High-Rate Recursive Convolutional Codes for Concatenated Channel Codes

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***Abstract*—This letter presents the results of the search for optimum punctured recursive convolutional codes (RCCs) of rate $k/k+1$, for $k = 2, \ldots, 8$, suitable for concatenated channel codes whose constituent encoders are recursive, systematic convolutional codes. The mother codes that are punctured are rate-1/2 RCCs proposed for use in parallel and/or serial concatenation schemes. Extensive tables of systematic and nonsystematic puncturing patterns, optimized relative to various objective functions suitable for concatenated channel codes, are presented for several mother codes.**

***Index Terms*—Convolutional codes, parallel concatenated convolutional codes (PCCCs), punctured, recursive convolutional codes (RCCs), serial concatenated convolutional codes (SCCCs), turbo codes, universal mobile telecommunications systems (UMTS) code.**

I. Introduction

FOR applications requiring high spectral efficiency, there is often a need for high-rate codes that satisfy the system requirements in terms of the required bit-error rate (BER) or frame-error rate (FER) at a target signal-to-noise ratio (SNR). To this end, high-rate punctured convolutional codes (CCs) or a suitable concatenation of such codes are among the most commonly used for forward error correction (FEC). Puncturing, introduced in [1], is the most widely used technique to obtain high-rate CCs, since the trellis complexity of the overall code is the same as the lower rate mother code whose output is punctured.

It is known that for soft-decision Viterbi decoding, the BER of a convolutional code of rate $R_c = k/n$ with binary phase-shift keying (BPSK) or quaternary phase-shift keying (QPSK) modulation in additive white Gaussian noise (AWGN), can be well upper bounded by the following expression:

$$P_b \le \frac{1}{k} \sum_{d=d_{\text{free}}}^{\infty} w_d Q\left(\sqrt{2\frac{E_b}{N_o} R_c d}\right) \qquad (1)$$

in which d_{free} is the minimum nonzero Hamming distance of the CC, w_d is the cumulative Hamming weight associated with all the paths that diverge from the correct path in the trellis of the code, and re-emerge with it later and are at Hamming distance d from the correct path, and finally $Q(\cdot)$ is the Gaussian integral function, defined as $Q(t_o) = (1/\sqrt{2\pi}) \cdot \int_{t_o}^{\infty} e^{-(t^2/2)} dt$. Note that (1) is valid for any linear code, provided that the summation is upper-limited to the block size of the block code.

TABLE I
RATE-1/2 SYSTEMATIC RECURSIVE CONSTITUENT ENCODERS USED IN THE DESIGN OF HIGH-RATE PUNCTURED ENCODERS

$R = \frac{1}{2}$	$G(D) = \left[1, \frac{g_1}{g_o}\right]$	Distance Spectra–d_i, m_i, w_i	$d_2,\ d_3$
$\nu = 3$	$G(D) = \left[1, \frac{15}{13}\right]$	$6,2,6-8,10,40-10,49,245-$ $12,241,1446-14,1185,8295$	$d_2 = 8$ $d_3 = 6$
$\nu = 4$	$G(D) = \left[1, \frac{33}{31}\right]$	$7,2,8-8,4,16-9,6,26-$ $10,15,76-11,37,201$	$d_2 = 12$ $d_3 = 7$
$\nu = 4$	$G(D) = \left[1, \frac{21}{37}\right]$	$6,1,2-7,1,5-8,3,10-$ $9,5,25-10,12,56$	$d_2 = 6$ $d_3 = \infty$
$\nu = 4$	$G(D) = \left[1, \frac{27}{31}\right]$	$7,2,8-8,3,12-9,4,16-$ $10,16,84-11,37,213$	$d_2 = 12$ $d_3 = 7$
$\nu = 4$	$G(D) = \left[1, \frac{37}{23}\right]$	$6,1,4-8,6,23-10,34,171-$ $12,174,1055-14,930,6570$	$d_2 = 12$ $d_3 = 8$
$\nu = 4$	$G(D) = \left[1, \frac{33}{23}\right]$	$7,2,8-8,4,16-9,6,26-$ $10,15,76-11,37,201$	$d_2 = 12$ $d_3 = 7$

A classic approach to the design of good punctured codes consists of finding the puncturing pattern (PP) that yields a code whose distance spectrum has the property of having the maximum minimum distance d_{free}. A better approach is to obtain the distance spectra of the punctured codes and to select the one which minimizes the BER upper bound based on the first few terms of the distance spectra.

