1/2}$ from tri inverse,
  - which is the forward channel.

$$Q_{wcn} \cdot R_{wcn}^{-1} \cdot H = \left[\begin{array}{cc} \underbrace{\mathbf{0}}_{(L_x - U^o) \times U^o} & \underbrace{R}_{U^o \times U^o} \end{array}\right] \left[\begin{array}{c} \boldsymbol{q} \\ \underbrace{Q^*}_{U^o \times L_x} \end{array}\right] = R \cdot Q^*$$

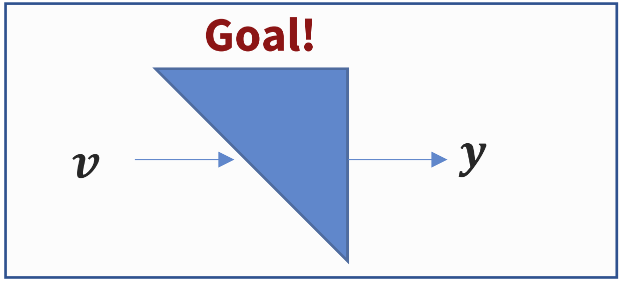
- Cholesky factorization the input.

$$\Phi \cdot \Phi^* = Q^* \cdot R_{\boldsymbol{xx}} \cdot Q \quad .$$

- A special square root!
  - "pre-triangularizes" the channel,
  - which becomes $R \cdot \Phi$.

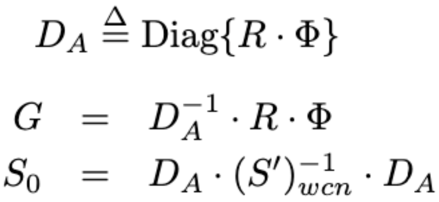
$$R_{\boldsymbol{xx}}^{1/2} = A = Q \cdot \Phi$$

Goal!



The precoder

- Design wants monic G for precoder:



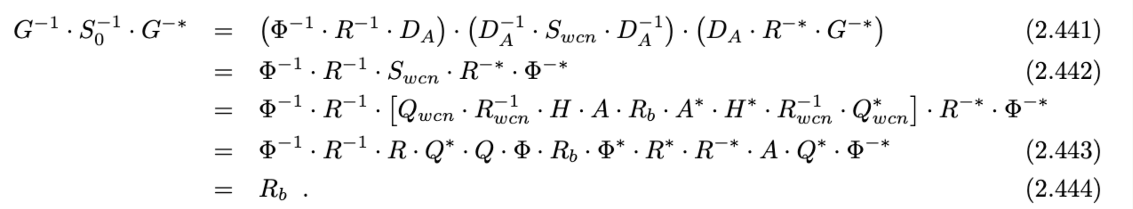
$$D_A \triangleq \text{Diag}\{R \cdot \Phi\}$$

$$\begin{aligned} G &= D_A^{-1} \cdot R \cdot \Phi \\ S_0 &= D_A \cdot (S')^{-1}_{wcn} \cdot D_A \end{aligned}$$

Find diagonal values

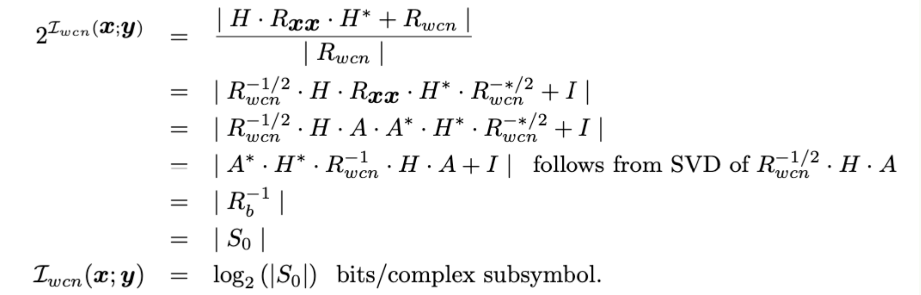
Monic Equivalent

- Check = R_b for G and S_0:



