



STANFORD

Lecture 3

Modulation Types

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JOHN M. CIOFFI

Hitachi Professor Emeritus (recalled) of Engineering

Instructor EE379A – Winter 2026

Announcements & Agenda

■ Announcements

- Homework due tomorrow
- Solutions distributed Friday, or earlier if no late.
 - Anything after that is 20%.
- PS2 is due 1/20

■ Problem Set 2 = PS2 due Tues Jan20 at 17:00

- | | | |
|----|------|-----------------------------|
| 1. | 1.14 | SQ odd-b QAM constellations |
| 2. | 1.19 | shaping gain |
| 3. | 1.22 | basic QAM design |
| 4. | 1.34 | hexagonal constellation QAM |
| 5. | 1.35 | Baseband equivalents |
| 6. | 1.38 | 2-tap channel |

■ Today (1.3)

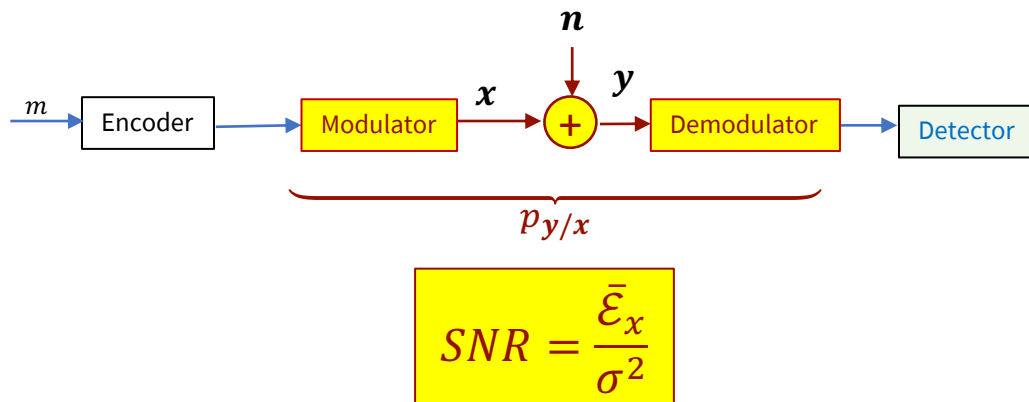
- Simple AWGN Modulation and SNR
- PAM
- QAM
- Filtered AWGN Channels



Simple AWGN Modulation & SNR

Section 1.3.3

AWGN Channel and SNR

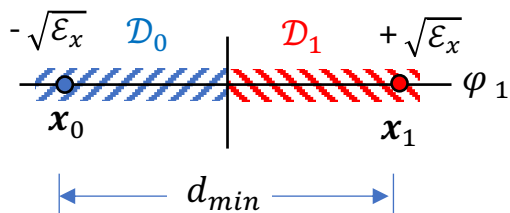


- The numerator and denominator must have **same number of dimensions!**
 - Automatic with barred quantities (σ^2 is the noise energy/dimension for any AWGN at MF output = power-spectral density, 2-sided).
- A constellation's SNR maps to P_e (like 10^{-6}) through the (possibly scaled) Q-function argument's square root.
- **margin** = the excess SNR above that required to meet a P_e target.
- **energy/bit** $\epsilon_b = \epsilon_x / b$; **caution, this measure confuses issues when $\bar{b} \geq 1$** (SNR works everywhere).
 - Energy quadruples with each additional bit/dim when $\bar{b} \geq 1$, an exponential growth.
 - So normalizing by linear-factor $1/b$ can create inconsistencies.
 - When $\bar{b} \leq 1$, the energy growth is more consistently linear, so this measure *only then* is self-consistent.



Binary Antipodal (NRZ) – simple Binary

- 2 constellation points $\pm\sqrt{\mathcal{E}_x}$ best one-dimensional binary code

