STANFORD

Supplementary Lecture 12B

Voronoi/Geometric Shaping Codes

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Announcements & Agenda

- Partition Chains & Binary Lattices
- The GS Encoder & Decoder
- Trellis Shaping

Intermediate shaping gain when N must be short (low delay).

Shaping Gain Review

- Coding gain is

$$\gamma \triangleq \frac{\left(d^2_{\min}(\boldsymbol{x})/\bar{\mathcal{E}}_{\boldsymbol{x}}\right)}{\left(d^2_{\min}(\breve{\boldsymbol{x}})/\bar{\mathcal{E}}_{\breve{\boldsymbol{x}}}\right)} = \underbrace{\frac{\left(\frac{d^2_{\min}(\boldsymbol{x})}{V^{2/N}(\Lambda)}\right)}{\left(\frac{d^2_{\min}(\breve{\boldsymbol{x}})}{V^{2/N}(\breve{\Lambda})}\right)}}_{\gamma_f \text{ fundamental gain}} \cdot \underbrace{\frac{\left(\frac{V^{2/N}(\Lambda)}{\bar{\mathcal{E}}_{\boldsymbol{x}}}\right)}{\left(\frac{V^{2/N}(\breve{\Lambda})}{\bar{\mathcal{E}}_{\breve{\boldsymbol{x}}}}\right)}}_{\gamma_s \text{ shaping gain}}$$

Shaping gain is this S12B's focus.

- Shaping gain relative to $\mathbb{Z}^N$ (SQ QAM) with $d = 1$ in reference simplifies to

$$\gamma_s = \frac{V^{2/N}(\Lambda) \cdot \left(2^{2\bar{b}} - 1\right)}{\tilde{\mathcal{E}}_{\boldsymbol{x}} \cdot 6}$$

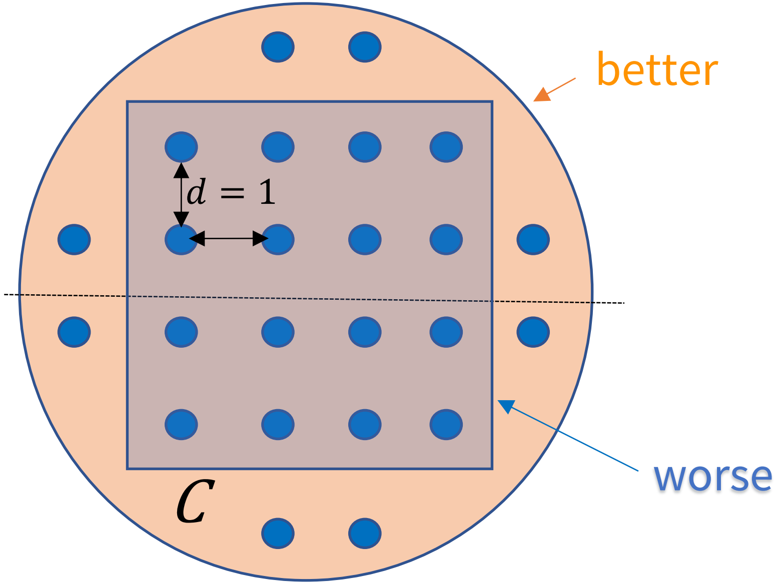
- When both constellations use the same number of 2D constellation points, this further simplifies to

$$\gamma_s = \frac{\left(2^{2\bar{b}} - 1\right)}{6 \cdot \tilde{\mathcal{E}}_{\boldsymbol{x}}}$$

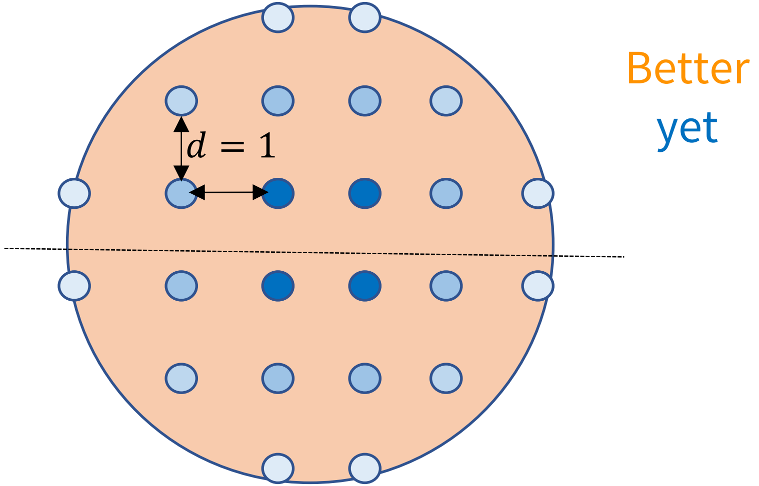
So shaping is basically the energy ratio to send same # of points.