$$\begin{aligned} G^{-1} \cdot S_0^{-1} \cdot G^{-*} &= \left(\Phi^{-1} \cdot R^{-1} \cdot D_A\right) \cdot \left(D_A^{-1} \cdot S_{wcn} \cdot D_A^{-1}\right) \cdot \left(D_A \cdot R^{-*} \cdot G^{-*}\right) && (2.441) \\ &= \Phi^{-1} \cdot R^{-1} \cdot S_{wcn} \cdot R^{-*} \cdot \Phi^{-*} && (2.442) \\ &= \Phi^{-1} \cdot R^{-1} \cdot \left[Q_{wcn} \cdot R_{wcn}^{-1} \cdot H \cdot A \cdot R_b \cdot A^* \cdot H^* \cdot R_{wcn}^{-1} \cdot Q_{wcn}^*\right] \cdot R^{-*} \cdot \Phi^{-*} \\ &= \Phi^{-1} \cdot R^{-1} \cdot R \cdot Q^* \cdot Q \cdot \Phi \cdot R_b \cdot \Phi^* \cdot R^* \cdot R^{-*} \cdot A \cdot Q^* \cdot \Phi^{-*} && (2.443) \\ &= R_b \ . && (2.444) \end{aligned}$$

- Check SNR and mutual-info:



$$\begin{aligned} 2^{\mathcal{I}_{wcn}(\boldsymbol{x};\boldsymbol{y})} &= \frac{|\, H \cdot R_{\boldsymbol{xx}} \cdot H^* + R_{wcn}\,|}{|\, R_{wcn}\,|} \\ &= |\, R_{wcn}^{-1/2} \cdot H \cdot R_{\boldsymbol{xx}} \cdot H^* \cdot R_{wcn}^{-*/2} + I\,| \\ &= |\, R_{wcn}^{-1/2} \cdot H \cdot A \cdot A^* \cdot H^* \cdot R_{wcn}^{-*/2} + I\,| \\ &= |\, A^* \cdot H^* \cdot R_{wcn}^{-1} \cdot H \cdot A + I\,| \quad \text{follows from SVD of } R_{wcn}^{-1/2} \cdot H \cdot A \\ &= |\, R_b^{-1}\,| \\ &= |\, S_0\,| \\ \mathcal{I}_{wcn}(\boldsymbol{x};\boldsymbol{y}) &= \log_2\left(|S_0|\right) \ \text{ bits/complex subsymbol.} \end{aligned}$$

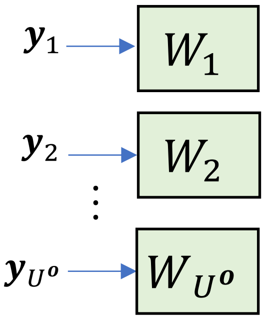
Works!

The Receiver

- The MMSE receiver is block diagonal(!)
  - for WCN only, but is
  - just what the BC needs

$$
\begin{aligned}
W &= \underbrace{S_0^{-1} \cdot G^{-*}}_{1-to-1} \cdot \underbrace{A^* \cdot H^* \cdot R_{wcn}^{-1}}_{\text{noise-white-match}} \cdot \underbrace{Q_{wcn}^* \cdot Q_{wcn}}_{I} \\
&= S_0^{-1} \cdot G^{-*} \cdot \Phi^* \cdot Q^* \cdot Q \cdot R^* \cdot Q_{wcn} \\
&= S_0^{-1} \cdot G^{-*} \cdot \Phi^* \cdot R^* \cdot Q_{wcn} \\
&= S_0^{-1} \cdot G^{-*} \cdot G^{-1} \cdot D_A \cdot Q_{wcn} \\
&= S_0^{-1} \cdot D_A \cdot Q_{wcn} \quad ,
\end{aligned}
$$

- Design has same bias removal as with all MMSE.