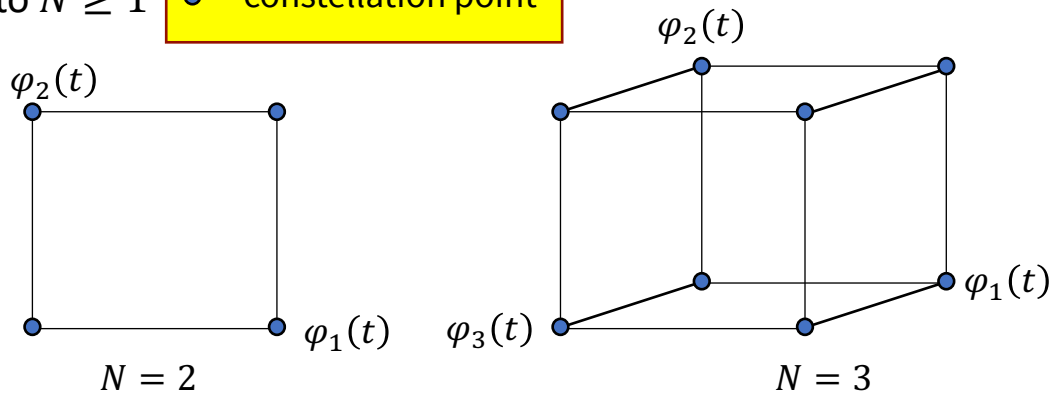


$$\sqrt{SNR} = \frac{\sqrt{\mathcal{E}_x}}{\sigma} = \frac{d_{min}}{2\sigma}$$

$$P_e = \bar{P}_e = 1 \cdot Q(\sqrt{SNR})$$

- Extend to $N \geq 1$

● = constellation point



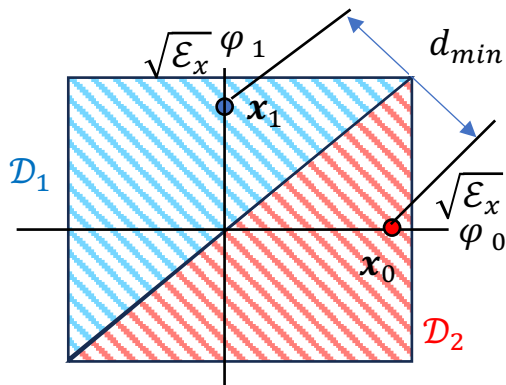
$$\bar{P}_e \leq 1 \cdot Q(\sqrt{SNR})$$

(bound when $N \geq 2$)



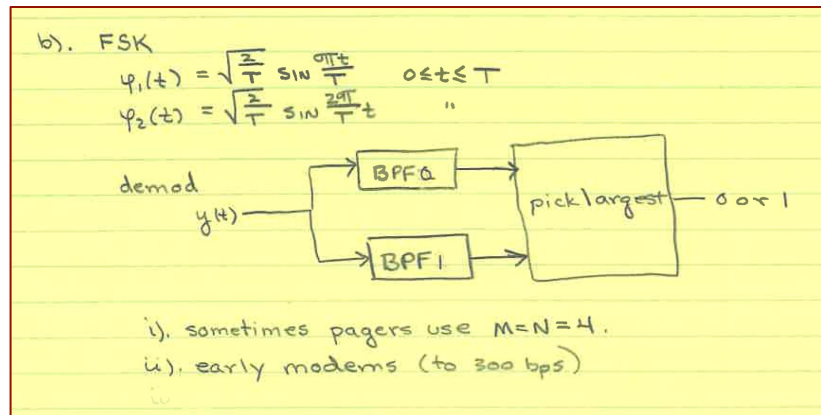
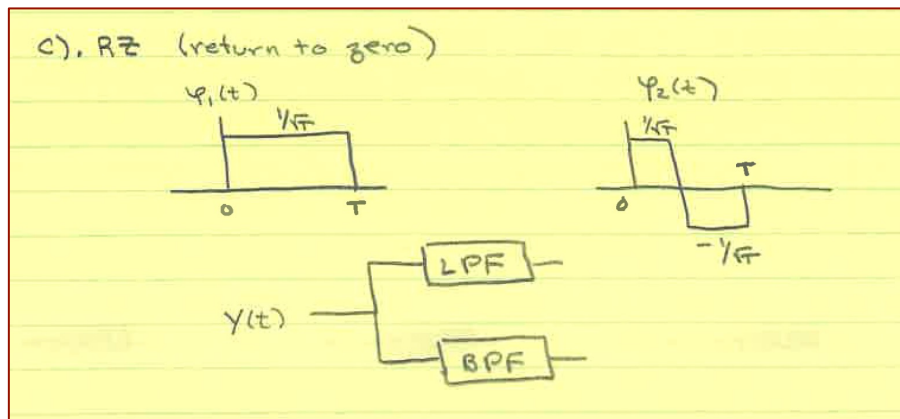
Binary-Orthogonal Constellations

- 2 constellation points $\pm\sqrt{\mathcal{E}_x}$ (antipodal) is the best one-dimensional binary code, but how about:



$$\frac{d_{min}}{2\sigma} = \frac{\sqrt{\mathcal{E}_x}}{\sqrt{2}\sigma}$$

$$P_e = \bar{P}_e = 1 \cdot Q\left(\frac{\sqrt{\mathcal{E}_x}}{\sqrt{2}\sigma}\right)$$



Block-Orthogonal Constellations

- Extending to more dimensions is wasteful of system resources (temporal in particular, time or freq).

