STANFORD

Lecture 4

Complex AWGN and Other Channels

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Announcements & Agenda

- Announcements
  - PS2 next Wed – Jan 24
    - See HWH if you are having difficulty before spending too much time
    - Or ask questions (homework feedback is good also for future students)
  - Uploading assignments issues?
    - So far, sending to Ethan works

- Today
  - Coding Gain
  - Signal Representations
    - The Phase-Splitting Demodulator
  - Noise and passband processes
  - SSB, VSB, CAP, other forms of QAM
  - Discrete Memoryless Channels

Coding Gain

Lattices and Codes (AWGN)

- **Lattice** $\Lambda = \{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \cdots\}$ that is closed under an operation "addition" (usually normal addition, but can also be over a finite field when $|\Lambda| < \infty$. (Appendix B)

- Examples include:
  - $\mathbb{Z}$ - the integers (think PAM),
  - $\mathbb{Z}^2$- 2D integer vectors (think QAM), and
  - $\mathbb{Z}^N$ - think codewords built from PAM/QAM.

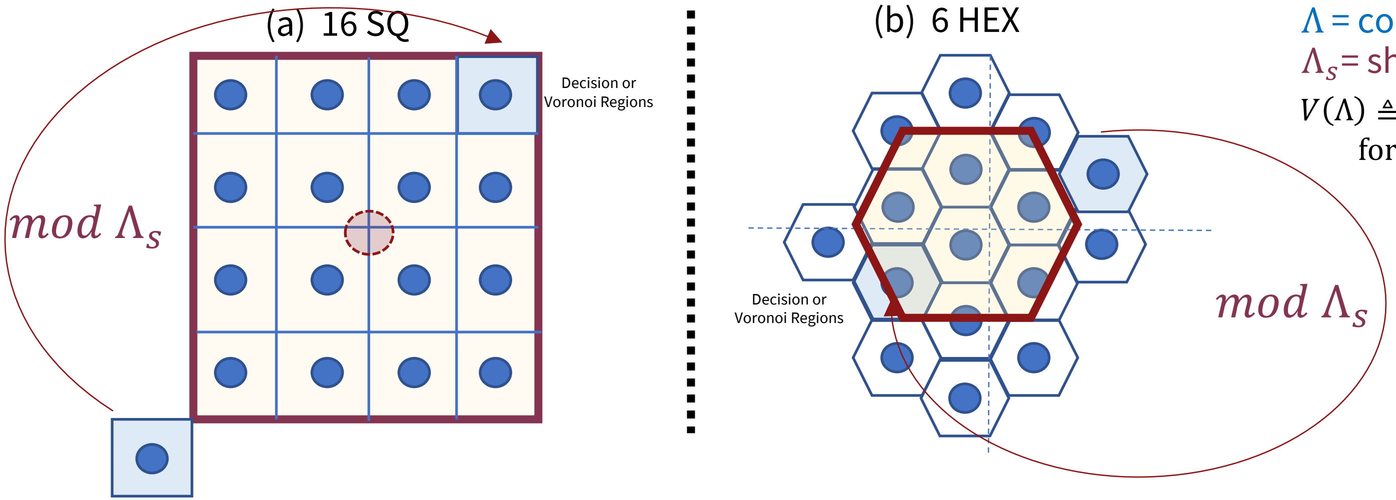
$D_2 = 2Z^2 + \{0,1\} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is odd-b square QAM

- **Coset** $\Lambda + \boldsymbol{a} = \{\boldsymbol{\lambda}_0 + \boldsymbol{a}, \boldsymbol{\lambda}_1 + \boldsymbol{a}, \cdots\}$ basically maintains all the lattice properties, but need to add constant $\boldsymbol{a}$ (or remove it) appropriately.

- Most constellations C are subsets of lattices Λ (or their cosets).
  - Designs choose M symbols from Λ, and subtract mean so that the set C has minimum average energy.

- Lattices are a nice way for code designers to pack points evenly into given volume or energy.

Coding Gain and Constellation/Code



Λ = coding lattice for d_{min}

Λ_s= shaping lattice for $\mathcal{E}_x$

$V(\Lambda) \triangleq$ decision − region volume for (any) lattice point in Λ

$$\gamma \triangleq \frac{\left(d^2_{\min}(\boldsymbol{x})/\bar{\mathcal{E}}_{\boldsymbol{x}}\right)}{\left(d^2_{\min}(\breve{\boldsymbol{x}})/\bar{\mathcal{E}}_{\breve{\boldsymbol{x}}}\right)} = \underbrace{\frac{\left(\frac{d^2_{\min}(\boldsymbol{x})}{V^{2/N}(\Lambda)}\right)}{\left(\frac{d^2_{\min}(\breve{\boldsymbol{x}})}{V^{2/N}(\breve{\Lambda})}\right)}}_{\gamma_f} \cdot \underbrace{\frac{\left(\frac{V^{2/N}(\Lambda)}{\bar{\mathcal{E}}_{\boldsymbol{x}}}\right)}{\left(\frac{V^{2/N}(\breve{\Lambda})}{\bar{\mathcal{E}}_{\breve{\boldsymbol{x}}}}\right)}}_{\gamma_s}$$

coding gain

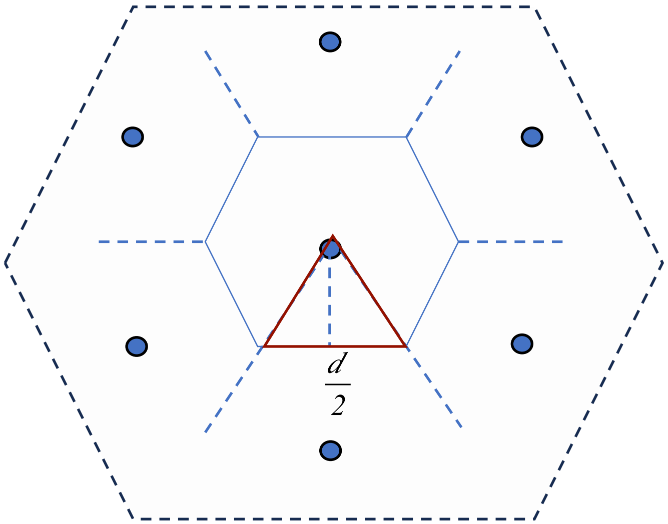
fundamental gain

shaping gain

Basic principle extends as $\bar{N} \to \infty$.
Hexagon→ hypersphere (Gaussian marginals).

Good codes can follow from $\Lambda_s / \Lambda = |C|$.