Hypersphere boundaries are good (Gaussian)

Subsymbol (ss)



- As N grows, harder to visualize but uniform spacing with fixed average energy leads to Gaussian marginals.
  - The 2D constellation below is a "marginal" from the N-D distribution.



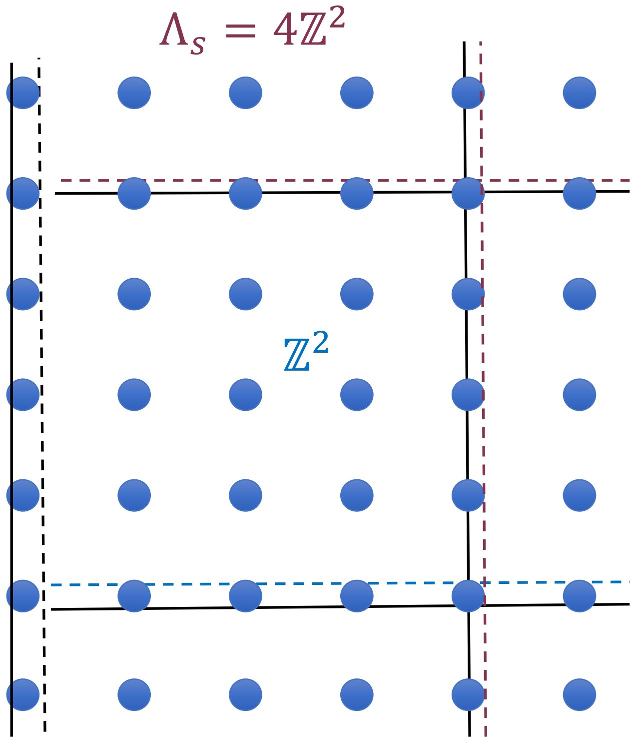
- So, a better boundary shape MARGINAL ~ non-equal probability (marginal) distribution
  - GS codes try to make the 2D constellation above a marginal (slice in 2D) of a N-dimensional Lattice's decision region.
  - Some of these lattices' **Voronoi Regions** also approximate a hypersphere for large N.

Partition Chains & Binary Lattices

Section 8.4.2

Binary Lattices Def & Partition Chain Review

- Place constellation points within geometric region:
  - This is trivial in 2D as →.
  - Decision regions of D_4 and E_8 lattices increasingly approximate hyperspheres.
- **Coding:** Pick a good lattice for inner code Λ_c ,
  - Example to right is $\mathbb{Z}^2$ (easy to illustrate not good code).
  - Good codes have high distance per volume ratios, γ_f.
- **Shaping:** Pick a good lattice for shaping code Λ_s
  - Decision regions approximate hyperspheres.
  - Need not be same code as inner code Λ_c .
- Binary lattices satisfy $\mathbb{Z}^N / \Lambda_s \, / 2^m \mathbb{Z}^N$:
  - $|\Lambda_s \, / 2^m \mathbb{Z}^N| = 2^k$
  - $|\mathbb{Z}^N \, / \Lambda_s| \; = 2^{n-k} = 2^p$
- Examples (Appendix B or Reed Muller Codes):
  - $\mathbb{Z}^2 / D_2 \, / 2\mathbb{Z}^2$ (not a good code, but easy to illustrate →).
  - $\mathbb{Z}^4 / D_4 / (D_2)^2 / R_4 D_4 \, / 2\mathbb{Z}^4$
  - $\mathbb{Z}^8 / D_8 \, / (R_4 D_4)^2 / DE_4 / E_8 / \, R_8 D_8 / \, (2D_4)^2 / R_8 D_8 / 2\mathbb{Z}^8$



Often good shaping codes are good for γ_f also, but not always.

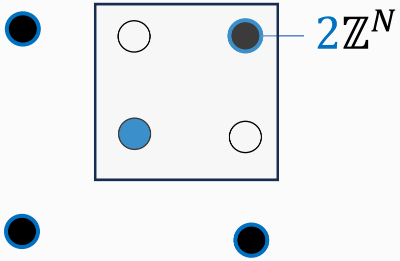
Binary-Code Lattice Descriptions

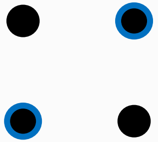
- G_s is the binary-code generator with dual-code generator H_s .

$$\underbrace{Z^n/\Lambda_s}_{H_s} \quad \underbrace{\Lambda_s/2\mathbb{Z}^n}_{G_s}$$

$$\Lambda_s = D_2 = 2\mathbb{Z}^N + \{\overbrace{[0\ 0], [1\ 1]}^{G_s = [1\ 1]}\}$$

- $\boldsymbol{\lambda} = \boldsymbol{u} \cdot G_s$
- $\boldsymbol{\lambda} \in \Lambda_s$
- $\boldsymbol{x} = \boldsymbol{\lambda} + 2\mathbb{Z}^N$





$$H_s^{-t} = [1\ \ 0]$$

$$\mathbb{Z}^N = D_2 + \{\overbrace{[0\ 0], [1\ 0]}^{\boldsymbol{s} \cdot H_s^{-t}}\}$$