$$x_m(t) = \sqrt{\mathcal{E}_x} \cdot \varphi_m(t)$$

$$\lim_{N \rightarrow \infty} \bar{\mathcal{E}}_x = 0$$

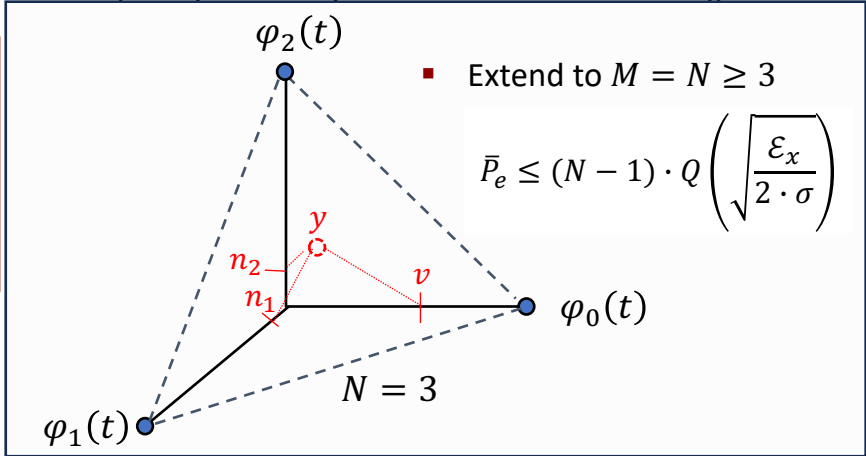
$$SNR = \frac{\mathcal{E}_x}{N \cdot \sigma^2}$$

Not good,
but simple MAP/ML.

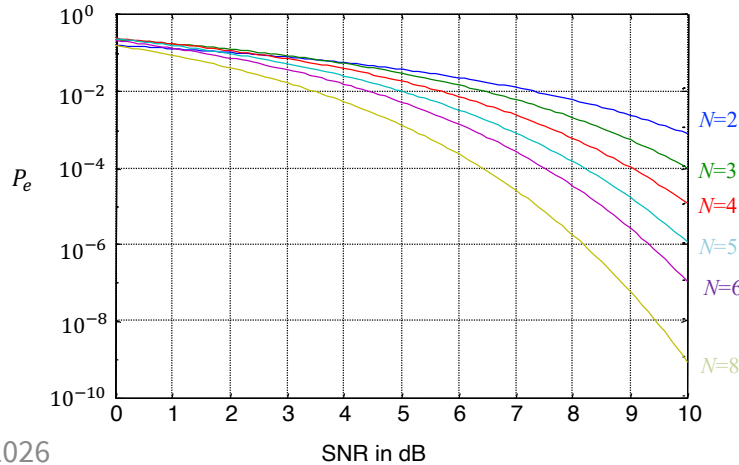
$$P_{c/0, y_0=v} = P\{n_i \leq v, \forall i \neq 0\}$$

$$= \prod_{i=1}^{N-1} P\{n_i \leq v\}$$

$$= [1 - Q(v/\sigma)]^{N-1}$$



$$P_e = 1 - \int_{-\infty}^{\infty} (\sqrt{2\pi\sigma^2})^{-1} \cdot e^{-\frac{1}{2\sigma^2}(v - \sqrt{\mathcal{E}_x}x)^2} \cdot [1 - Q(v/\sigma)]^{N-1} dv$$

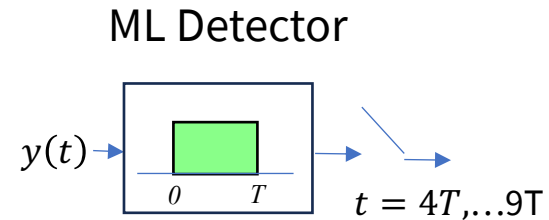
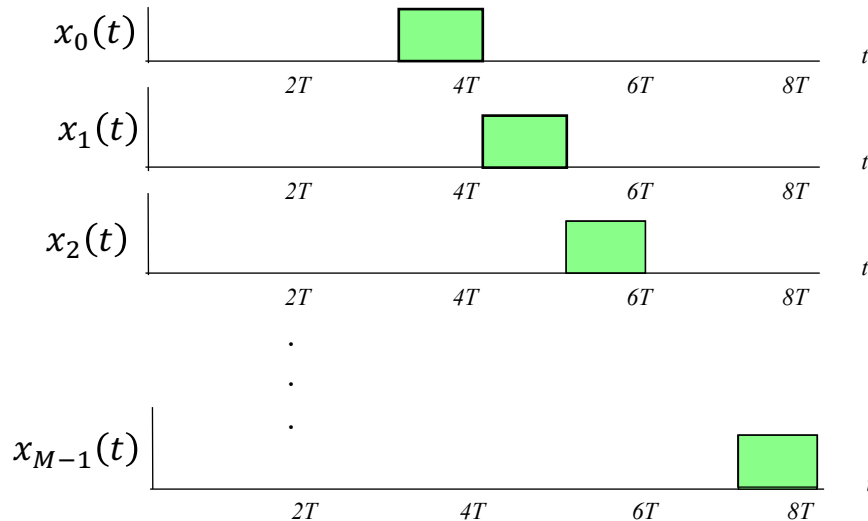


$$\frac{\mathcal{E}_x}{\sigma^2} = N \cdot SNR$$

So energy increases at fixed SNR with N , while \bar{b} decreases.



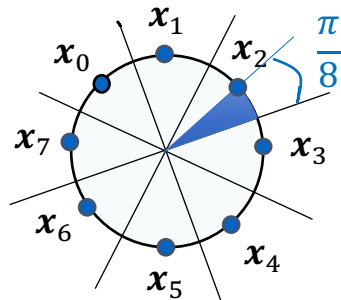
Example: Pulse Position Modulation (PPM)



- For instance, radar and lidar where the message is the delay/distance (so position).
- Visible Light Communication (VLC) systems (inside rooms, “Li-Fi”) – wide bandwidth (10’s Mbps)
 - IEEE 802.15.7.
- Similarly, pulse-duration modulation, where detector accumulates receiver input energy.