BC WCN-Design Steps Summary (2.8.3.3)

Special Square Root

- Find R_{wcn} - this step also finds $\mathcal{S}_{wcn}$ and also the primary/secondary users and $b_{max}(R_{\boldsymbol{xx}})$.
  - Delete rows/columns (secondary sub user dimensions) with zeros from $\mathcal{S}_{wcn}$, and correspondingly then in R_{wcn}.
- If $\mathcal{S}_{wcn}$ is non-trivial (block diagonal), form $\mathcal{S}_{wcn}$=$Q^*_{wcn} \cdot \mathcal{S}'_{wcn} \cdot Q_{wcn}$ (eigen decomp).
- Perform QR factorization on $Q_{wcn} \cdot R^{-1}_{wcn} \cdot H = R \cdot Q^*$ where R is upper triangular, and Q is unitary.
- Perform Cholesky Factorization on $Q^* \cdot R_{\boldsymbol{xx}} \cdot Q$ = $\Phi \cdot \Phi^*$ where Φ is also upper triangular.
- And now, the special square root is $R^{1/2}_{\boldsymbol{xx}} = Q \cdot \Phi$ (see diagram L10:22 = A).

Precoder and Diagonal Receiver

- Find the diagonal matrix D_A =Diag$\{R \cdot \Phi\}$.
- Find the (primary sub-user) precoder $G = D_A^{-1} \cdot R \cdot \Phi$ (monic upper triangular).
- Find the backward MMSE (block) diagonal matrix $S_0 = D_A \cdot (S')^{-1}_{wcn} \cdot D_A$ (note, $R_b^{-1} = G^* \cdot S_0 \cdot G$).
- Block diagonal (unbiased) receiver is $W_{unb} = (S_0^{-1} - I)^{-1} \cdot D_A \cdot Q_{wcn}$.
- Can check, but $b_{max}(R_{\boldsymbol{xx}})$ from WCN will be $\mathcal{I}_{wcn}(\boldsymbol{x};\boldsymbol{y})$=$\log_2|S_0| = \sum_{u=1}^{U^o} \log_2\left(1 + SNR_{BC,wcn,u}\right)$.

Other data rate vectors $\boldsymbol{b}$ then share this system between primary/secondary.

Example – all primary

- Energy $\mathcal{E}_x$ =2 , $L_x = 2$

```
>>  H = [80 70 ; 50 60];
>>Rxx=[1 .8 ; .8 1];
```

```
>> [Rwcn,b]=wcnoise(Rxx,H,1)
Rwcn =
    1.0000    0.0232
    0.0232    1.0000
b = 9.6430
>> Swcn = inv(Rwcn)-inv(H*Rxx*H'+Rwcn) =
    0.9835    0.0000
    0.0000    0.9688
>> Htilde=inv(Rwcn)*H =
   78.8817   68.6440
   48.1687   58.4064
>> [R,Q,P]=rq(Htilde)
R =
  -12.4389  -74.6780
         0    -104.5673
Q =
    0.6565   -0.7544
   -0.7544   -0.6565
P =     2     1
ORDER IS REVERSED SO SWITCH USERS!
J=[0 1; 1 0];
```

Nonsingular Rwcn

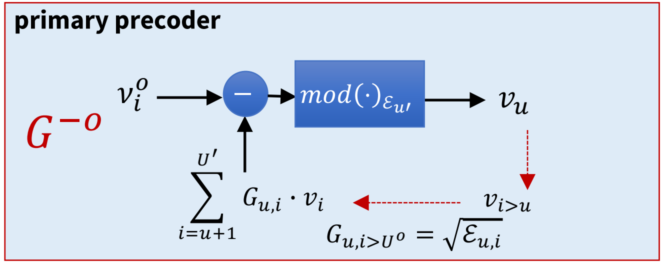
```
>> Rxxrot=Q'*Rxx*Q;
>> Phi=lohc(Rxxrot) =
    0.4482    0.0825
         0       1.3388
>> DA=diag(diag(R*Phi));
>> G=inv(DA)*R*Phi  =
    1.0000   18.1182
         0        1.0000
>> A=Q*inv(R)*DA*G =
    0.2942   -0.9557
   -0.3381   -0.9411
>> S0=DA*inv(Swcn)*DA =   1.0e+04 *
    0.0032   -0.0000
   -0.0000    2.0229
Wunb=inv((S0)-eye(2))*DA*J
-0.0000   -0.1822
-0.0069   -0.0000
Indeed diagonal with order switch!
>> Gunb=eye(2)+S0*inv(S0-eye(2))*(G-eye(2)) =
    1.0000   18.7103
         0        1.0000
>> b=0.5*log2(diag(S0))' =   2.4909    7.1521
>> sum(b) =    9.6430 (checks
```