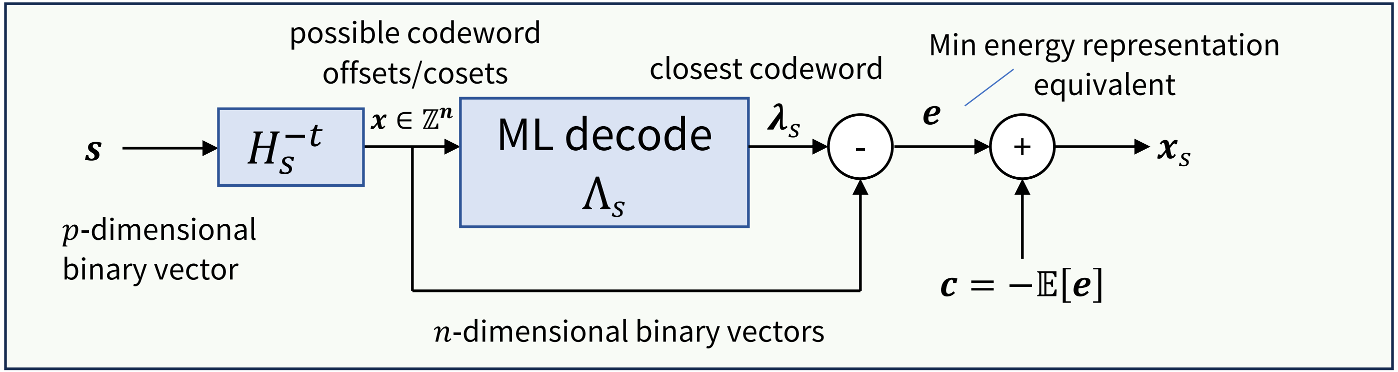
- $\boldsymbol{s} \cdot H_s^{-t}$ corresponds to 2^p codeword offsets/cosets from the possible (think $\mathbf{0}$) codeword to $\mathbb{Z}^N$.
  - "Completes" the spatial points to $\mathbb{Z}^N$.
  - $H_s = [1\ 1]$; $H_s^{-t} = [0\ 1]$ $(p = 1)$.
  - $\boldsymbol{s}$ is one dimensional here (1 bit).

$$\mathbb{Z}^n = \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{e} + \Lambda_s \ \wedge\ \boldsymbol{e} = \boldsymbol{s} \cdot H_s^{-t}\}$$

- Constellation points ($\mathbb{Z}^N$) can be constructed from G_s's codewords by adding offsets computed as $\boldsymbol{s} \cdot H_s^{-t}$!
  - But we'd like the ones that are in the zero-mean-centered Voronoi region of D_2.

The Voronoi/GS encoder & decoder

- Has a decoder in it!



- The decoder is simpler and inverts this process

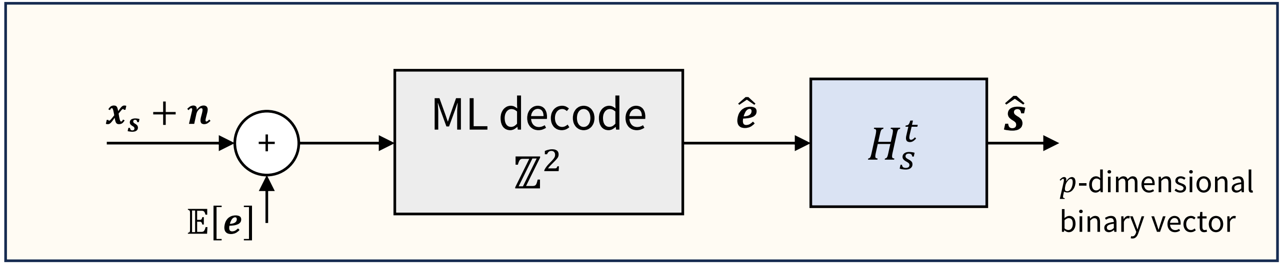
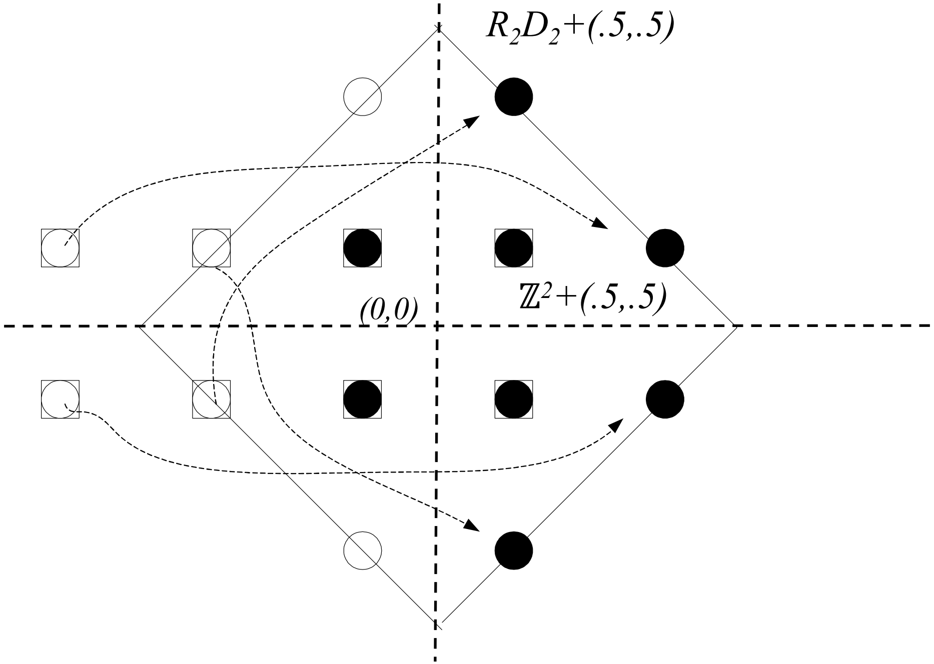