Phase-Shift Keying – Circular Constellations



$$radius = \sqrt{\mathcal{E}_x}$$

- PSK is shown here with $M = 8$, but generally M is any positive integer.
 - All points equal energy \mathcal{E}_x – this can simplify energy driver/receiver-amplifier circuits' implementation and overall energy consumption.
- Minimum distance is $d_{min} = 2 \cdot \sqrt{\mathcal{E}} \cdot \sin \frac{\pi}{M}$.
- Error Prob is $P_e < 2 \cdot Q(\sqrt{SNR} \cdot \sin \frac{\pi}{M})$.

Often used in satellite transmission
e.g., LEO QPSK, 8PSK (some 16 QAM)
 $1/T = 10 - 400$ MHz, roughly
carriers are typically above 10 GHz

Low peak-to-average can simplify design.

Performance degrades (low d_{min} for given energy) when $M > 4$.

BPSK ($M=2$) basically wastes a dimension
(although 3 dB larger distance than QPSK)

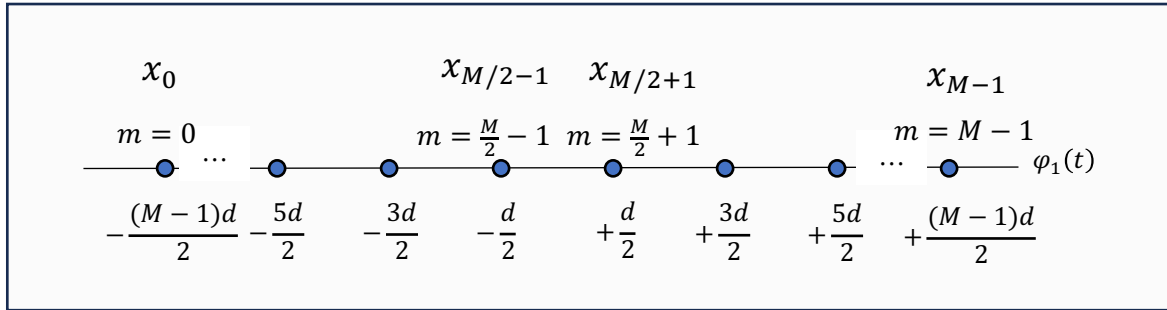


Pulse Amplitude Modulation (PAM)

Section 1.3.4.1

M'ary PAM (Sec 1.3.4.1)

- **Pulse Amplitude Modulation (PAM)** has $M = 2^b$ symbol values equally spaced in $N = 1$ dimension.



$$\varphi_1(t) = \frac{1}{\sqrt{T}} \cdot \text{sinc}\left(\frac{t}{T}\right)$$

- PAM symbol energy is

$$\mathcal{E}_x = \frac{M^2 - 1}{12} \cdot d^2 \quad d = \sqrt{\frac{12 \cdot \mathcal{E}_x}{M^2 - 1}}$$

- PAM bits/dim are

$$b = \frac{1}{2} \cdot \log_2 \left(1 + 12 \cdot \frac{\mathcal{E}_x}{d^2} \right)$$

- PAM error prob is

$$P_e = \underbrace{2 \cdot \left(1 - \frac{1}{M}\right)}_{N_e} \cdot \underbrace{Q\left(\sqrt{\frac{3 \cdot \text{SNR}}{M^2 - 1}}\right)}_{Q(d/2\sigma)}$$



PAM Table for $Pe=1e-6$

$b = \bar{b}$	M	$\frac{d_{min}}{2\sigma}$ for $\bar{P}_e = 10^{-6} \approx$ $2Q\left(\frac{d_{min}}{2\sigma}\right)$	$SNR =$ $\frac{(M^2-1) \cdot 10^{1.37}}{3}$	SNR increase = $\frac{M^2-1}{(M-1)^2-1}$
1	2	13.7dB	13.7dB	—
2	4	13.7dB	20.7dB	7dB
3	8	13.7dB	27.0dB	6.3dB
4	16	13.7dB	33.0dB	6.0dB
5	32	13.7dB	39.0dB	6.0dB

Table 1.2: **PAM constellation energies.**

- **PAM** is often cited as needed **6dB/bit** (last column, large b).
- **PCIe 6.0** Uses PAM4 with $1/T \approx 28, 44, 56, 112, 228$ GHz (for 56, 88, 112, 228, and 456 Gbps)
 - Per wire (PCIe allows up to 8-16 wires in parallel).