Hexagon Constellation, fund gain



$\Lambda = A_2$, the **"hexagonal" lattice.**

$$V(A_2) = \underbrace{6}_{of\ \Delta's} \cdot \frac{1}{2} \cdot \underbrace{\frac{d}{\sqrt{3}}}_{base} \cdot \underbrace{\frac{d}{2}}_{altitude} = \frac{\sqrt{3}}{2} \cdot d^2 \ .$$

$$\gamma_f = \frac{d^2}{\frac{\sqrt{3}d^2}{2}} = \frac{2}{\sqrt{3}} = .625 \text{ dB}$$

- A_2 is (up to) .625 dB better than $\mathbb{Z}^2$ $(d = 1, V = 1)$ in fundamental gain.
- It's also better as a shaping lattice (closer to a circle).

Maximum Shaping Gain

- Let $N \to \infty$, then **best shape** is a **hypersphere**.

- For hypersphere: $$\frac{\bar{\mathcal{E}}_x}{V^{2/N}} = \underbrace{\frac{r^2}{N+2}}_{2nd\ moment} \cdot \underbrace{\frac{\left(\frac{N}{2}!\right)^{2/N}}{\pi \cdot r^2}}_{1/area} = \frac{\left(\frac{N}{2}!\right)^{2/N}}{\pi \cdot (N+2)}$$

$r^2/4$ $\quad$ $1/\pi \cdot r^2$

Problem 1.19

- Limit, relative to $\mathbb{Z}^N$, is $\frac{\pi e}{6}$ = 1.53 dB .
  - Proof in text.

BEST SHAPING GAIN IS 1.53 dB

- Fundamental gain can be infinite – see Chapter 2.

Peak-to-Average Ratio (PAR)

- Can be important for amplifiers (see PSK discussion).

Definition 1.3.23 [Discrete Peak Energy] *A constellation's N-dimensional* **discrete peak energy** *is $\mathcal{E}_{peak}$.*

$$\mathcal{E}_{peak} \stackrel{\Delta}{=} \max_i \sum_{n=1}^{N} x_{in}^2 \quad . \tag{1.328}$$

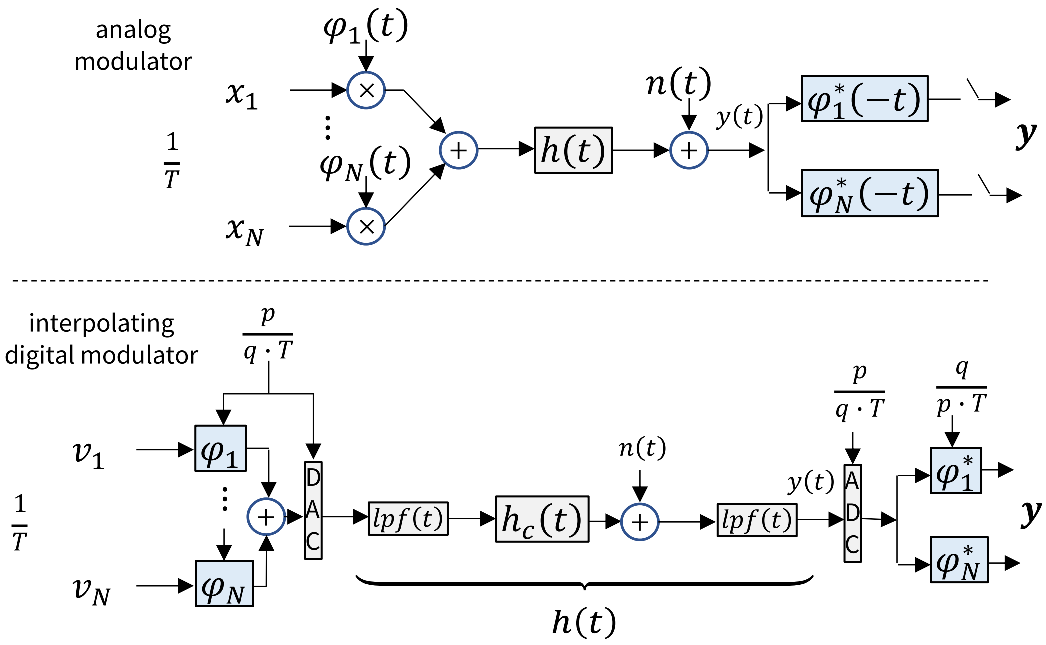
A modulated signal's **continuous-time peak energy** *is*

$$\mathcal{E}_{cont} \stackrel{\Delta}{=} max_{i,t} |x_i(t)|^2 \geq \mathcal{E}_{peak} \quad . \tag{1.329}$$

- PARs could be an additional measure:
  - At symbol instants: $PAR = {}^{\mathcal{E}_{peak}}/_{\mathcal{E}_x}$
  - In continuous time for an overall $PAR = {}^{\mathcal{E}_{cont}}/_{\mathcal{E}_x}$ - this one is always at least as large.
  - Example – simple sinusoid symbol-rate sampled at peaks has symbol-rate PAR =1 while any continuous sinusoid has PAR 3dB.

Signal Representations

Revisit the filtered AWGN, more detail



$fixed/given$

$designed/optimized$

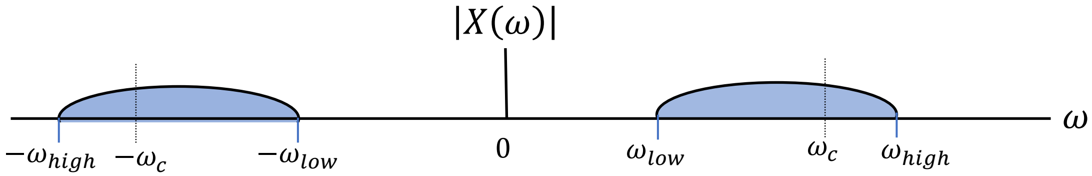
$p, q \in \mathbb{Z}^+,$
$p \geq q$

noise whitening (L13) absorbed into receiver matched filters

- Often the modulator basis functions are designed with DSP (379B).
- They may be optimized for the channel $h(t)$ - as opposed to some fixed filter.
- Channels often limit to a band centered at f_c , the ``carrier'' or ``center'' frequency.
- If not familiar with multi-rate filters, just set $q = 1$, insert $p - 1$ zeros between symbol values at xmit.
  - Filter is designed at p times sample rate ; similarly, receiver just accepts inputs at $\frac{p}{T}$ and discards $p - 1$ samples/symbol-period at filter output.

Carrier Modulated Waveform

- Most channels don't use DC or some lower (and higher) frequencies – they exist in a band.