Illustration of Encoder Action

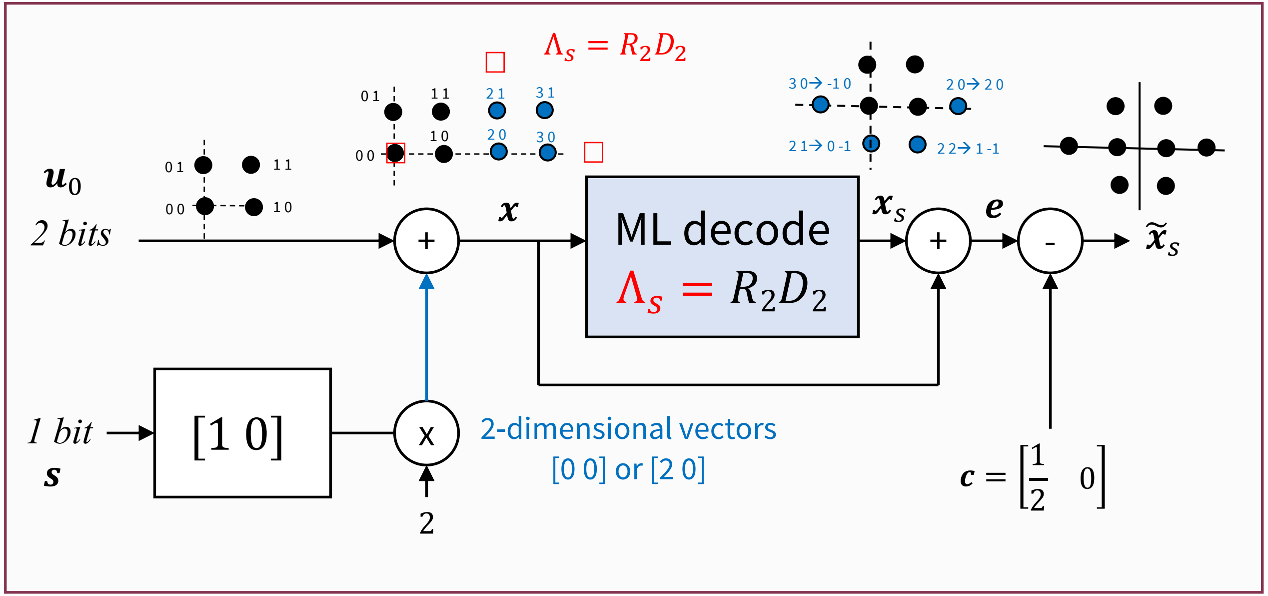


$\boldsymbol{e}$ point	$\bar{\mathcal{E}}_{\boldsymbol{e}}(i)$	$p_{\boldsymbol{e}}$	$\mathbb{E}[\boldsymbol{e}(i)]$
$\frac{1}{2}[\pm 1,\ \pm 1]$	$\frac{1}{4}$	1/2	$[0,\ 0]$
$\frac{1}{2}[3,\ \pm 1]$	$\frac{5}{4}$	1/4	$[\frac{3}{2},\ 0]$
$\frac{1}{2}[1,\ \pm 3]$	$\frac{5}{4}$	1/4	$[\frac{1}{2},\ 0]$
average	$\frac{3}{4}$	-	$[\frac{1}{2},\ 0]$

- Input subsymbols are the squared points (in an 2x4 rectangle (or any really 3 binary bits).
- The encoder maps them into the circle points, with scaled/translated D_2 decision region
  - See the large diamond
- There is no shaping gain here, but illustrates the operation.