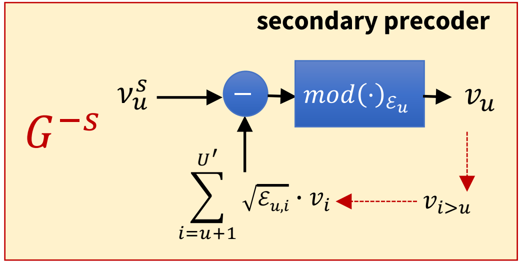
```
>> J*Wunb*H*A*inv(Gunb)  =

   0.0386  >>   -0.0004
   0.0236  <<  25.4902

Diagonally dominant.
```

Try different
Input Rxx,
See text, Ex 2.8.7

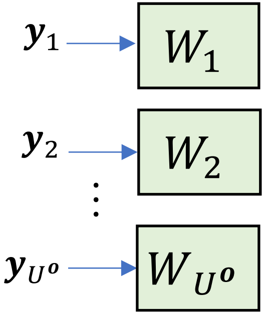
Return to Design

- The design can allocate R_{xx} energy to secondary and primary users as





- The receivers are easy



Another example – singular 3x3 BC (Ex 2.8.8)

```
>> H=[80 60 40
60 45 30
20 20 20];
>> rank(H)  =     2
>> Rxx=diag([3 4 2]);
>> [Rwcn1, b]=wcnoise(Rxx, H, 1, 1e-5 , 1e-4);
>> Rwcn1
    1.0000    0.7500    0.0016
    0.7500    1.0000    0.0012
    0.0016    0.0012    1.0000
>> b =    11.3777
>> Swcn=inv(Rwcn1)-inv(H*Rxx*H'+Rwcn1) =
    0.9995    0.0000    0.0000
    0.0000   -0.0000    0.0000
    0.0000    0.0000    0.9948
```

User 2 is secondary – remove for now

```
>> H1=[H(1,1:3)
H(3,1:3)] =
   80   60   40
   20   20   20
>> [Rwcn, b]=wcnoise(Rxx, H1, 1, 1e-5 , 1e-4);
>> Rwcn =
   1.0000   0.0016
   0.0016   1.0000
>> b =  11.3777
>> Swcn=inv(Rwcn)-inv(H1*Rxx*H1'+Rwcn) =
   0.9995   0.0000
   0.0000   0.9948
```

Primary/Secondary

```
>> [R,Q,P]=rq(inv(Rwcn)*H1)
R =
      0   9.1016  -33.2537
      0        0 -107.6507
Q =
   0.4082  -0.5306  -0.7429
  -0.8165   0.1517  -0.5571
   0.4082   0.8340  -0.3713
P =    2    1
```

ORDER IS REVERSED (Here it is order of users 1 and 3 since 2 was eliminated)

```
>> R1=R(1:2,2:3);
>> Q1=Q(1:3,2:3);
>> Rxxrot=Q1' *Rxx*Q1  =
   2.3275   0.2251
   0.2251   3.1725
>> Phi=lohc(Rxxrot);
>> DA=diag(diag(R1*Phi)) =
  13.8379        0
        0 -191.7414
>> G=inv(DA)*R1*Phi  =
   1.0000  -4.1971
        0   1.0000
>> A=Q1*inv(R1)*DA*G =
  -0.8067  -1.3902
   0.2306  -0.9730
   1.2679  -0.5559
>> A*A'  =
   2.5833   1.1667  -0.2500
   1.1667   1.0000   0.8333
  -0.2500   0.8333   1.9167
```