Definition 1.3.25 **[Carrier-Modulated Signal]** *A* **carrier-modulated signal** *is any passband signal that satisfies*

$$x(t) = a(t) \cdot \cos\left(\omega_c t + \theta(t)\right) \quad , \tag{1.333}$$

where $a(t)$ is the modulated signal's time-varying **amplitude** *or* **envelope** *and $\theta(t)$ is its time-varying* **phase**. *$\omega_c = 2\pi f_c$ is the* **carrier frequency** *(in radians/sec; f_c is in Hz).*

$A(\omega) = 0$ for $\omega \geq \omega_c$ avoids positive/neg translations' overlap.

Quadrature Decomposition

Definition 1.3.26 [Quadrature Decomposition] *The* **quadrature decomposition** *of a carrier modulated signal is*

$$x(t) = x_I(t) \cdot \cos(\omega_c t) - x_Q(t) \cdot \sin(\omega_c t) \quad , \qquad (1.334)$$

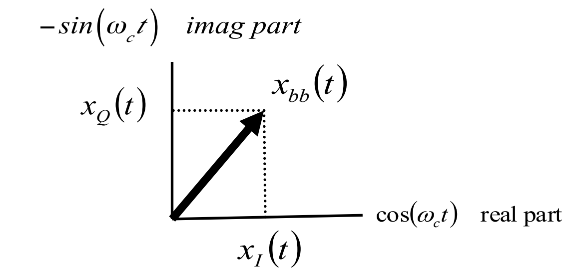
where $x_I(t) = a(t) \cdot \cos(\theta(t))$ *is the modulated signal's time-varying* **inphase component**, *and* $x_Q(t) = a(t) \cdot \sin(\theta(t))$ *is its time-varying* **quadrature component**.

Previous slide's

$$a(t) = \sqrt{x_I^2(t) + x_Q^2(t)}$$

$$\theta(t) = \text{Tan}^{-1}\left[\frac{x_Q(t)}{x_I(t)}\right]$$

- Note capital T on Tan^{-1}, so must know each of I's and Q's signs.
  - Baseband components are at/near DC, as if frequency re-indexes to $\omega \to \omega - \omega_c$.



Complex Equivalents

- **Baseband equivalent** (carrier frequency not included)

Definition 1.3.27 [Baseband-Equivalent Signal] *The complex* **baseband-equivalent signal** *for* $x(t)$ *in (1.333) is*

$$x_{bb}(t) \triangleq x_I(t) + \jmath x_Q(t) \quad , \tag{1.337}$$

where $\jmath = \sqrt{-1}$.

  - Baseband components are at or near DC, as if frequency re-indexes so that $\omega \rightarrow \omega - \omega_c$.

- **Analytic equivalent** (carrier frequency included):

Definition 1.3.28 [Analytic-Equivalent Signal] *The* **analytic-equivalent signal** *for* $x(t)$ *in (1.333) is*

$$x_A(t) \triangleq x_{bb}(t) \cdot e^{\jmath \omega_c t} \quad . \tag{1.338}$$

  - Translates components up to positive frequencies. (Analytic has zero neg-freq energy.)

Example

$$x(t) = \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + 3 \cdot \text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t)$$

$$\begin{aligned} x_I(t) &= \text{sinc}(10^6 t) \\ x_Q(t) &= -3 \cdot \text{sinc}(10^6 t) \end{aligned}$$

$$x_{bb}(t) = (1 - 3\jmath) \cdot \text{sinc}(10^6 t)$$

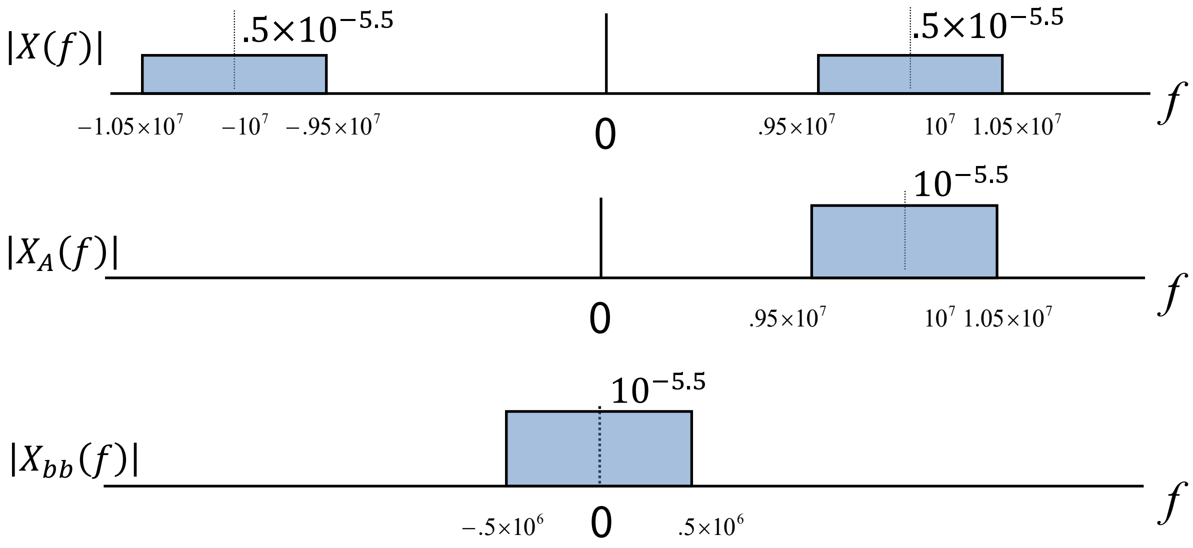
$$\begin{aligned} a(t) &= \sqrt{10} \cdot \text{sinc}(10^6 t) \\ \theta(t) &= \text{Tan}^{-1}\left[\frac{-3}{1}\right] = -71.6^o \end{aligned}$$

$$x(t) = \sqrt{10} \cdot \text{sinc}(10^6 t) \cdot \cos(\omega_c t - 71.6^o)$$

- and

$$x_A(t) = (1 - 3\jmath) \cdot \text{sinc}(10^6 t) \cdot e^{\jmath 2\pi 10^7 t}$$