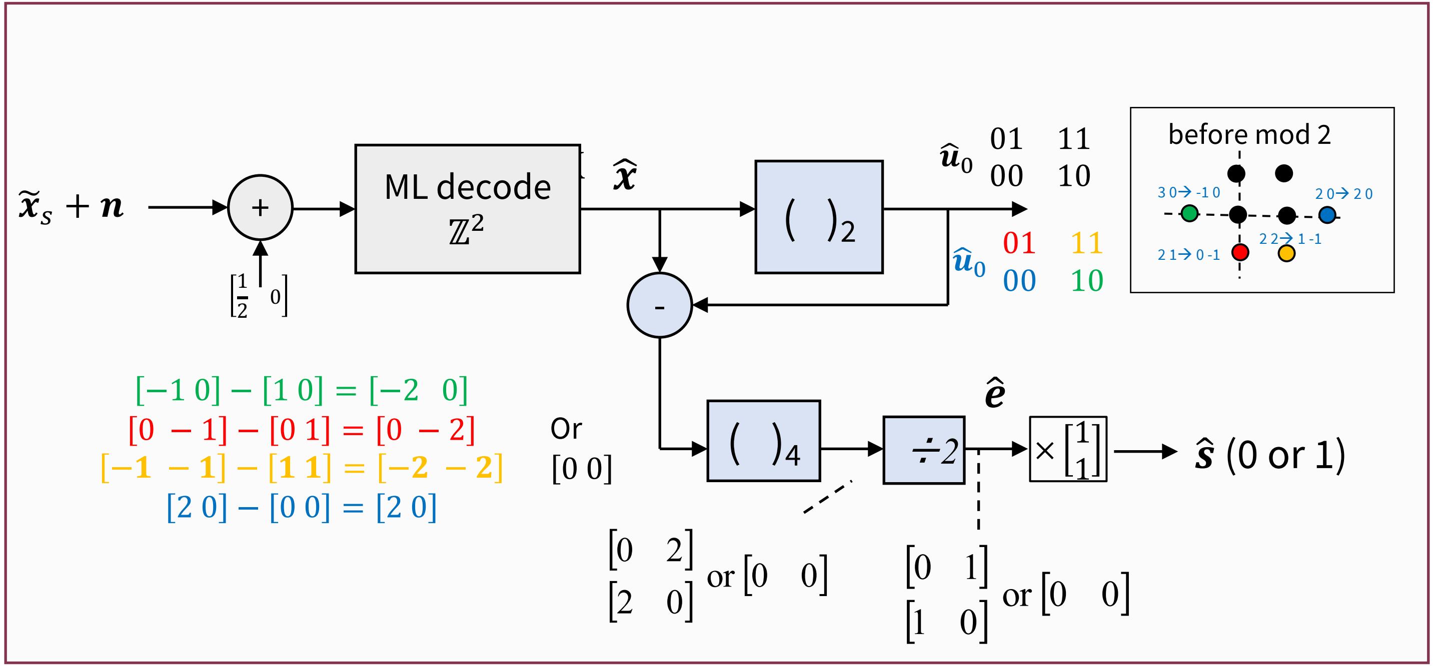
Specific Encoder for the Example

- 1 bit is syndrome
- Other 2 bits carry directly
  - They scale the D_2 .
- The factor 2^{m-1} is scaling.
- So the ML decode (in encoder) finds the closest point in R_2D_2
- Difference is transmitted