Sq Root & Precoder

Not equal to Rxx
Energy not inserted into null space (same on part that is in pass space)

Rcvr & Data Rate

```
>> S0=DA*inv(Swcn)*DA =   1.0e+04 *
   0.0192   0.0000
   0.0000   3.6957
>> MSWMFunb=inv((S0)-eye(2))*DA*J  =
    0.0000   0.0726
   -0.0052  -0.0000
>> Gunb=eye(2)+S0*inv(S0-eye(2))*(G-eye(2))  =
   1.0000  -4.2191
  -0.0000   1.0000
>> b=0.5*log2(diag(S0))'  =
   3.7909   7.5868
>> sum(b) =   11.3777 checks
>> H*A =
   0.0219 -191.8333
   0.0164 -143.8749
  13.8379  -58.3825
```

See Example 2.8.8 or details of below

Assign 1 energy unit to User 1, 1/3 to user 3, and now squeeze in 2/3 energy on user 2

```
>> b=0.5*log2(diag([ 1 1/3]) *diag(S0))  =
   3.7909
   6.7943
```

Crosstalk is >> ct=1/3*143.9^2 = 6.8928e+03

```
>> b2=0.5*log2(1+(2/3)*60^2/6892.8) =    0.2155
>> b2+sum(b) = 10.8007 < 11.3777
```

Energy on secondary reduces rate sum

System Diagram for this WCN design

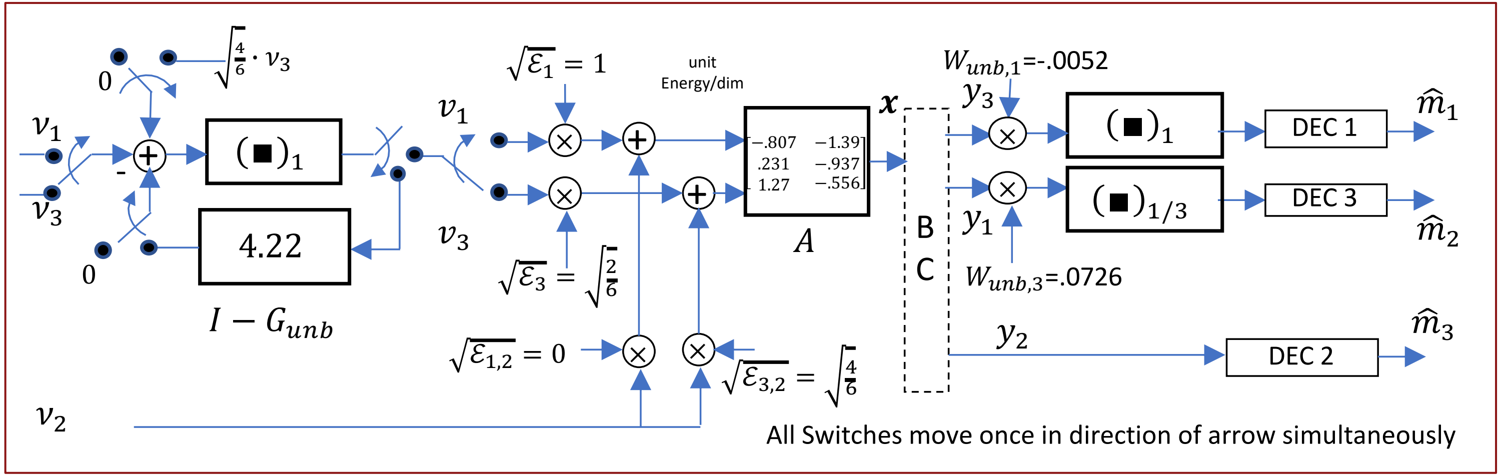
$$
\begin{aligned}
v_1 &= \sqrt{\mathcal{E}_1}\cdot v_1^o + \sqrt{\mathcal{E}_{1,2}}\cdot v_2 \\
v_3 &= \sqrt{\mathcal{E}_3}\cdot v_3^o + \sqrt{\mathcal{E}_{3,2}}\cdot v_2
\end{aligned}
$$

```
GU=   1.0000   -4.2191
     -0.0000    1.0000

MSWMFU=
 -13.7657    0.0000
  -0.0000  191.7362
```