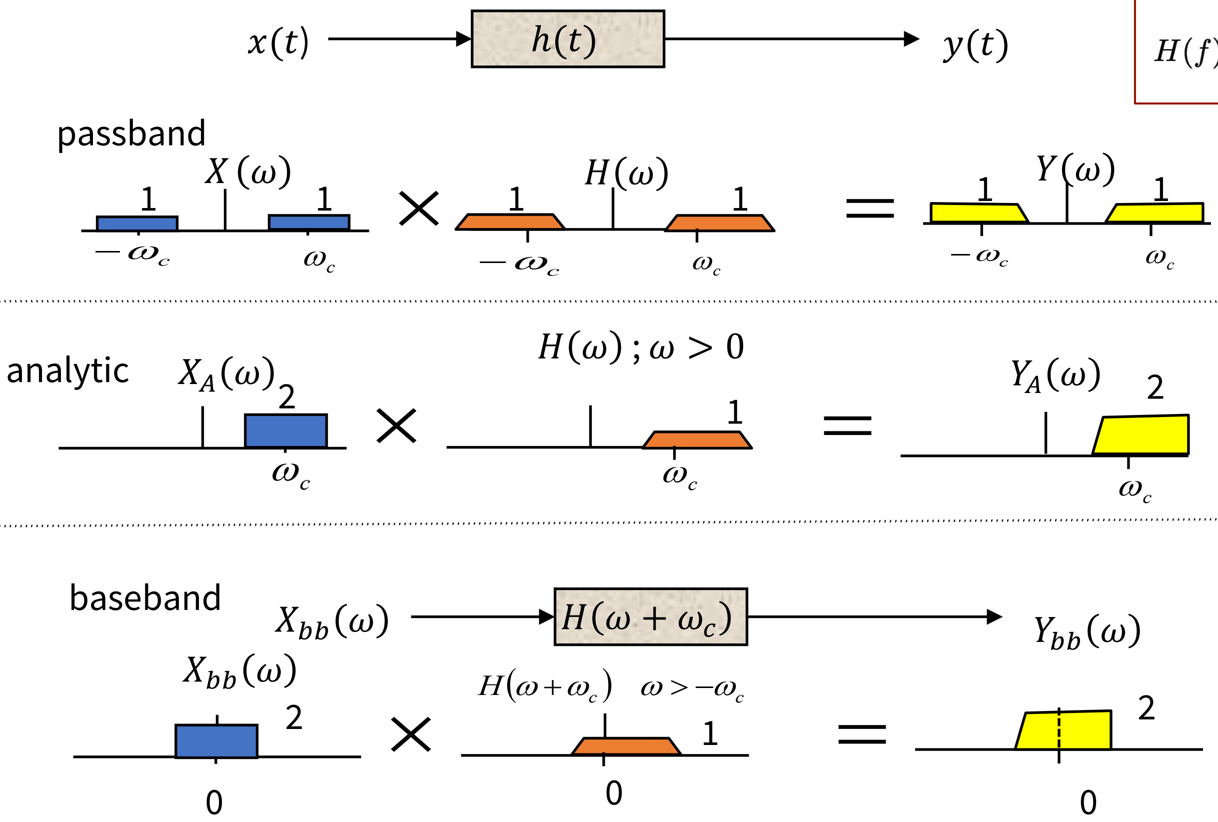
Example's Spectra



- The sinc function causes the brickwall nature of the signals (in practice never quite this perfect).

Example revisited in terms of channel $H(\omega)$

$$H(f) = \begin{cases} 1 & |f \pm 10^7| < .5 \times 10^6 \\ 0 & \text{elsewhere} \end{cases}$$



- The baseband view basically centers the channel carrier frequency at 0.

Analytic-Signal Generation (App A.3.1/2)

- **Hilbert Transform** – convolve signal $x(t)$ with $\hbar(t) = \begin{cases} 1/_{\pi \cdot t} & t \neq 0 \\ 0 & t = 0 \end{cases}$.

$$\check{x}(t) = \hbar(t) * x(t)$$

- Hilbert's Fourier Transform is $-j \cdot sgn(\omega)$, so HT:
  - shifts positive frequencies by +90^o and
  - shifts negative frequencies by -90^o ,
  - so that means cos into sin , and sin into –cos.

$$x(t) = x_I(t) \cdot \cos(\omega_c \cdot t) - x_Q(t) \cdot \sin(\omega_c \cdot t) \quad = \Re\{x_A(t)\}$$

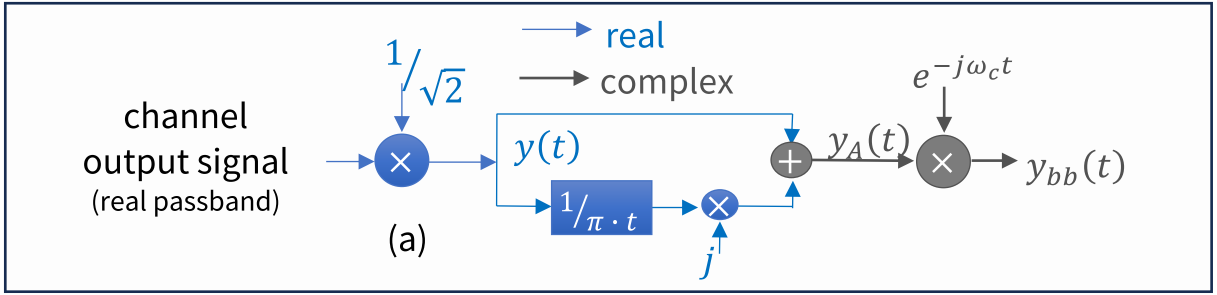
$$\check{x}(t) = x_I(t) \cdot \sin(\omega_c \cdot t) + x_Q(t) \cdot \cos(\omega_c \cdot t) = \Im\{x_A(t)\}$$

- Algebra leads to $x_A(t) = x(t) + j \cdot \check{x}(t)$.

- Use HT to demodulate passband to a complex baseband signal, any carrier.

Complex Demodulator Types

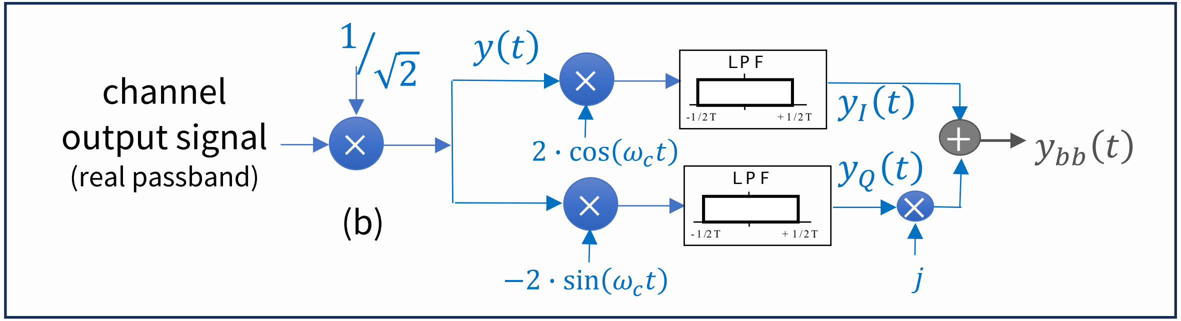
- Use Hilbert Transform? (usually implemented, with delay, in digital signal processing with one sampler at input).