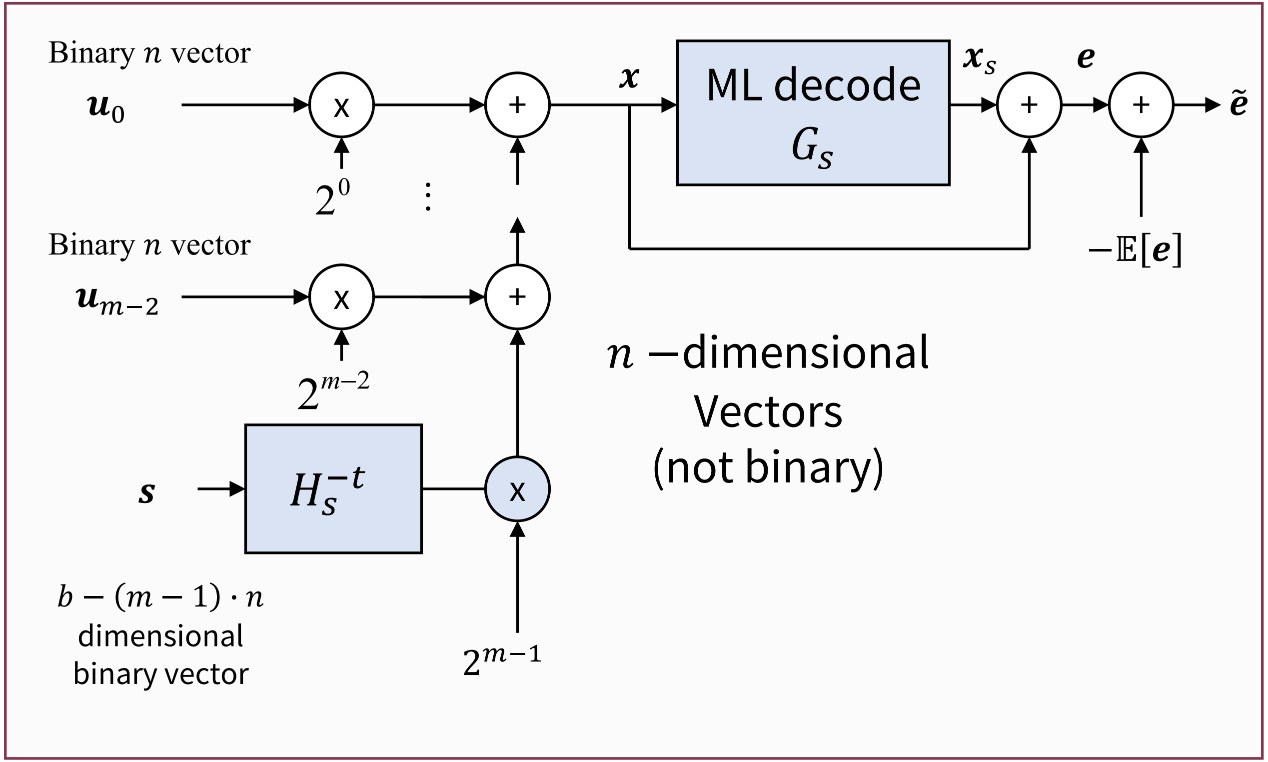
Example's Decoder

- Upper $(\ )_2$ decodes the extra 2 trivial bits.
- Their constellation points are adjusted to create last offset.
- The factor 2 removes the encoder scaling.
- Compute Syndrome
- Difference is transmitted



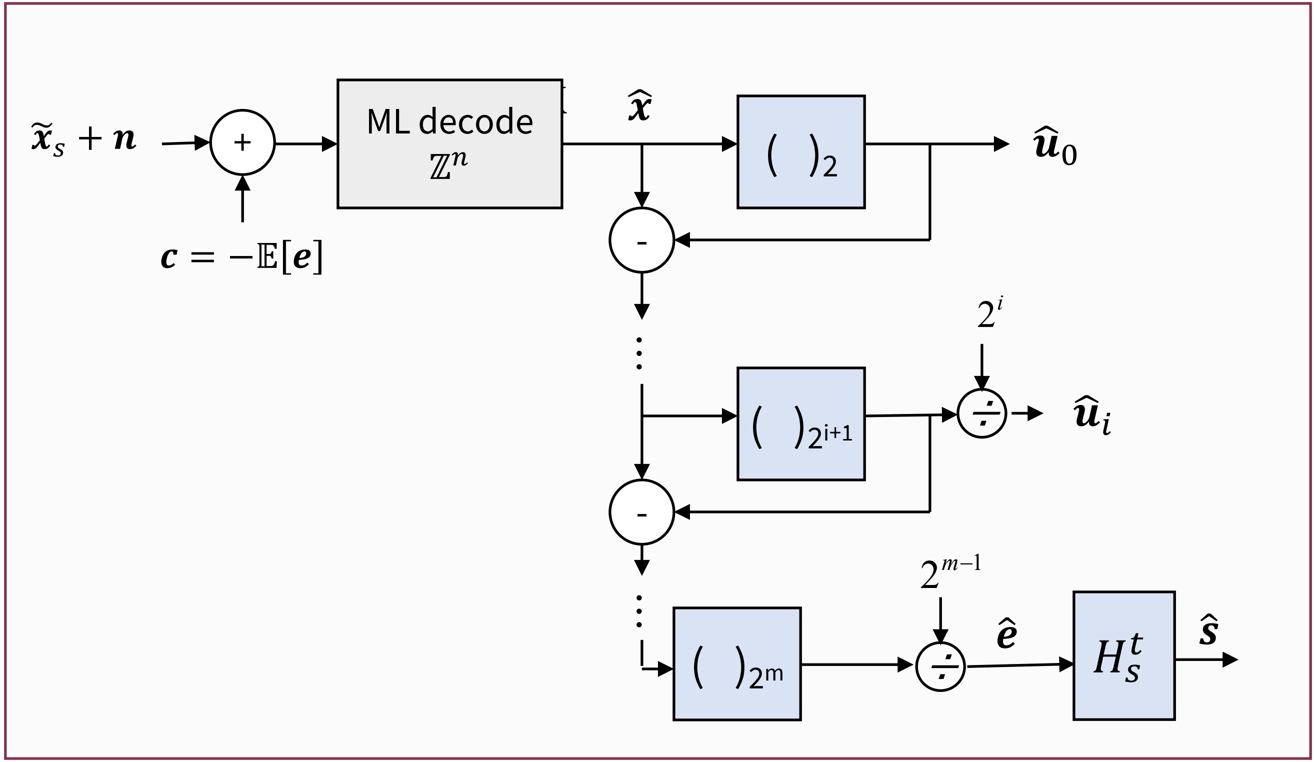
Generalize encoder to larger m (constellations)

- It follows same process.
  - Scaling squeezes more points into constellation.
  - Larger m
- Use up all bits
- Top $\boldsymbol{u}_0$'s bits can be zeroed to match b.