Matlab Commands

- Modulator is **pammod.m**

```
>> real(pammod(0:3,4)) % = -3 -1 1 3 - lists the 4-PAM outputs (d=2 default)
>> real(pammod(0:3,8)) % = -7 -5 -3 -1 - lists first four 8-PAM outputs
>> real(pammod([3 0 7],8)) % = -1 -7 7 - some message sequence to 8-PAM
>> d=1;
>> (d/2)*real(pammod([3 0 7],8)) % = -0.5000 -3.5000 3.5000 - change d
>> randi(4,1,5) % = 2 1 4 1 3 - random (uniform) messages
>> (d/2)*real(pammod((randi(4,1,5)-1),8)) % Put it all together, 5 successive 8-PAM symbols
= 0.5000 1.5000 -2.5000 0.5000 1.5000
```

- ML detector is **pamdemod.m**

```
>> message=randi(8,1,10000)-1; %1 x 10k ∈ [1: 8]
>> x=real(pammod(message,8)); % create PAM
>> SNR=23; % dB
>> Ex=63/3; % from L3:11
>> sig2=10^(-SNR/10)*Ex; % find noise
>> n=sqrt(sig2)*randn(1,10000); % make noise
>> y=x+n; % AWGN
>> xhat = pamdemod(y,8);
>> sum(xhat ~= message) % = 16
>> Pe=ans/10000 % = 0.0016
> 1.5*qlfunc(sqrt((3/63)*10^(SNR/10))) % = 0.0015
```

Can run this with longer data sequences, and test several SNR.
Generate P_e versus SNR

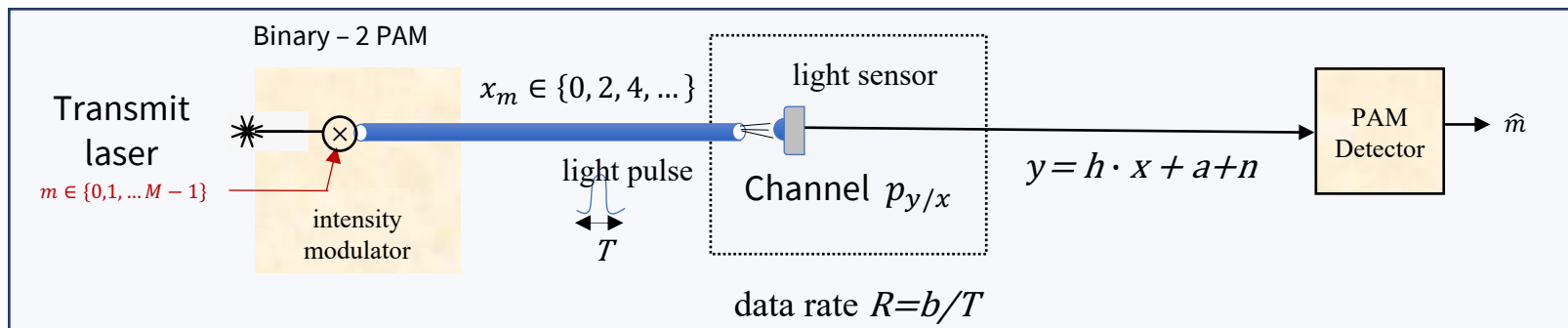


Avoid matlab “awgn.m” program – can be confusing on “SNR”

```
>> real(pammod([3 0 7],8,0,'gray')) % = -3 -7 3 - the 0 is initial phase
% gray code bit-mapping to constellation differs only by 1 bit in adjacent points, so then
Pb = Pe
>> bits = [ 0 1 3 2 6 7 5 4];
% 000 001 011 010 110 111 101 100
>> real(pammod(bits,8,0,'gray')) % = -7 -5 -3 -1 1 3 5 7
>> pamdemod(real(pammod(bits,8,0,'gray')),8) % = 0 1 2 3 4 5 6 7
>> pamdemod(real(pammod(bits,8,0,'gray')),8,0,'gray') % = 0 1 3 2 6 7 5 4
```



Example: Noncoherent fiber transmission



- Modulates light intensity – typically all positive so effectively adds a constant $a = \frac{1-M}{2} d$ (& $x \geq 0$).
 - The average energy additional loss caused by adding the non-zero mean a is $\frac{4 \cdot M^2 - 6 \cdot M + 2}{M^2 - 1}$.
 - For $M = 2, 4, 8, \dots$ this is 3, 4.5, 5.2, ... 6 dB.
- The light's wavelength is part of modulation, so not relevant to this **Offset-PAM**.



Some interconnect uses (Copper or Fiber)

- **PCIe (Peripheral Component Interconnect Express)** has $\frac{1}{T} = 32 \text{ GHz}$; $b = 4$ (4PAM) $R = 64 \text{ Gbps}$:
 - PCIe can have up to 16 lanes (each at this speed) – 128 GBYTES/sec,
 - PCIe helps connect computer processor to peripheral components,
 - PCIe is typically (short) copper wires.
 - Both falling/rising edges of a 16 GHz clock determine the symbol transmission times
- **GDDR6 (Graphics Double Data Rate)** - $\frac{1}{T} = 12 \text{ GHz}$; $b = 4$ (4PAM) $R = 24 \text{ Gbps}$:
 - GDDR6 is a memory interface (version 6 went to 4 PAM) that is used in gaming,
 - GDDR6 is also for copper wires.
 - The “double data rate” refers also here to using both falling/rising edges of a 6 GHz clock.
- **Ethernet 100 and 200 Gbps (noncoherent fiber)**:
 - Fiber - $\frac{1}{T} = 50 \text{ or } 100 \text{ GHz}$; $b = 4$ (4PAM) $R = 100 \text{ or } 200 \text{ Gbps}$,
 - 200 Gbps is relatively new – just entering market.
- **Multiwavelength combinations (ITU standards)** use $\frac{1}{T} \approx (\text{up to}) k \cdot 12.5 \text{ GHz}$ (ITU G.964.1) with PAM4 (contemplating PAM8 and PAM16), with $k = 1, 2, \dots, 8$ (8 probably is part of a DMT/OFDM system, see 379B); noncoherent fiber.
- **GPON (broadband fiber access, G.989.2 with G.sup64)** – 4PAM with $\frac{1}{T} \approx 25, 50 \text{ GHz}$.
 - Noncoherent fiber