The factor $1/\sqrt{2}$ maintains signal energy.

This is useful for theory, but not implemented

- Use cos/sin demodulation (usually analog implementation, with 2 samplers at I and Q outputs).



- Typically symbol rate and carrier are locked (rational fraction, Chapter 6) to same source – HT version is then much more amenable to DSP implementation.

Inner Product Generalization

1. The inner product becomes

$$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^* \boldsymbol{y} = \int_{-\infty}^{\infty} x^*(t) \cdot y(t) dt \quad , \tag{1.408}$$

   ($\boldsymbol{x}^*$ means conjugate transpose of $\boldsymbol{x}$).

2. The matched filter is conjugated, that is $\varphi(T-t) \rightarrow \varphi^*(T-t)$.

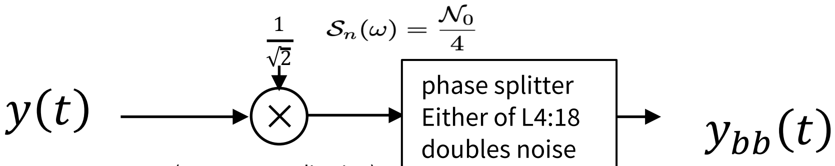
3. Energies of complex scalars are $\mathcal{E}_{\boldsymbol{x}} = E\left\{|x(t)|^2\right\}$, or the expected squared magnitude of the complex scalar, and $\bar{\mathcal{E}}_{\boldsymbol{x}} = \mathcal{E}_{\boldsymbol{x}}/2$.

- It is rare that transpose by itself is used or needed any longer – almost always conjugate transpose
- Throughout rest of EE379A, B …. your design careers!

Noise and Passband Processes

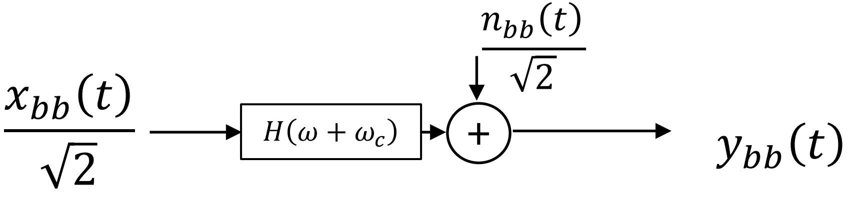
Section 1.3.5.2

AWGN through demodulators



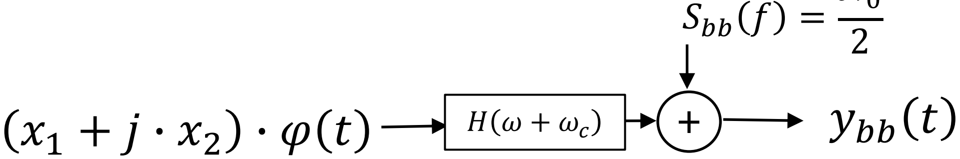
(energy normalization)

$$S_A(\omega) = \begin{cases} \mathcal{N}_0 & \omega > 0 \\ \frac{\mathcal{N}_0}{2} & \omega = 0 \\ 0 & \omega < 0 \end{cases}$$

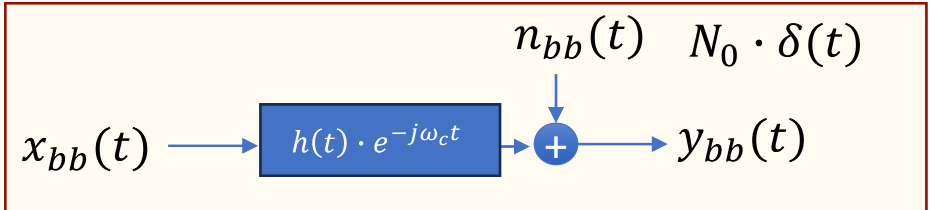


(pushed back to noise and channel input)

$$S_{bb}(\omega) = \begin{cases} \mathcal{N}_0 & \omega > \omega_c \\ \frac{\mathcal{N}_0}{2} & \omega = -\omega_c \\ 0 & \omega < -\omega_c \end{cases}$$



- Scaling removes the $\sqrt{2}$ that was in the real basis functions for QAM and resets the noise to $\frac{\mathcal{N}_0}{2}$ in each of the two dimensions.

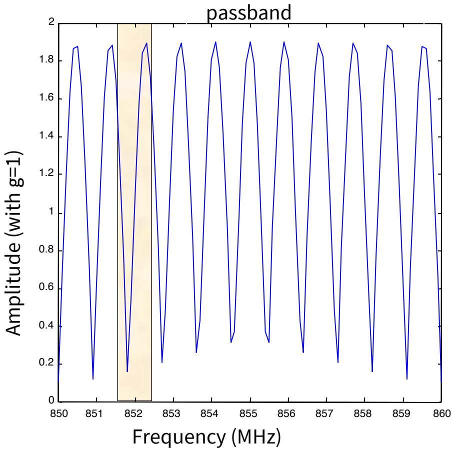


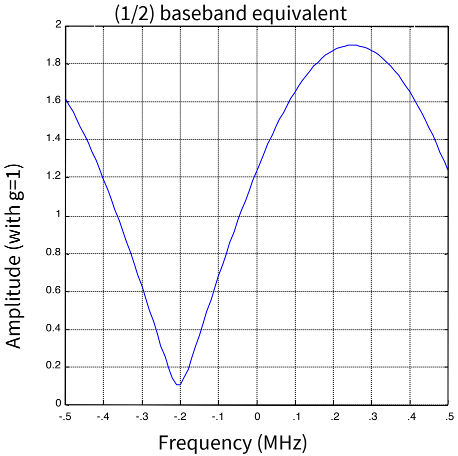
Complex BB AWGN Channel

Wireless Multipath Example

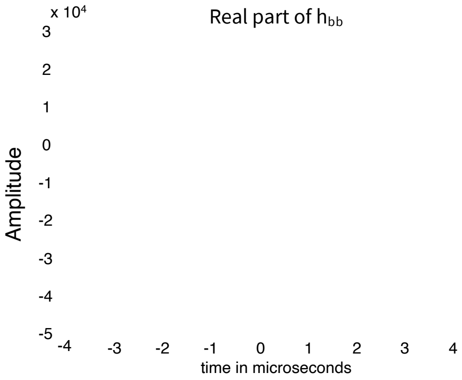
$$h(t) = g \cdot [\delta(t) - .9 \cdot \delta(t - \tau)]$$