General Decoder

- The ML is only for inner code
- The inner code's errors will change outputs
- Otherwise, this is 1-to-1 mapping.



Shaping Gains for different lattices

- These are for large b.
- Same scaling and descaling, but the ML in encoder gets more complicated.
- The rest is the same.

n	Λ_s	γ_s (dB)
8	$1/2$	1.05
8	$D_8^\perp$	.47
8	E_8	.65
16	Λ_{16}	.86
24	Λ_{24}	1.03

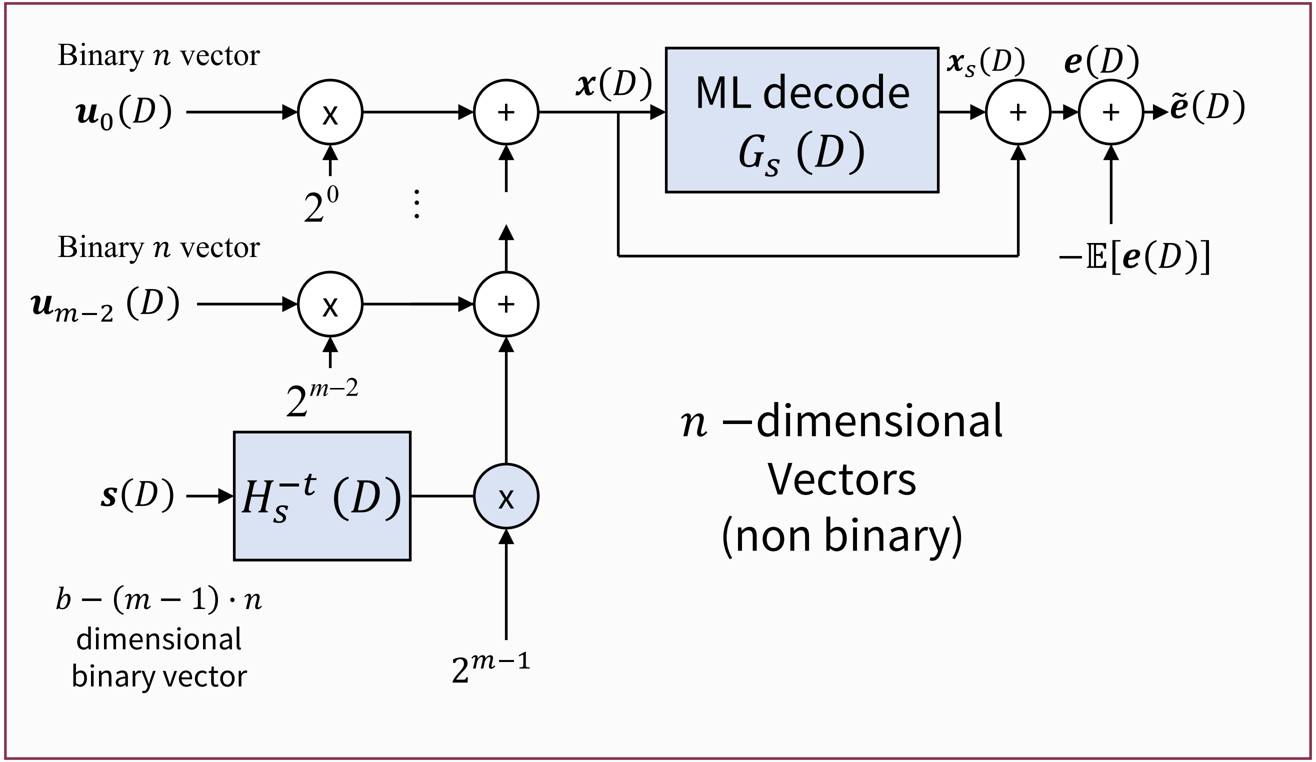
Not bad gains, and $for\ short\ N = n$.