In this letter, the emphasis is on the use of punctured CC in serially concatenated convolutional codes (SCCCs) [2] and parallel concatenated convolutional codes (PCCCs) [3]. Because of the inherent difficulty in finding good PPs for concatenated channel codes, usually it is preferable to search for the optimal punctured constituent encoders of the concatenated codes satisfying some specific requirements. Several authors have already considered the problem of obtaining good PPs for PCCCs [4]–[7] and SCCCs [8]–[11], while many others have addressed the code-search problem for optimum punctured nonrecursive convolutional codes. There is ample literature in this area [12]–[16]. This letter presents the results of our exhaustive search for good punctured, rate-$k/k+1$, *recursive* convolutional codes (RCCs) to be used in the construction of PCCCs and SCCCs.

II. Code-Search Technique

Mother codes selected for puncturing in this letter are the best recursive rate-1/2 CCs with 4, 8, 16, and 32 states proposed in the literature for the construction of both PCCCs and SCCCs. Matrix generators of the considered codes are shown in the extensive PP tables presented in the letter. The first few terms of the distance spectra of the rate-1/2 recursive CCs chosen for puncturing are shown in Tables I and II. The tables also list the effective distance d_2, that is the minimum Hamming weight of

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TABLE II
OPTIMIZED RATE-1/2 SYSTEMATIC RECURSIVE CONSTITUENT ENCODERS FOR CONCATENATED CHANNEL CODES

$R=\frac{1}{2}$	$G(D)=\left[1,\frac{g_1}{g_o}\right]$	Distance Spectra–d_i, m_i, w_i	$d_2,\ d_3$
$\nu=2,\ d_2$	$G(D)=\left[1,\frac{5}{7}\right]$	$5,1,3-6,2,6-7,4,14-$ $8,8,32-9,16,72$	$d_2=6$ $d_3=5$
$\nu=2,\ snr$	$G(D)=\left[1,\frac{7}{5}\right]$	$5,1,2-6,2,6-7,4,14-$ $8,8,32-9,16,72$	$d_2=5$ $d_3=\infty$
$\nu=3,\ d_2$	$G(D)=\left[1,\frac{17}{13}\right]$	$6,1,4-7,3,9-8,5,20-$ $9,11,51-10,25,124$	$d_2=8$ $d_3=7$
$\nu=3,\ snr$	$G(D)=\left[1,\frac{15}{17}\right]$	$6,1,2-7,3,12-8,5,20-$ $9,11,48-10,25,126$	$d_2=6$ $d_3=\infty$
$\nu=4,\ d_2$	$G(D)=\left[1,\frac{35}{23}\right]$	$7,2,8-8,3,12-9,4,16-$ $10,16,84-11,37,213$	$d_2=12$ $d_3=7$
$\nu=4,\ snr$	$G(D)=\left[1,\frac{23}{35}\right]$	$7,2,6-8,3,12-9,4,20-$ $10,16,76-11,37,194$	$d_2=7$ $d_3=\infty$
$\nu=5,\ d_3$	$G(D)=\left[1,\frac{71}{53}\right]$	$8,3,12-10,16,84-$ $12,68,406-14,860,6516-$ $16,3812,30620$	$d_2=12$ $d_3=\infty$
$\nu=5,$ $snr,\ d_2$	$G(D)=\left[1,\frac{67}{51}\right]$	$8,2,7-10,20,110-$ $12,68,398-14,469,3364-$ $16,2560,20864$	$d_2=20$ $d_3=8$

the codewords generated by weight-2 input patterns, and the distance d_3 generated by weight-3 input patterns of the considered codes. To the best of our knowledge, all the mother CCs we have used are among the best RCCs obtained by using primitive feedback polynomials for the code generators [3], [17].

For clarity of presentation, in the following, we shall distinguish between the design of mother encoders and PPs for PCCCs and SCCCs.

A. Design of High-Rate Constituent Encoders for PCCCs

In this section, the focus is on the design of good high-rate constituent encoders for PCCCs.

In addition to the mother codes presented in Table I, we have conducted a search for good constituent recursive rate-1/2 convolutional encoders to be used in PCCCs. Note that the design of good mother encoders and PPs for PCCCs follows the same general rules.

In connection with the PCCCs, it is known that the constituent encoders must be recursive and systematic in order for the interleaver to yield a gain [3]. Furthermore, in PCCCs, the dominant patterns yielding the lowest terms of the distance spectra are due to input patterns with weight-2, especially for large interleaver sizes. In fact, it is known that the performance of the PCCC with large interleavers [17] for moderate-to-high SNR can be expressed as

$$P_b \approx \frac{1}{N} Q\left(\sqrt{2\frac{E_b}{N_o}R_c d_2}\right) \tag{2}$$

where R_c is the code rate of the PCCC, and N is the interleaver length. For this reason, a good criteria for obtaining both good mother encoders and PPs in a PCCC consists of maximizing the effective distance d_2.