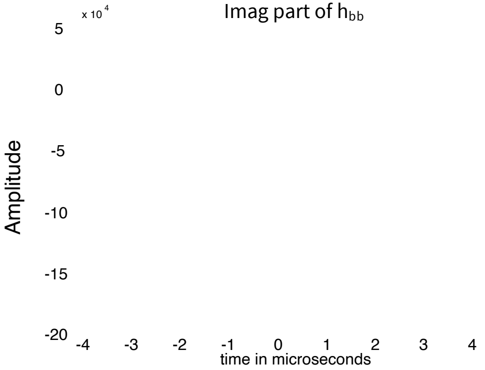
$$\tau = 1.1ns$$
$$f_c = 852 \text{ MHz}$$





- The baseband equivalent need not be symmetric around zero frequency (it is complex).

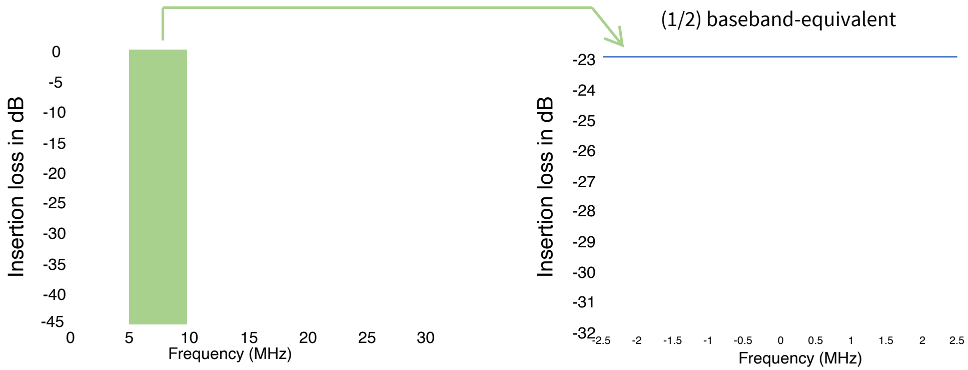




- Baseband complex channel in time domain is shown:
- Channel adds white noise.

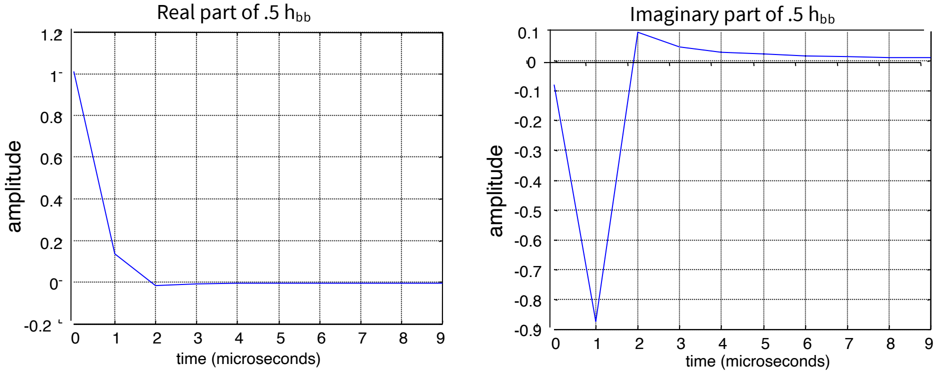
Transmission Line Example

- 26-gauge twisted pair:
  - "Cat 3"
  - 300 meters length
  - See "linemod" software at website.
  - https://cioffi-group.stanford.edu/linemod/
  - Happy to hear of matlab replacements



- Baseband complex channel in time domain:

- Channel adds white noise.



Divide $H(f)$ by $\bar{S}_n^{-1/2}(f)$ to get noise-equivalent channel for analysis (rcvr needs to implement this filter), see L13.

SSB, VSB, CAP, other forms of QAM

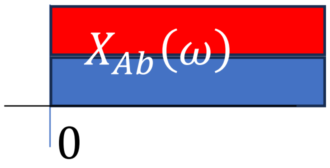
Section 1.3.6

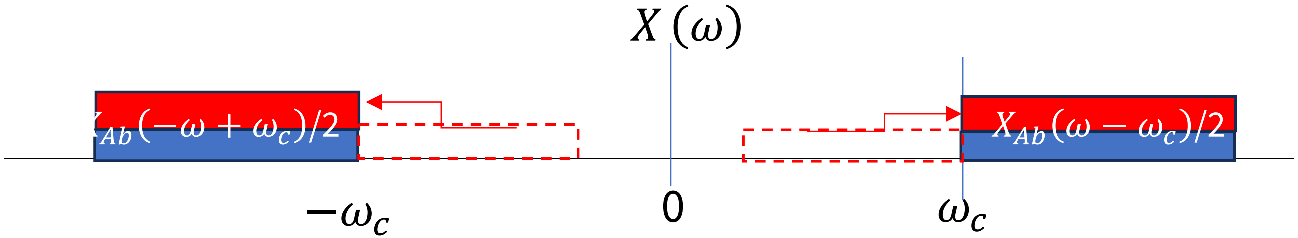
Single Side Band (SSB)

- **SSB** has inphase and quadrature from same source.
  - Upper side band is $x\,(t) = x_I(t)\cdot\cos(\omega t) - \check{x}_I(t)\cdot\sin(\omega t)$.

$$x_{Ab}(t) = x_{bb}(t) = x_I(t) + j\cdot\check{x}_I(t)$$

$$x_A(t) = \left(x_I(t) + j\cdot\check{x}_I(t)\right)\cdot e^{j\omega_c t}$$





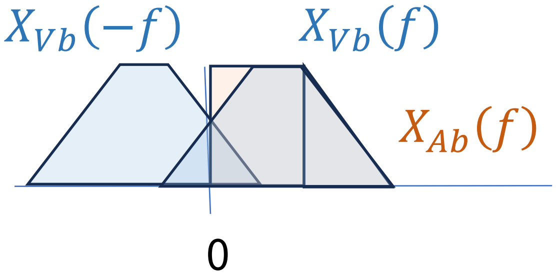
- SSB, effectively in same bandwidth, can have twice the symbol rate of QAM.
  - But quadrature derives from inphase, so the data rate remains the same.
  - SSB was used in analog radio and especially TV to half bandwidth of a real continuous-time signal.
    Double side band does not exploit the quadrature dimension.