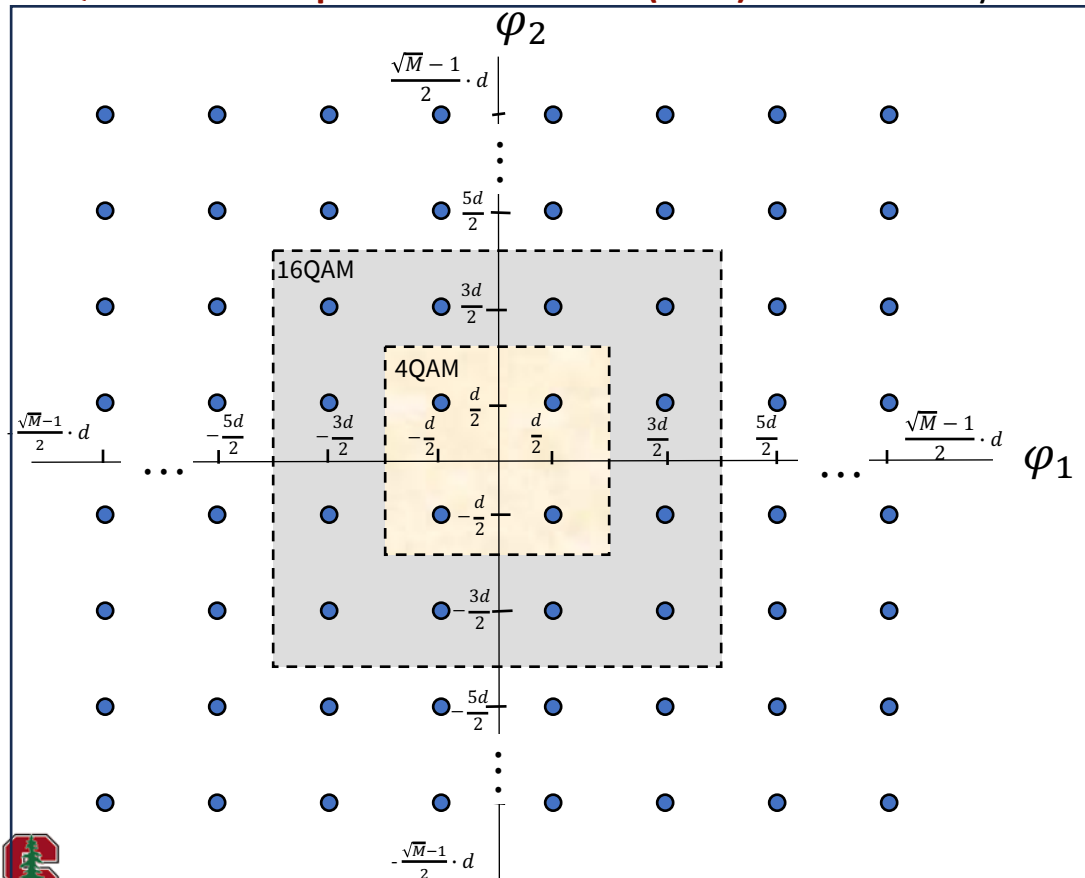


Quadrature Amplitude Modulation (QAM)

[Section 1.3.4.2](#)

M'ary QAM

- Quadrature Amplitude Modulation (PAM) has $M = 2^b$ symbol values in $N = 2$ dimension, “squares PAM”



Cartesian product of 2 PAMS

$$\varphi_1(t) = \sqrt{\frac{2}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right) \cdot \cos \omega_c t$$

$$\varphi_2(t) = -\sqrt{\frac{2}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right) \cdot \sin \omega_c t$$

- QAM symbol energy

$$\mathcal{E}_x = \frac{M-1}{6} \cdot d^2 \quad \bar{\mathcal{E}}_x = d^2 \left(\frac{M-1}{12} \right)$$

- QAM bits/sym are

$$b = \log_2 \left(1 + 6 \cdot \frac{\mathcal{E}_x}{d^2} \right)$$

- QAM error prob is

$$P_e = \underbrace{4 \cdot \left(1 - \frac{1}{\sqrt{M}} \right)}_{N_e} \cdot \underbrace{Q \left(\sqrt{\frac{3 \cdot \text{SNR}}{M-1}} \right)}_{Q(d/2\sigma)}$$