We note that the maximum effective distance d_2 achievable with a recursive systematic rate-1/2 encoder with generator matrix $G(D) = [1, (g_1(D)/g_o(D))]$ can be obtained from [17], $d_2 \leq 4 + 2^{\nu-1}$. In particular, equality is achieved when the denominator polynomial $g_o(D)$ is primitive and under two additional conditions which require that $g_o(d) \neq g_1(D)$ and that $\deg[g_i(D)] \leq \nu$ for $i = 0, 1$. The search for good mother encoders having highest d_2 has been conducted by considering the above conditions on the polynomials associated with $G(D)$. We have used these codes as mother codes by following the generally accepted rule that "good mother codes" lead to "good punctured codes." Indeed, practical documented results show that PPs with maximum possible d_2 are derived from mother encoders having maximum d_2.

In connection with the use of punctured encoders in a PCCC, the possible puncturing strategies can be different. Due to the complexity of finding good PPs for the overall PCCC codewords [6], a viable solution is to use punctured constituent encoders. For example, in reference to the general scheme of a PCCC, a rate-$k/k+1$ recursive systematic encoder can be used as the upper encoder of the PCCC, whereas the lower encoder can be punctured so that the systematic bits and some of its parity bits are completely eliminated, in order to achieve the desired rate for the overall PCCC.

Considerations above motivated us to design both systematic mother encoders and PPs by using, as objective function, the maximization of the effective distance d_2. In a second phase, among the encoders yielding the same d_2 (if several), we chose the one requiring the minimum SNR for achieving the target BER $= 10^{-6}$, and then the one with maximum d_3. As a cost function for optimization in connection with the minimization of SNR, we have used the inverse of the BER upper bound, as expressed in (1), using the first few terms of the distance spectra of the codes. In the following, we shall identify this design criteria for PPs as criterion C_1.

The results of the search for good mother encoders are shown in Table II labeled with the acronym d_2 for constituent encoders with memory ν equal to 2, 3, 4, and 5 (the column heading shows the number of states). The second column shows the generator matrices of the optimal encoders, the third column lists the code distance spectra up to the fifth term (each triplet represents the Hamming weight of the codewords d_i, the multiplicity m_i of all the input patterns with overall weight w_i leading to codewords with weight d_i and the input weight w_i), and the last column shows the effective distance d_2 and the distance generated by weight-3 sequences denoted d_3 (the entry $d_3 = \infty$ is used to signify the fact that there are no weight-3 input patterns leading to low-distance codewords).

B. Design of High-Rate Constituent Encoders for SCCCs

In this section, the emphasis is on the design of good high-rate constituent encoders for SCCCs.

For serial concatenation of CCs, the optimization criteria adopted is somewhat different, depending on whether the punctured codes have to be used as outer or as inner codes. As discussed in [2], the asymptotic BER of an SCCC for very large interleaver sizes N is given by

$$P_b < C_e N^{-(d_f^o/2)} Q\left(\sqrt{d_f^o d_2^i R_s \frac{E_b}{N_o}}\right) \tag{3}$$

for even values of d_f^o, and

$$P_b \leq C_o N^{-(d_f^o+1/2)} Q\left(\sqrt{\left[\left(d_f^o - 3\right) d_2^i + 2d_{\min}^{(3)}\right] R_s \frac{E_b}{N_o}}\right) \tag{4}$$

for odd values of d_f^o. In both equations, the terms C_e and C_o do not depend on the interleaver length N, d_f^o is the free distance of the outer code, d_2^i is the effective distance of the inner code, R_s is the rate of the SCCC, and $d_{\min}^{(3)}$ is the minimum weight of the inner code codewords generated by weight-3 input sequences.