- This is really just complex QAM with carrier frequency selected at the top (LSB) or bottom (USB).
  - It has no fundamental advantage over QAM.

Vestigial Sideband (VSB)

- Old analog TV places carrier at 1/3 point in 6 MHz channel.
- **VSB** looks like SSB but the $X_I(\omega)$ has vestigial symmetry about ω_c .
  - ➢ $x\,(t) = x_{VSB}(t) \cdot \cos(\omega t) - \check{x}_{VSB}(t) \cdot \sin(\omega t)$

$$X_{Vb}(f) + X_{Vb}(-f) = X_{Ab}(f)\ \forall\ f > 0$$



$$x_{Vb}(t) = x_{bb}(t) = x_{VSB}(t) + j \cdot \check{x}_{VSB}(t)$$

- Again, design might have just placed carrier where desired in baseband QAM
  - ➢ But then not compatible with transition from analog to digital in TV.

Carrierless Amplitude Phase modulation (CAP)

- **CAP** is really just QAM that exploits the carrier and symbol clock are locked in modern digital-signal-processing-based transmitters.

$$
\begin{aligned}
x_A(t) &= \sum_k x_k \cdot \varphi(t-kT) \cdot e^{\jmath\omega_c t} \\
&= \sum_k x_k \cdot \varphi(t-kT) \cdot e^{\jmath\omega_c t} \cdot e^{-\jmath\omega_c kT} \cdot e^{+\jmath\omega_c kT} \\
&= \sum_k (x_k \cdot e^{+\jmath\omega_c kT}) \cdot \varphi(t-kT) \cdot e^{\jmath\omega_c (t-kT)} \\
&= \sum_k \breve{x}_k \cdot \varphi_A(t-kT)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_A(t) &= \varphi(t) \cdot e^{\jmath\omega_c t} \text{ and} \\
\breve{x}_k &= x_k \cdot e^{+\jmath\omega_c kT}
\end{aligned}
$$

- Synthesize directly the full bandwidth signal without any carrier.
  - Although it is hidden in the $\breve{x}_k$, often the rotation is simply implemented.

Suggestion

- Just stay with QAM baseband analysis.
- It is exactly the same for any of the SSB, VSB, CAP systems.
- Precise transmit/receiver implementation can exploit any of the techniques depending on the specific design
  - Fundamentally, the analysis and performance are the same.

Discrete Memoryless Channels

Section 1.4

DMC Definition

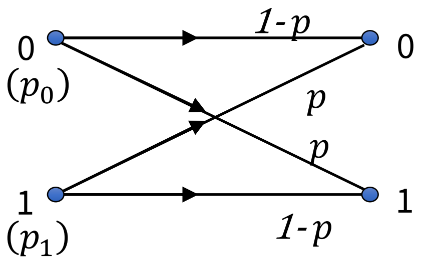
Definition 1.4.1 *[Discrete Memoryless Channel (DMC)]* *A* **discrete memoryless channel (DMC)** *has* $M' \geq M = |C| < \infty$ *with ordered transmitted message group* $\boldsymbol{X} \triangleq \{\boldsymbol{x}_n,\ n = 1, ..., N\}$, *with each message* $\boldsymbol{x}_n \in \{i = 0, ..., M-1\}$, *and with corresponding outputs* $\boldsymbol{Y} \triangleq \{\boldsymbol{y}_n,\ n = 1, ..., N\}$ *with each* $\boldsymbol{y}_n \in \{j = 0, ..., M'-1\}$ *that satisfy*

$$p_{\boldsymbol{Y}/\boldsymbol{X}}(j,i) = \prod_{n=1}^{N} p_{\boldsymbol{y}_n/\boldsymbol{x}_n}(j,i) \quad . \tag{1.461}$$

The integer n *here is a dimensional index (typically reflecting successive time-based DMC uses, but not necessarily so). The indices* j *and* i *reflect instead particular (output, input) sample values from the discrete distribution. A* **stationary DMC** *has* $p_{\boldsymbol{y}_n/\boldsymbol{x}_n}(j,i) = p_{\boldsymbol{y}/\boldsymbol{x}}(j,i)\ \forall\ j,i$, *or is thus independent of the dimensional index* n.

- **Memoryless** – the channel dimensions don't interfere with each other.

The Binary Symmetric Channel (BSC)



$P_e = p$

- ML Detector?
  - Easy, 0→ 0 , 1→ 1 .
- Error prob?
  - $P_e = p$.
- Often used to model an “inner detector” so $\bar{P}_b \rightarrow p$.

- Probability transition matrix example:

$$P_{y/x} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

Probability Transition Matrix

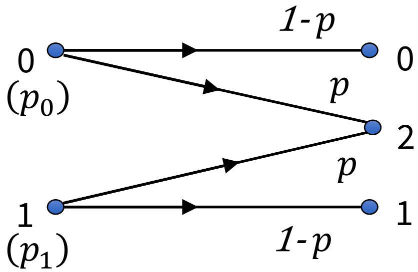
$$P_{\boldsymbol{y}/\boldsymbol{x}} = \begin{bmatrix} p_{M'-1\,/M-1} & \cdots & p_{M'-1\,/0} \\ \vdots & \ddots & \vdots \\ \underbrace{p_{0\,/M-1}}_{\triangleq P_{\boldsymbol{y}/\boldsymbol{x}(M-1)}} & \cdots & \underbrace{p_{0\,/0}}_{\triangleq P_{\boldsymbol{y}/\boldsymbol{x}(0)}} \end{bmatrix}$$

- $P_{\boldsymbol{y}/\boldsymbol{x}}$ is essentially just a table of the probabilities $p_{y/x}$.
- y has a discrete distribution.

Example 2: Binary Erasure Channel (BEC)

$M = 2 \;\; \& \;\; M' = 3$

$$\boldsymbol{P}_{y/x} = \begin{bmatrix} p & p \\ 1-p & 0 \\ 0 & 1-p \end{bmatrix}$$



$P_e = p$

- ML decoder always correct on 0→ 0 or 1 → 1, but 2?
- An **erasure** channel self reports it cannot decide.
  - ➢ This really corresponds to an "inner channel" like AWGN when y is on/close to the decision boundary.

