Double-rate and Half-rate Transmission over the $0.9\delta(t+\tau)+\delta(t)$ Channel

This example offers an additional perspective to the famous $1+.9D^{-1}$ channel, discussing the construction of the pulse response from the channel impulse response $0.9\delta(t+\tau)+\delta(t)$, as well as looking at equalization when data is transmitted at a symbol period that is different from τ, which represents the delay between the two taps of the channel impulse response.

1. Constructing $p(t)$ from $\varphi(t)$ and $h(t)$

The channel pulse response $p(t)$ can be constructed as:

$$p(t) = \varphi(t) \otimes h(t)$$

in which,

$$h(t) = 0.9\delta(t+\tau)+\delta(t);$$

$$\varphi(t) = \sqrt{T}\,\mathrm{sinc}\left(\frac{t}{T}\right), \quad T = \alpha\tau$$

Therefore,

$$P(\omega) = H(\omega) \bullet \Phi(\omega) = \sqrt{\alpha\tau}(1+0.9e^{j\omega\tau}), \quad \left(|\omega| \le \frac{\pi}{\alpha\tau}\right); \text{ 0 otherwise.}$$

When $\alpha = 1$, this is the familiar $1+.9D^{-1}$ channel. What about the cases where $\alpha \neq 1$?

2. A look at the frequency domain

Using Matlab, we can plot the function $H(\omega)$ in the frequency domain. As shown in Fig. 1, this channel is periodic, with notches at $\pm 0.5MHz$, $\pm 1.5MHz$, $\pm 2.5MHz$...

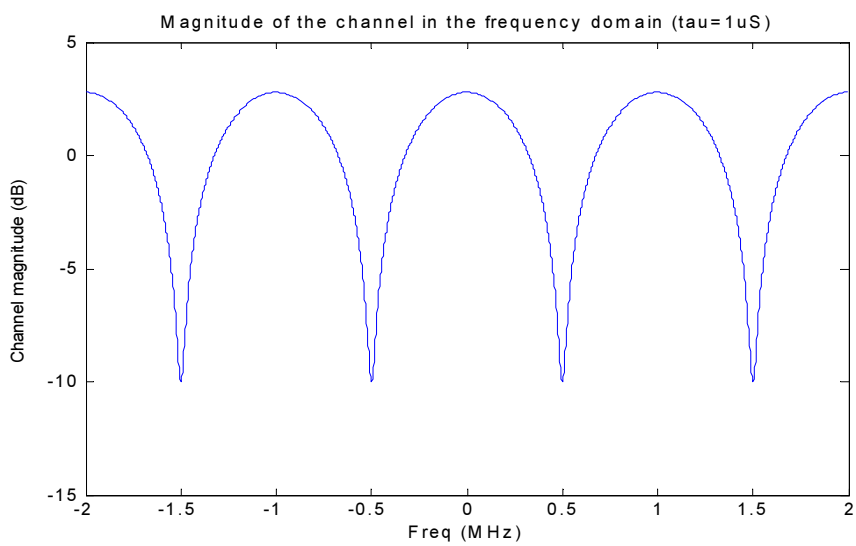

Figure 1: Magnitude of the channel in the frequency domain

We can also plot $P(\omega)$ using Matlab, and compare the spectrum for $\alpha = 0.5, 1, 2$:

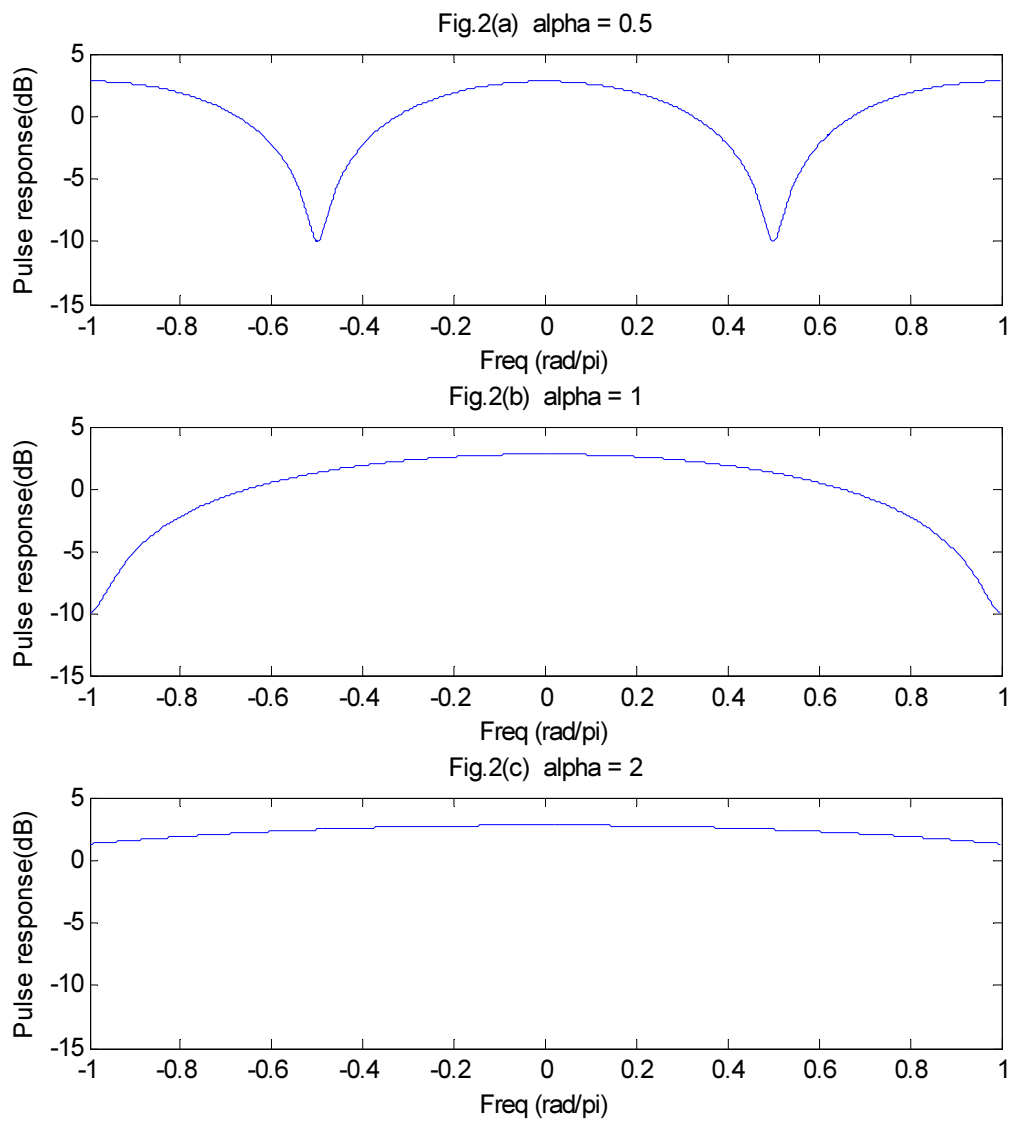

Figure 2: Pulse response for α=0.5, 1 and 2

As we can see in Figure 2, the pulse response shows different characteristics depending on different symbol rate. We are looking at the same channel, but through windows of different sizes ($\varphi(t)$).

3. Performance expectations in terms of SNR_{MFB}

Intuitively, we expect the channel in Figure 2(c) to perform the best among the three, in terms of SNR performance, since it has the flattest spectrum. Between Figure 2(a) and Figure 2(b), we

expect to see similar performance, since these two cases have the same amount of notches on the average considering the periodicity.

We can verify the above observations by deriving the norm of the pulse response, then the SNR_{MFB} of the three cases, given the conditions that for PAM transmission, $\varepsilon_x = 1$ and $\sigma^2 = 0.181$.

$$||p||^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\sqrt{\alpha\tau}(1 + 0.9e^{j\omega\tau})]^2 d\omega$$

$$= \frac{\alpha}{2\pi}\int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}}(1 + 0.9e^{j\Omega})^2 d\Omega = 1.81 + 1.8\frac{\alpha}{\pi}\sin\frac{\pi}{\alpha}$$