Equations (3) and (4) can be used to deduce some useful design criteria for constituent encoders of SCCCs. First of all, we note that the inner encoder in an SCCC must be a recursive convolutional encoder, no matter if it is systematic or not, while the outer encoder should have maximum free distance. It is not necessary for the outer encoder to be either recursive or systematic, as is evident from the asymptotic interleaving gain given by $N^{-(d_f^o+1/2)}$ for odd values of d_f^o, and $N^{-(d_f^o/2)}$ for even values of d_f^o. In particular, compatible with the desired rate R_s of the SCCC, it is better to choose outer encoders with odd values of d_f^o. In summary, for the outer encoder in an SCCC, classical design methodologies for design of optimal CCs [12]–[16] are adequate. In the moderate-to-high SNRs where an interleaver gain is observed, the outer encoder should simply possess a good distance spectrum, i.e., highest minimum distance and low weight of the associated error-sequence combination not just for the minimum-distance term, but also for other low-distance terms of the distance spectrum. Let us focus on the design criteria for inner encoders in an SCCC. Equations (3) and (4) suggest that for outer encoders with even values of d_f^o, a suitable criteria to design good inner encoders is to maximize their effective distance d_2. In the case where the outer encoder has an odd value of d_f^o, it is also better to choose inner encoders with the greatest possible d_3 (recall that d_3 is the smallest weight of the inner encoder codeword generated by weight-3 input sequences). In this case, an inner encoder with the feedback polynomial containing the factor $(1+D)$ can be chosen, thus avoiding altogether terminating error patterns, and hence, low-weight codewords due to input sequences of weight-3. Once the outer and inner encoders are chosen in accordance to the previous criteria, the interleaver design should focus on optimizing the matching between the outer and the inner encoders in such a way that for each Hamming weight d_i^o of the outer codewords, the Hamming weight $d_{d_i^o}^i$ of the inner codewords are maximized (here, $d_{d_i^o}^i$ is the minimum Hamming weight of the inner codewords generated by inner input patterns with the same weight d_i^o of the outer codewords). This maximization is then applied to increasing weights of the outer codewords.

In addition to the mother codes presented in Table I, we have conducted a search for good constituent recursive rate-1/2 convolutional outer encoders in SCCCs, by using as an objective function the minimization of SNR at a target BER $= 10^{-6}$ [16]. In order to resolve potential ties, in a second phase between all the encoders yielding similar performance, we choose the encoder with the best d_2 and, subsequently, the best d_3. The results of this search are shown in Table II labeled with the acronym SNR for constituent encoders with memory ν equal to 2, 3, 4, and 5.

The results in Table II obtained for the 32-state recursive encoders are slightly different. During our search, we found an encoder satisfying both objective functions mentioned above (listed in the last line of Table II for $\nu = 5$ codes). For this reason, in the upper line of the same row, we show the best encoder having maximum d_3 while simultaneously satisfying the minimum SNR requirement and achieving maximum d_2. This encoder can be useful, for instance, as the inner encoder of an SCCC when the outer encoder is punctured so that its minimum distance is equal to three. In this case, because of the absence of inner encoder codewords generated by weight-3 input patterns, the overall SCCC yields better performance.

Following the general guidelines above, let us outline the design criteria adopted in this letter for obtaining good punctured encoders for SCCCs. As far as the inner encoder of the SCCC is concerned, the design rule consists of searching for the PP leading to the maximum possible effective distance d_2 for a given rate-$k/k+1$. Between all PPs having the same maximum d_2, we choose the one yielding the minimum SNR requirements for achieving the target BER $= 10^{-6}$, and finally, between all the PPs yielding the same SNR requirement, we chose the one with maximum d_3. We shall identify this design criteria as criterion C_2.

In connection with the design of punctured encoders to be used as outer codes in an SCCC, we conducted a search for the best PPs yielding the minimum SNR requirements for achieving the target BER $= 10^{-6}$. This target is such that the optimization acts to maximize the minimum distance of the punctured encoder and minimize the overall weight associated with the input pattern yielding the minimum distance, followed by maximization of the successive low-distance terms and minimization of their corresponding input weights for the first four minimum-distance terms used in the formulation of the cost function. Between various PPs satisfying the requirement with the same SNR, we chose the one having first of all the maximum d_2, and then the maximum d_3. We shall identify this design criteria as criterion C_3.

III. Code-Search Results

The results of our search for the best PPs are presented in Tables III–VIII.

In particular, for any encoder with memory ν having 2^ν states (the column heading shows the number of states) in any given column, Tables III–V show the best PPs resulting in punctured encoders of rate-$k/k+1$ under criteria C_1, C_2, and C_3.