Trellis Shaping

Section 8.4.2

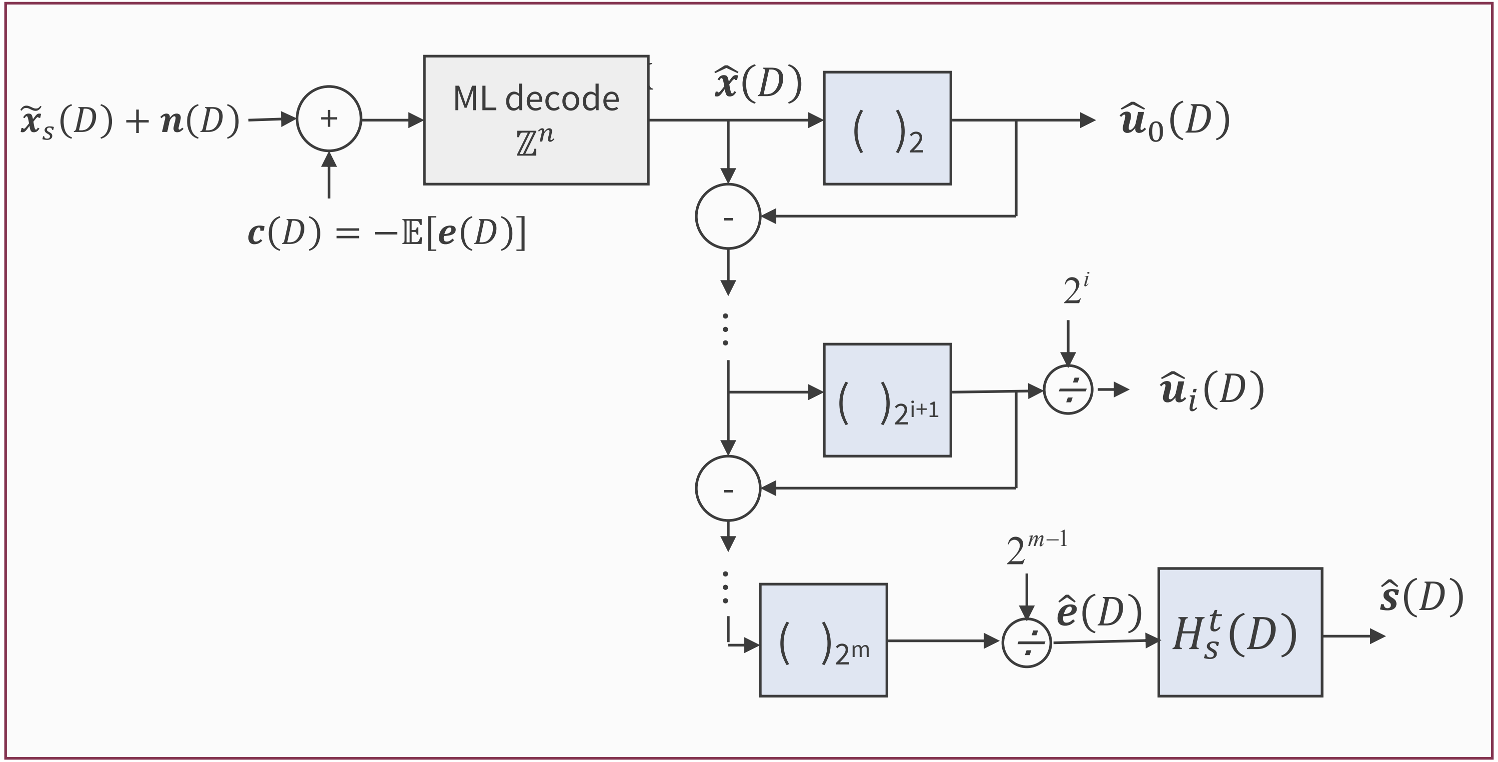
Encoder uses a Viterbi Decoder inside

- Done for subsymbols
- ML is more complex
  - Adds delay
  - Survivor
- Otherwise functions the same



Trellis Shaping Decoder

- Also follows Voronoi case
- Decoder is not complex
- Want FIR parity polynomial to limit error propagation



Some Trellis Shaping Codes and gains

- Intermediate to Voronoi and GS.
- Higher gains
- Not aware of their use.

2^ν	γ_s (dB)	$H(D)$	$H(D)^{-*}$	delay
2	0.59	$[1+D \;\; 1]$	$[1 \;\; D]$	8
4	0.97	$[1+D^2 \;\; 1+D+D^2]$	$[D \;\; 1+D]$	26
8	1.05	$\left[\begin{array}{cc} 1+D^2+D^3 & 1+D+D^3 \end{array}\right]$	$\left[\begin{array}{cc} 1+D & D \end{array}\right]$	34
8	1.06	$\left[\begin{array}{ccc} 1+D^3 & 0 & D \\ 0 & 1+D^3 & D^2 \end{array}\right]$	$\left[\begin{array}{ccc} \frac{1}{1+D^3} & 0 & 0 \\ 0 & \frac{1}{1+D^3} & 0 \end{array}\right]$	42
16	1.14	$\left[\begin{array}{ccc} 1+D+D^4 & 0 & D+D^2+D^3 \\ 0 & 1+D+D^4 & D^2 \end{array}\right]$	$\left[\begin{array}{ccc} \frac{1}{1+D+D^4} & 0 & 0 \\ 0 & \frac{1}{1+D+D^4} & 0 \end{array}\right]$	100?

End Lecture S12B