QAM Table for $P_e=1e-6$

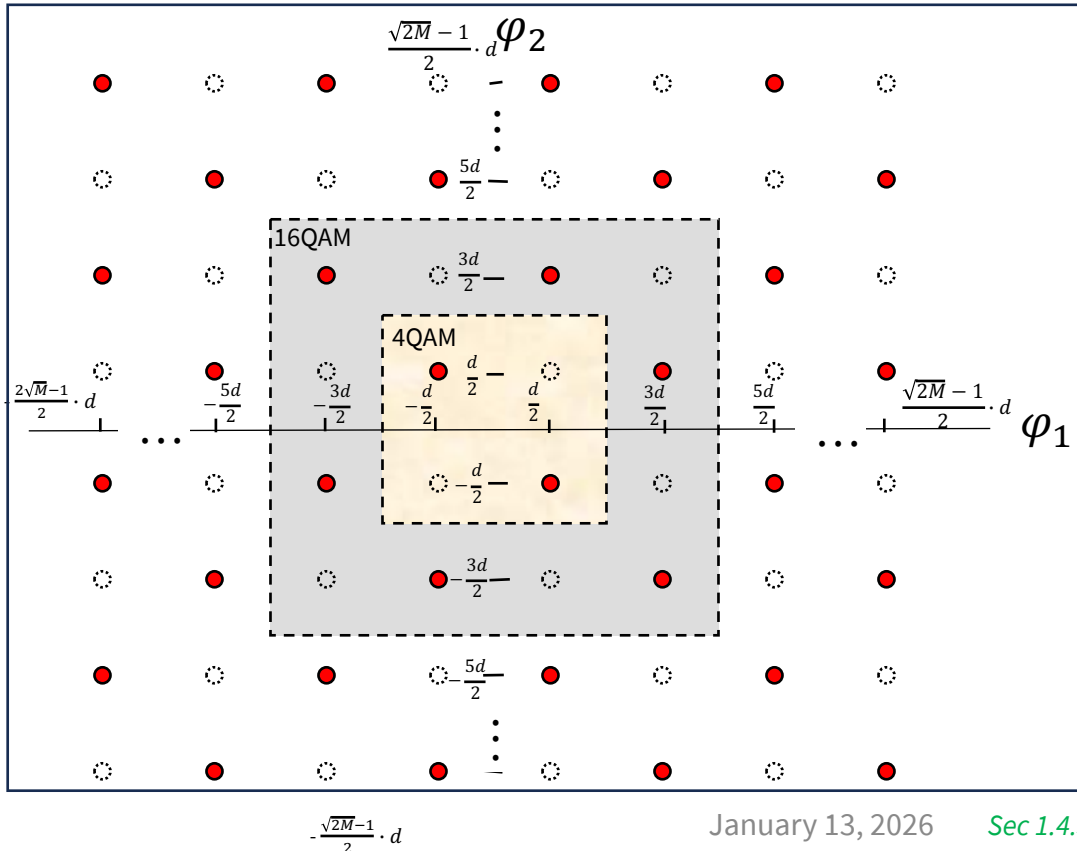
$b = 2\bar{b}$	M	$\frac{d}{2\sigma}$ for $\bar{P}_e = 10^{-6} \approx 2Q\left(\frac{d_{\min}}{2\sigma}\right)$	SNR = $\frac{(M-1) \cdot 10^{1.37}}{3}$	SNR increase = $\frac{M-1}{(M-1)-1}$	dB/bit
2	4	13.7dB	13.7dB	$M_{QAM} = M_{PAM}^2$	—
4	16	13.7dB	20.7dB	7.0dB	3.5dB
6	64	13.7dB	27.0dB	6.3dB	3.15dB
8	256	13.7dB	33.0dB	6.0dB	3.0dB
10	1024	13.7dB	39.0dB	6.0dB	3.0dB
12	4096	13.7dB	45.0dB	6.0dB	3.0dB
14	16,384	13.7dB	51.0dB	6.0dB	3.0dB

- **QAM** is often cited as needed **3dB/bit** (last column, large b) – same as PAM, but over 2 dimensions
- Some wireless (satellite 16QAM, cellular 1G,2G,3G, Wi-Fi 802.11b, early US digital TV)
- Wireline – coherent fiber, $1/T \approx k \cdot 32$ GHz or $1/T \approx k \cdot 12.5$ GHz $k = 1,2,3,4 \dots 8$ so far



SQ QAM for odd b ?