See Ex 2.8.8

$$
\boldsymbol{x} = \underbrace{\begin{bmatrix} -0.8067 & -1.3902 \\ 0.2306 & -0.9370 \\ 1.2679 & -0.5559 \end{bmatrix}}_{A} \cdot \begin{bmatrix} \sqrt{\mathcal{E}_1} & \sqrt{\mathcal{E}_{12}} & 0 \\ 0 & \sqrt{\mathcal{E}_{23}} & \sqrt{\mathcal{E}_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
$$

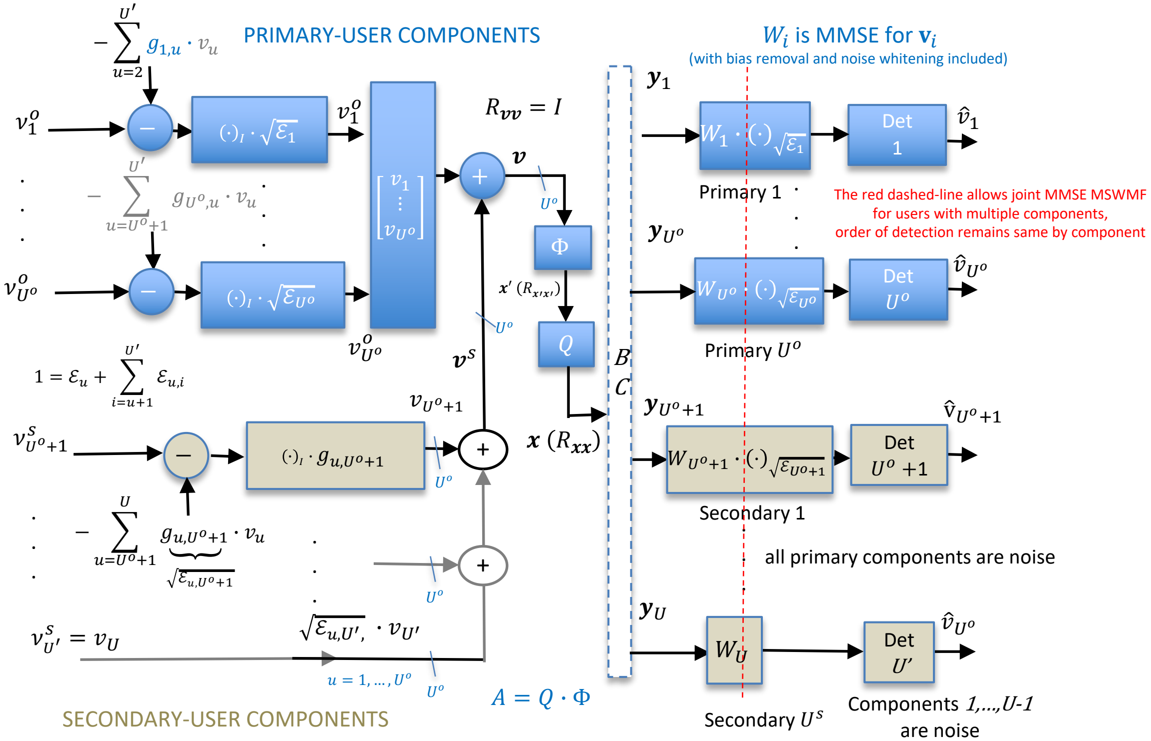
Try:
$$
\mathcal{E}_1 = 1 \;\; and \;\; \mathcal{E}_{12} = 0
$$
$$
\mathcal{E}_3 = \frac{2}{6} \;\; and \;\; \mathcal{E}_{32} = \frac{4}{6}
$$