For any given memory size ν and code rate, Tables VI–VIII show the best PPs under criteria C_1, C_2, and C_3, respectively. Since there was no single punctured encoder of a given memory outperforming all other encoders with the same memory over all code rates examined, we developed three global cost metrics as follows. For codes obtained using criterion C_3, for any given code rate, we evaluated the loss in SNR between the performance of each encoder and the one achieving the desired target with the minimum SNR. Then, we summed the SNR losses of the punctured encoders over all the code rates. This was repeated for other memory sizes independently. These PPs are shown in Table VIII. In relation to the criteria C_1 and C_2, for any given

TABLE III
BEST PPs FOR VARIOUS CODE RATES UNDER CRITERION C_1

k	4-S	8-S	16-S	32-S
2	$[1,5/7]$ p.13, (3,1,3) $d_2=4$, $d_3=3$	$[1,15/13]$ p.13, (4,3,10) $d_2=5$, $d_3=4$	$[1,37/23]$ p.13, (4,2,6) $d_2=7$, $d_3=4$	$[1,67/51]$ p.13, (5,2,7) $d_2=9$, $d_3=5$
3	$[1,5/7]$ p.56, (3,4,10) $d_2=3$, $d_3=3$	$[1,15/13]$ p.53, (3,2,5) $d_2=3$, $d_3=3$	$[1,37/23]$ p.53, (3,1,3) $d_2=4$, $d_3=3$	$[1,67/51]$ p.53, (4,2,7) $d_2=7$, $d_3=4$
4	$[1,5/7]$ p.253, (2,1,2) $d_2=2$, $d_3=3$	$[1,15/13]$ p.253, (3,9,24) $d_2=3$, $d_3=3$	$[1,37/23]$ p.253, (3,3,9) $d_2=4$, $d_3=3$	$[1,67/51]$ p.253, (3,1,3) $d_2=5$, $d_3=3$
5	$[1,5/7]$ p.1253, (2,2,4) $d_2=2$, $d_3=3$	$[1,17/13]$ p.1253, (3,15,40) $d_2=3$, $d_3=3$	$[1,27/31]$ p.1272, (3,2,6) $d_2=4$, $d_3=3$	$[1,71/53]$ p.1272, (4,108,406) $d_2=4$
6	$[1,5/7]$ p.5352, (2,22,44) $d_2=2$, $d_3=3$	$[1,17/13]$ p.5253, (2,1,2) $d_2=2$, $d_3=3$	$[1,27/31]$ p.5253, (3,12,33) $d_2=3$, $d_3=3$	$[1,71/53]$ p.5253, (3,3,6) $d_2=3$
7	$[1,5/7]$ p.25253, (2,7,14) $d_2=2$, $d_3=3$	$[1,15/17]$ p.25253, (2,7,14) $d_2=2$	$[1,33/23]$ p.25253, (2,1,2) $d_2=2$, $d_3=3$	$[1,67/51]$ p.25253, (2,1,2) $d_2=3$, $d_3=3$
8	$[1,5/7]$ p.125253, (2,9,18) $d_2=2$, $d_3=3$	$[1,15/13]$ p.125253, (2,4,8) $d_2=2$, $d_3=3$	$[1,37/23]$ p.125253, (2,1,2) $d_2=2$, $d_3=3$	$[1,67/51]$ p.125253, (3,17,49) $d_2=3$, $d_3=3$

TABLE IV
BEST PPs FOR VARIOUS CODE RATES UNDER CRITERION C_2

k	4-S	8-S	16-S	32-S
2	$[1,5/7]$ p.7, (3,1,3) $d_2=5$, $d_3=3$	$[1,15/13]$ p.7, (4,3,10) $d_2=7$, $d_3=4$	$[1,27/31]$ p.7, (4,1,5) $d_2=11$, $d_3=5$	$[1,67/51]$ p.7, (5,2,7) $d_2=19$, $d_3=6$
3	$[1,5/7]$ p.27, (2,1,4) $d_2=4$, $d_3=3$	$[1,15/13]$ p.27, (3,2,9) $d_2=6$, $d_3=3$	$[1,27/31]$ p.27, (3,1,5) $d_2=10$, $d_3=4$	$[1,67/51]$ p.65, (4,7,51) $d_2=18$, $d_3=5$
4	$[1,5/7]$ p.67, (2,4,13) $d_2=4$, $d_3=2$	$[1,15/13]$ p.127, (3,9,52) $d_2=6$, $d_3=3$	$[1,27/31]$ p.351, (4,16,176) $d_2=10$, $d_3=4$	$[1,67/51]$ p.325, (4,22,191) $d_2=18$, $d_3=5$
5	$[1,5/7]$ p.527, (2,6,26) $d_2=4$, $d_3=2$	$[1,15/13]$ p.527, (2,1,6) $d_2=6$, $d_3=3$	$[1,27/31]$ p.635, (3,8,43) $d_2=10$, $d_3=4$	$[1,67/51]$ p.1525, (3,3,21) $d_2=18$, $d_3=5$
6	$[1,5/7]$ p.3525, (2,84,2693) $d_2=4$, $d_3=2$	$[1,17/13]$ p.2527, (2,4,20) $d_2=6$, $d_3=4$	$[1,37/23]$ p.6525, (2,4,22) $d_2=10$, $d_3=5$	$[1,67/51]$ p.6525, (3,4,34) $d_2=18$, $d_3=5$
7	$[1,5/7]$ p.12527, (2,15,74) $d_2=4$, $d_3=2$	$[1,17/13]$ p.12527, (2,7,42) $d_2=6$, $d_3=4$	$[1,33/23]$ p.32525, (2,6,51) $d_2=10$, $d_3=4$	$[1,67/51]$ p.32525, (3,14,135) $d_2=18$, $d_3=5$
8	$[1,5/7]$ p.72525, (2,153,5216) $d_2=4$, $d_3=2$	$[1,15/13]$ p.52527, (2,4,32) $d_2=6$, $d_3=3$	$[1,33/23]$ p.72525, (2,6,42) $d_2=10$, $d_3=4$	$[1,67/51]$ p.152525, (3,23,299) $d_2=18$, $d_3=5$