Some Properties of Transition Matrix

- Convenient description
- Can be helpful for matlab/other implementations/simulations

1. **unit column sum** - Each column sums to unity:

$$1 = \sum_{j=0}^{M'} p_{j/i} = [\mathbf{1}]^* \boldsymbol{p_{y/x}}(i) \; \forall \; i = 0, ..., M-1 \; . \tag{1.464}$$

2. **weighted row-sum is** $p_{\boldsymbol{y}}(j)$ - Each row sums to the corresponding $\boldsymbol{y}$-value's probability:

$$p_{\boldsymbol{y}}(j) = \sum_{i=0}^{M} p_{j/i} \cdot p_{\boldsymbol{x}}(i) \; \forall \; j = 0, ..., M'-1 \; . \tag{1.465}$$

Equivalently, If $\boldsymbol{p_y}$ and $\boldsymbol{p_x}$ are row vectors that stack $\boldsymbol{y}$ probability values $\boldsymbol{p_y} \triangleq [p_{\boldsymbol{y}}(M'-1)...p_{\boldsymbol{y}}(0)]^*$ and $\boldsymbol{p_x} = [p_{\boldsymbol{x}}(M-1)...p_{\boldsymbol{x}}(0)]^*$ respectively, then there is an input/output matrix-multiply relation

$$\boldsymbol{p_y} = P_{\boldsymbol{y}/\boldsymbol{x}} \cdot \boldsymbol{p_x} \; . \tag{1.466}$$

3. **Joint Probability Distribution** - The joint distribution is

$$P_{\boldsymbol{y},\boldsymbol{x}} = P_{\boldsymbol{y}/\boldsymbol{x}} \cdot \text{Diag}\{\boldsymbol{p_x}\} \quad . \tag{1.467}$$

4. **Á Posteriori Distribution** - The à priori distribution is

$$P_{\boldsymbol{x}/\boldsymbol{y}} = \left[\text{Diag}\left\{\boldsymbol{p_y}\right\}\right]^{-1} \cdot \underbrace{P_{\boldsymbol{y}/\boldsymbol{x}} \cdot \text{Diag}\{\boldsymbol{p_x}\}}_{P_{\boldsymbol{y},\boldsymbol{x}}} \; . \tag{1.468}$$

5. **ML Detector** - An ML detector selects for any specific received DMC channel output $\boldsymbol{y} = j$ or thus row j:

$$\hat{\boldsymbol{x}}_i = \hat{i} = \arg\left\{\max_{i\in\{0,...,M-1\}} [p_{j/i}]\right\} \quad , \tag{1.469}$$

the index of row j's largest element. The ML decision region $\mathcal{D}_i$ is the set of all row indices $\{j\}$ for which element i maximizes those rows' probabilities in $P_{\boldsymbol{y}/\boldsymbol{x}}$.

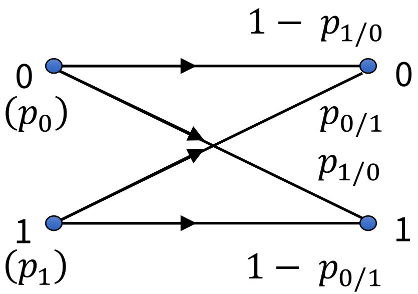
6. **MAP Detector** - An MAP detector selects for any specific received DMC channel output $y = j$:

$$\hat{\boldsymbol{x}}_i = \hat{i} = \arg\left(\max_{i\in\{0,...,M-1\}} \left\{ [P_{\boldsymbol{y}/\boldsymbol{x}} \cdot \text{Diag}\,(\boldsymbol{p_x})](j,i) \right\}\right) \quad . \tag{1.470}$$

The MAP decision region $\mathcal{D}_i$ is the set of all row indices $\{j\}$ for which element i maximizes those rows' probabilities in $P_{\boldsymbol{y}/\boldsymbol{x}} \cdot \text{Diag}\{\boldsymbol{p_x}\}$.

Binary Asymmetric Channel (BAC)

$$P_{y/x} = \begin{bmatrix} 1 - p_{1/0} & p_{0/1} \\ p_{1/0} & 1 - p_{0/1} \end{bmatrix}$$



$$P_e = p_{0/1} \cdot p_1 + p_{1/0} \cdot p_0$$

- **BAC** can model optical (fiber) transmission as well as some disk channels.
  - Nonlinear effects or data-dependent noise effects can cause the asymmetry.

Symmetric DMC $M = M'$

Definition 1.4.2 [**Symmetric Channel**] *A* **symmetric channel** *has MAP-detector P_e that is independent of input distribution.*

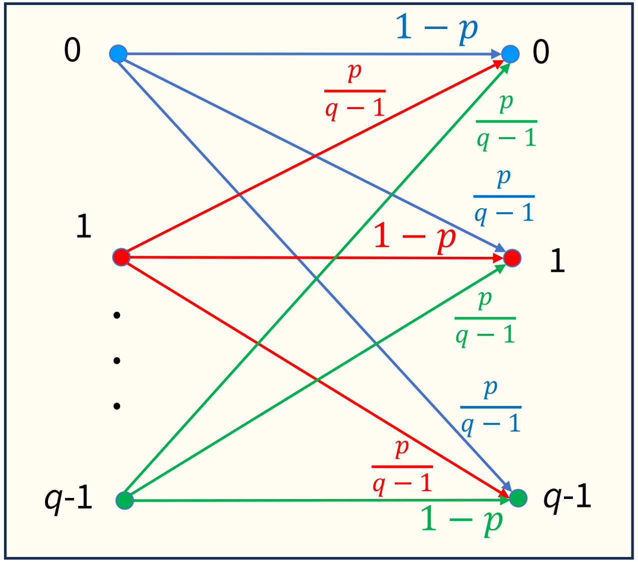
Theorem 1.4.1 [**Symmetric DMC Properties**] *The following statements are equivalent:*

1. The DMC is symmetric.
2. The MAP and ML dectectors' error probability P_e is invariant to input distribution $\boldsymbol{p_x}$.
3. Any column of $P_{\boldsymbol{y}/\boldsymbol{x}}$ is a permutation of another column.
4. For any 1-to-1 self-reversible permutation $\pi = \pi^{-1}$ on discrete $\boldsymbol{y}$, then $p_{\boldsymbol{y}/\boldsymbol{x}}(i) = P_{\pi(\boldsymbol{y})/\boldsymbol{x}}(i')$ for some $i' \neq i$.

involution

- SDMC is useful as the channel for outer code designs.
  - There is already an "inner detector" (example is ML for symbols on AWGN).
  - BSC and BEC are symmetric DMCs.

Example: The q-ary Symmetric Channel



- This is used with “bytes” (blocks) of inner-channel detected bits.
  - $q > 2$ codes can be much more powerful than best binary codes.
- This model can have erasures in various modifications.
- Typically models an “inner channel” for application of outer cyclic codes over finite field (will see in Lecture 11).

End Lecture 4