$T = \tau$:	$\alpha = 1$	$\|\|p\|\|^2 = 1.81$	$\mathrm{SNR}_{MFB} = 10 = 10dB$
Double-rate transmission:	$\alpha = 0.5$	$\|\|p\|\|^2 = 1.81$	$\mathrm{SNR}_{MFB} = 10 = 10dB$
Half-rate transmission:	$\alpha = 2$	$\|\|p\|\|^2 = 2.96$	$\mathrm{SNR}_{MFB} = 16.35 = 12.14dB$

4. MMSE-DFE design for the double-rate transmission

We can now derive the MMSE-DFE design for $\alpha = 2$, and check the $SNR_{MMSE\text{-}DFE,U}$ that can be achieved:

$$P(D) = 1 + 0.9D^{-2}$$

$$\dot{\bar{Q}}(D) = \frac{1}{||p||^2}\left[P(D)P^*(D^{-*}) + \frac{1}{SNR_{MFB}}\right] = \frac{1}{1.81}(0.9D^{-2} + 1.991 + 0.9D^2)$$

$$= 0.785(1 + 0.633D^2)(1 + 0.633D^{-2})$$

$$W(D) = \frac{0.9469}{1 + 0.633D^{-2}}$$

$$B(D) = G(D) = 1 + 0.633D^2$$

$$SNR_{\mathrm{MMSE\text{-}DFE,U}} = 6.85 = 8.4dB$$

Therefore, the MMSE-DFE for the double-rate transmission achieves the same performance as for the course reader example for the $1+.9D^{-1}$ channel.

5. MMSE-DFE design for the half-rate transmission

Designing the MMSE-DFE for the half-rate transmission is a little more complex. First, since the channel response is now faster than the symbol rate, we would need to sample faster than the sym-

bol rate. Obviously we need to set the over-sampling factor to at least 2. Second, the pulse response is now an infinite series in the time domain. Therefore, we need to truncate it in order to use the Stanford DFE program.

To find the proper length for truncation, we set our target such that the truncated series should at least retain 95% of the original energy ($\|p\|^2 = 2.96$). The finite length pulse response can be obtained by adding up the time-domain series energy and comparing it with our target:

```
%Find out the time response of the over-sampled case
length=70;
t=[-length/2:length/2];
t=t/2;
p_t=sinc(t)+0.9*sinc(t+0.5);
p_energy=0.5*sum(p_t*p_t');
per_cent = p_energy/2.96*100

per_cent =

   99.5134
```

Now, with the pulse response series that is close enough, we can obtain the equalizer design. Intuitively, for a channel with such a relatively flat response, we can get a good performance by just using the MMSE-LE equalizer:

```
% MMSE-LE design for the half-rate transmission, nff = 50, nbb = 0
[SNR1, wt] = dfecolor(2,p_t,70,0,52,1,[2*0.181,zeros(1,2*70-1)]);
SNR1

SNR =

   11.8822
```

This is only 0.26 dB below the SNR_{MFB}!

Now try a MMSE-DFE design:

```
% MMSE-DFE design for the half-rate tranmission
% Oversample by 2
% nff=54, nbb=4
[SNR,wt]=dfecolor(2,p_t,70,4,54,1,[2*0.181,zeros(1,2*70-1)]);
SNR

SNR =

   11.9940
```

With a design of 70 feed-forward taps and 4 feed-back taps, we get an unbiased SNR of 11.994, which is very close to the SNR_{MFB} of 12.14dB. With a more reasonable design of 20 feed-forward taps and just 1 feed-back tap, we get an SNR of 11.892dB, which is still pretty close to the SNR_{MFB}.

```
% Try a more reasonable design
[SNR,wt]=dfecolor(2,p_t,20,1,27,1,[2*0.181,zeros(1,2*20-1)]);
SNR

SNR =

    11.8920
```

Therefore, by transmitting at half the original rate, we see a much flatter channel, and get an improvement in SNR of around 3dB. Of couse this is at the expense of reduced data rate!

6. Summary

With the above example, we can see the following facts:

i) Given the physical channel and the basis function, we can see channel characteristics that are vastly different, by varying the symbols rate.

ii) In actual equalizer designs, we often need to truncate the pulse response to a finite time-domain series.

iii) In actual equalizer designs, we often need to use over-sampling. Part of the reason is the actual channel characteristics is often not known precisely.