TABLE V
BEST PPs FOR VARIOUS CODE RATES UNDER CRITERION C_3

k	4-S	8-S	16-S	32-S
2	$[1,7/5]$ p.15, (3,1,2) $d_2=3$	$[1,15/13]$ p.13, (4,3,10) $d_2=5$, $d_3=4$	$[1,33/31]$ p.13, (5,7,25) $d_2=6$, $d_3=5$	$[1,71/53]$ p.13, (6,15,60) $d_2=6$
3	$[1,5/7]$ p.56, (3,4,10) $d_2=3$, $d_3=3$	$[1,15/17]$ p.33, (4,29,126) $d_2=4$	$[1,23/35]$ p.17, (4,29,150) $d_2=4$	$[1,71/53]$ p.36, (4,1,4) $d_2=8$
4	$[1,5/7]$ p.253, (2,1,2) $d_2=2$, $d_3=3$	$[1,15/17]$ p.136, (3,5,16) $d_2=3$	$[1,27/31]$ p.351, (4,16,176) $d_2=10$, $d_3=4$	$[1,71/53]$ p.133, (4,28,192) $d_2=8$
5	$[1,5/7]$ p.1253, (2,2,4) $d_2=2$, $d_3=3$	$[1,17/13]$ p.1253, (3,15,40) $d_2=3$, $d_3=3$	$[1,33/31]$ p.653, (4,98,436) $d_2=4$, $d_3=4$	$[1,71/53]$ p.1272, (4,108,406) $d_2=4$
6	$[1,7/5]$ p.3247, (2,4,8) $d_2=2$	$[1,17/13]$ p.5253, (2,1,2) $d_2=2$, $d_3=3$	$[1,27/31]$ p.3352, (3,7,24) $d_2=4$, $d_3=3$	$[1,71/53]$ p.5253, (3,3,6) $d_2=3$
7	$[1,7/5]$ p.15247, (2,6,12) $d_2=2$	$[1,15/17]$ p.15652, (2,2,4) $d_2=2$	$[1,23/35]$ p.13632, (3,13,52) $d_2=3$	$[1,71/53]$ p.13172, (3,4,18) $d_2=5$
8	$[1,7/5]$ p.65247, (2,9,18) $d_2=2$	$[1,15/17]$ p.65256, (2,3,6) $d_2=2$	$[1,33/31]$ p.123255, (3,21,75) $d_2=4$, $d_3=3$	$[1,67/51]$ p.124672, (3,11,36) $d_2=5$, $d_3=3$

number of states of the considered mother encoders, we selected the encoders having the maximum d_2 over the largest possible set of code rates. Then, between these encoders, we chose the one yielding the minimum SNR losses evaluated as above for criterion C_3. The results are shown in Tables VI and VII.