v_1
v_3
0
0
$\sqrt{\frac{4}{6}}\cdot v_3$
-
$(\blacksquare)_1$
4.22
$I - G_{unb}$
v_1
v_3
$\sqrt{\mathcal{E}_1} = 1$
$\sqrt{\mathcal{E}_3} = \sqrt{\frac{2}{6}}$
$\sqrt{\mathcal{E}_{1,2}} = 0$
$\sqrt{\mathcal{E}_{3,2}} = \sqrt{\frac{4}{6}}$
unit Energy/dim
$\begin{bmatrix} -.807 & -1.39 \\ .231 & -.937 \\ 1.27 & -.556 \end{bmatrix}$
A
$\boldsymbol{x}$
B
C
$W_{unb,1}$=-.0052
y_3
y_1
$W_{unb,3}$=.0726
$(\blacksquare)_1$
$(\blacksquare)_{1/3}$
DEC 1
DEC 3
DEC 2
y_2
$\hat{m}_1$
$\hat{m}_2$
$\hat{m}_3$
v_2

All Switches move once in direction of arrow simultaneously

Gaussian Vector BC System Diagram



PRIMARY-USER COMPONENTS
$-\sum_{u=2}^{U'} g_{1,u} \cdot v_u$
v_1^o
$(\cdot)_I \cdot \sqrt{\mathcal{E}_1}$
v_1^o
$-\sum_{u=U^o+1}^{U'} g_{U^o,u} \cdot v_u$
$v_{U^o}^o$
$(\cdot)_I \cdot \sqrt{\mathcal{E}_{U^o}}$
$v_{U^o}^o$
$\begin{bmatrix} v_1 \\ \vdots \\ v_{U^o} \end{bmatrix}$
$R_{vv} = I$
$\boldsymbol{v}$
U^o
Φ
$\boldsymbol{x}'\ (R_{x'x'})$
U^o
Q
$\boldsymbol{v}^s$
$1 = \mathcal{E}_u + \sum_{i=u+1}^{U'} \mathcal{E}_{u,i}$
v_{U^o+1}
$v_{U^o+1}^s$
$(\cdot)_I \cdot g_{u,U^o+1}$
U^o
$\boldsymbol{x}\ (R_{\boldsymbol{xx}})$
$-\sum_{u=U^o+1}^{U} \underbrace{g_{u,U^o+1}}_{\sqrt{\mathcal{E}_{u,U^o+1}}} \cdot v_u$
U^o
$\sqrt{\mathcal{E}_{u,U',}} \cdot v_{U'}$
$v_{U'}^s = v_U$
$u = 1, \ldots, U^o$
U^o
$A = Q \cdot \Phi$
SECONDARY-USER COMPONENTS
B
C
W_i is MMSE for $\mathbf{v}_i$
(with bias removal and noise whitening included)
$\boldsymbol{y}_1$
$W_1 \cdot (\cdot)\sqrt{\mathcal{E}_1}$
Det
1
$\hat{v}_1$
Primary 1
The red dashed-line allows joint MMSE MSWMF
for users with multiple components,
order of detection remains same by component
$\boldsymbol{y}_{U^o}$
$W_{U^o} \cdot (\cdot)\sqrt{\mathcal{E}_{U^o}}$
Det
U^o
$\hat{v}_{U^o}$
Primary U^o
$\boldsymbol{y}_{U^o+1}$
$W_{U^o+1} \cdot (\cdot)\sqrt{\mathcal{E}_{U^o+1}}$
Det
U^o +1
$\hat{v}_{U^o+1}$
Secondary 1
all primary components are noise
$\boldsymbol{y}_U$
W_U
Det
U'
$\hat{v}_{U^o}$
Secondary U^s
Components *1,…,U-1*
are noise

- This design is for any $R_{\boldsymbol{xx}}$, but the square root $Q \cdot \Phi$ is very special and unique; this design is for the R_{wcn}, no matter the real correlation between receiver noises; U' is number of user components.

STANFORD

End Lecture 10