TABLE VI
OPTIMAL PUNCTURED ENCODERS OVER ALL CODE RATES UNDER CRITERION C_1

k	4-S	8-S	16-S	32-S
2	$[1,5/7]$ p.13, (3,1,3) $d_2=4$ $d_3=3$	$[1,15/13]$ p.13, (4,3,10) $d_2=5$, $d_3=4$	$[1,37/23]$ p.13, (4,2,6) $d_2=7$, $d_3=4$	$[1,67/51]$ p.13, (5,2,7) $d_2=9$, $d_3=5$
3	$[1,5/7]$ p.56, (3,4,10) $d_2=3$, $d_3=3$	$[1,15/13]$ p.53, (3,2,5) $d_2=3$, $d_3=3$	$[1,37/23]$ p.53, (3,1,3) $d_2=4$, $d_3=3$	$[1,67/51]$ p.53, (4,2,7) $d_2=7$, $d_3=4$
4	$[1,5/7]$ p.253, (2,1,2) $d_2=2$, $d_3=3$	$[1,15/13]$ p.253, (3,9,24) $d_2=3$, $d_3=3$	$[1,37/23]$ p.253, (3,3,9) $d_2=4$, $d_3=3$	$[1,67/51]$ p.253, (3,1,3) $d_2=5$, $d_3=3$
5	$[1,5/7]$ p.1253, (2,2,4) $d_2=2$, $d_3=3$	$[1,15/13]$ p.1253, (2,1,2) $d_2=3$, $d_3=3$	$[1,37/23]$ p.1352, (4,407,1491) $d_2=4$, $d_3=4$	$[1,67/51]$ p.1253, (3,4,12) $d_2=4$, $d_3=3$
6	$[1,5/7]$ p.5352, (2,22,44) $d_2=2$, $d_3=3$	$[1,15/13]$ p.5253, (2,3,6) $d_2=2$, $d_3=3$	$[1,37/23]$ p.5253, (2,1,2) $d_2=2$, $d_3=3$	$[1,67/51]$ p.5253, (3,10,28) $d_2=3$, $d_3=3$
7	$[1,5/7]$ p.25253, (2,7,14) $d_2=2$, $d_3=3$	$[1,15/13]$ p.25272, (2,37,74) $d_2=2$, $d_3=3$	$[1,37/23]$ p.25253, (2,2,4) $d_2=2$, $d_3=3$	$[1,67/51]$ p.25253, (2,1,2) $d_2=3$, $d_3=3$
8	$[1,5/7]$ p.125253, (2,9,18) $d_2=2$, $d_3=3$	$[1,15/13]$ p.125253, (2,4,8) $d_2=2$, $d_3=3$	$[1,37/23]$ p.125253, (2,1,2) $d_2=2$, $d_3=3$	$[1,67/51]$ p.125253, (3,17,49) $d_2=3$, $d_3=3$

TABLE VII
OPTIMAL PUNCTURED ENCODERS OVER ALL CODE RATES UNDER CRITERION C_2

k	4-S	8-S	16-S	32-S
2	$[1,5/7]$ p.7, (3,1,3) $d_2=5$ $d_3=3$	$[1,15/13]$ p.7, (4,3,10) $d_2=7$, $d_3=4$	$[1,27/31]$ p.7, (4,1,5) $d_2=11$, $d_3=5$	$[1,67/51]$ p.7, (5,2,10) $d_2=19$, $d_3=6$
3	$[1,5/7]$ p.27, (2,1,4) $d_2=4$, $d_3=3$	$[1,15/13]$ p.27, (3,2,9) $d_2=6$, $d_3=3$	$[1,27/31]$ p.27, (3,1,5) $d_2=10$, $d_3=4$	$[1,67/51]$ p.65, (4,7,51) $d_2=18$, $d_3=5$
4	$[1,5/7]$ p.67, (2,4,13) $d_2=4$, $d_3=2$	$[1,15/13]$ p.127, (3,9,52) $d_2=6$, $d_3=3$	$[1,27/31]$ p.351, (4,16,176) $d_2=10$, $d_3=4$	$[1,67/51]$ p.325, (4,22,191) $d_2=18$, $d_3=5$
5	$[1,5/7]$ p.527, (2,6,26) $d_2=4$, $d_3=2$	$[1,15/13]$ p.527, (2,1,6) $d_2=6$, $d_3=3$	$[1,27/31]$ p.635, (3,8,43) $d_2=10$, $d_3=4$	$[1,67/51]$ p.1525, (3,3,21) $d_2=18$, $d_3=5$
6	$[1,5/7]$ p.3525, (2,84,2693) $d_2=4$, $d_3=2$	$[1,15/13]$ p.2527, (2,3,22) $d_2=6$, $d_3=3$	$[1,27/31]$ p.6525, (2,6,42) $d_2=10$, $d_3=4$	$[1,67/51]$ p.6525, (3,4,34) $d_2=18$, $d_3=5$
7	$[1,5/7]$ p.12527, (2,15,74) $d_2=4$, $d_3=2$	$[1,15/13]$ p.13525, (2,37,1518) $d_2=6$, $d_3=3$	$[1,27/31]$ p.12565, (2,56,1765) $d_2=10$, $d_3=4$	$[1,67/51]$ p.32525, (3,14,135) $d_2=18$, $d_3=5$
8	$[1,5/7]$ p.72525, (2,153,5216) $d_2=4$, $d_3=2$	$[1,15/13]$ p.52527, (2,4,32) $d_2=6$, $d_3=3$	$[1,27/31]$ p.72525, (2,7,50) $d_2=10$, $d_3=4$	$[1,67/51]$ p.152525, (3,23,299) $d_2=18$, $d_3=5$

TABLE VIII
OPTIMAL PUNCTURED ENCODERS OVER ALL CODE RATES UNDER CRITERION C_3

k	4-S	8-S	16-S	32-S
2	$[1,7/5]$ p.15, (3,1,2) $d_2=3$	$[1,15/17]$ p.7, (4,3,12) $d_2=4$	$[1,27/31]$ p.13, (4,1,3) $d_2=6$, $d_3=4$	$[1,71/53]$ p.13, (6,15,60) $d_2=6$
3	$[1,7/5]$ p.33, (3,6,18) $d_2=3$	$[1,15/17]$ p.33, (4,29,126) $d_2=4$	$[1,27/31]$ p.17, (4,29,152) $d_2=6$, $d_3=4$	$[1,71/53]$ p.36, (4,1,4) $d_2=8$
4	$[1,7/5]$ p.147, (2,1,2) $d_2=2$	$[1,15/17]$ p.136, (3,5,16) $d_2=3$	$[1,27/31]$ p.351, (4,16,176) $d_2=10$, $d_3=4$	$[1,71/53]$ p.133, (4,28,192) $d_2=8$
5	$[1,7/5]$ p.647, (2,2,4) $d_2=2$	$[1,15/17]$ p.656, (3,15,50) $d_2=3$	$[1,27/31]$ p.1272, (3,7,6) $d_2=4$, $d_3=3$	$[1,71/53]$ p.1272, (4,108,406) $d_2=4$
6	$[1,7/5]$ p.3247, (2,4,8) $d_2=2$	$[1,15/17]$ p.3256, (2,1,2) $d_2=2$	$[1,27/31]$ p.3352, (3,7,24) $d_2=4$, $d_3=3$	$[1,71/53]$ p.5253, (3,3,6) $d_2=3$
7	$[1,7/5]$ p.15247, (2,6,12) $d_2=2$	$[1,15/17]$ p.15652, (2,2,4) $d_2=2$	$[1,27/31]$ p.13172, (3,13,59) $d_2=5$, $d_3=3$	$[1,71/53]$ p.13172, (3,4,18) $d_2=5$
8	$[1,7/5]$ p.65247, (2,9,18) $d_2=2$	$[1,15/17]$ p.65256, (2,3,6) $d_2=2$	$[1,27/31]$ p.53156, (3,24,129) $d_2=6$, $d_3=3$	$[1,71/53]$ p.64752, (3,11,50) $d_2=4$

The tables are organized as follows. For any given memory size shown in the column with the respective number of states and any rate shown in a given row, we show the polynomial generators of the mother encoder yielding the best PP in the first line, the triplet (d_m, M_m, W_m) identifying the minimum distance d_m, the number of nearest neighbors yielding the minimum distance M_m, and the total weight of these input patterns W_m as the second line, and the effective distance d_2 and the weight-3 distance d_3 of the punctured codes as the third line. Where not specified, d_3 does not exist in the distance spectra of the punctured codes. As an example of how to read the table entries, consider the rate-2/3 puncturing pattern PP-13 related to the mother encoder $G(D) = [1, (5/7)]$ shown in Table III yielding the best PPs under criterion C_1. This PP, represented in octal form, leads to a code whose minimum distance three is due to one input pattern with input weight three. The effective distance d_2 of the code is four, and the weight-3 distance is three.

As noted above, the PPs are represented in octal form. A given PP should be read from right to left by collecting k pairs of systematic-parity bits. As an example, the PP in Table III, which yields a code with rate-2/3 for the 4-state code, should be interpreted as follows: $p = 13_8 = 1011_2 = \langle x_1, y_1, x_2, y_2 \rangle$ (the subscript denotes the base of the numbers). In this case, the PP leaves the encoder systematic and deletes the first parity bit associated with every two input bits.

IV. Conclusions

In this letter, we have presented extensive optimized PP tables for recursive CCs to be used in the design of parallel and serially concatenated CCs. The optimization was conducted using three different objective functions, each one of which is suited to a certain application in connection with the design of PCCCs and SCCCs. We have further conducted exhaustive searches for mother encoders of rate-1/2 to be used for puncturing using two different selection criteria. These encoders, and several other encoders reported in the literature, were then used as mother encoders to which puncturing is applied.

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