- Most typical today is “every other point” from $b + 1$ size constellation
 - Use SQ QAM formulas, but increase $d_{min} \rightarrow \sqrt{2} \cdot d$ - below



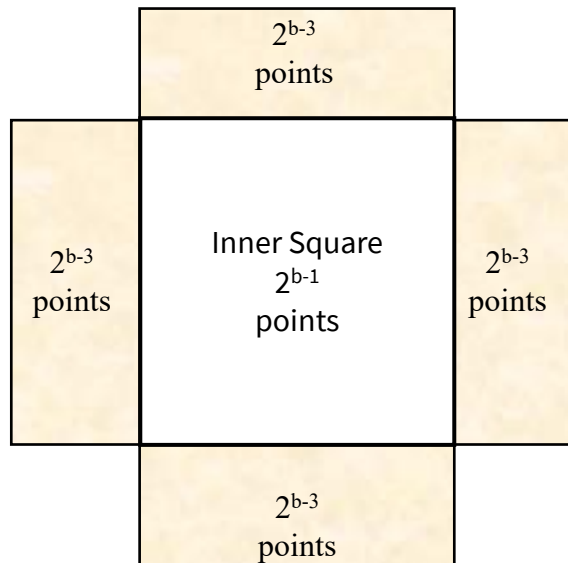
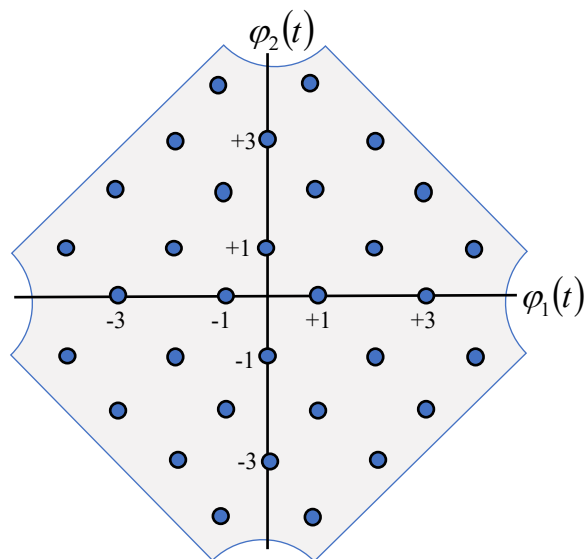
PS2.1 (1.14) - your turn to prove formulas

$$P_e \approx 4 \cdot \left(1 - \frac{3}{2 \cdot M}\right) \cdot Q \left[\sqrt{\frac{6 \cdot SNR}{2^{b+1} - 1}} \right] \quad b = 1, 3, 5, \dots$$



CR constellations Slightly Better for odd b QAM

- More symmetrically removes high-energy corner points



$$\varepsilon_x = \frac{d^2}{6} \cdot \left(\frac{31}{32} M - 1 \right)$$

$$P_e = \underbrace{4 \cdot \left(1 - \frac{1}{\sqrt{2M}} \right)}_{N_e} \cdot \underbrace{Q \left(\sqrt{\frac{3 \cdot \text{SNR}}{\frac{31}{32} M - 1}} \right)}_{d/2\sigma}$$

- Used on some (of many carriers) in wireline systems (xDSL, G.fast, G.mgfast, etc)



Some interconnect uses (Copper or Fiber)

- **Satellite TV** –
 - Carriers between 12.2 and 12.7 GHz downlink
 - 17.3 and 17.8 GHz uplink
 - Symbol rate is 19.151 MHz (fit into 24 MHz wide channels)
 - SQ QAM only, so data rates are multiples of 38.302 Mbps

- **Coherent Fiber** – 16 **QAM** at $\frac{1}{T} = 130 \text{ GHz} \rightarrow 400 \text{ Gbps}$ of actual information (some code overhead).
 - The wavelength's corresponding frequency (c/λ) is the carrier
 - Coherent fiber uses two polarizations per wavelength (so 2 16-QAMs that are spatially orthogonal, so 2x2 channel).
 - Called Coherent DSP (two former students of this class P. Voois/N. Swenson started the company, ClariPhy that began this whole passage to QAM in coherent fiber – now part of Marvell).



Matlab Commands

- Modulator is **qammod.m**

```
>> reshape(qammod(0:3,4),2,2) %=
-1.0000 + 1.0000i  1.0000 + 1.0000i
-1.0000 - 1.0000i  1.0000 - 1.0000i
```

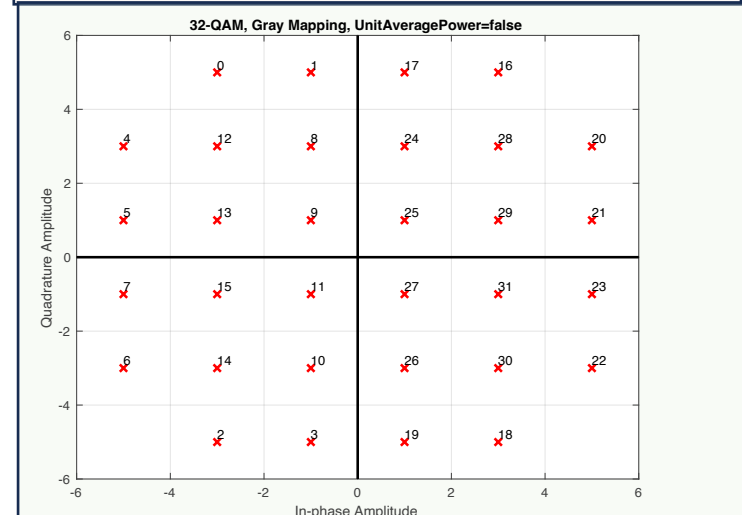
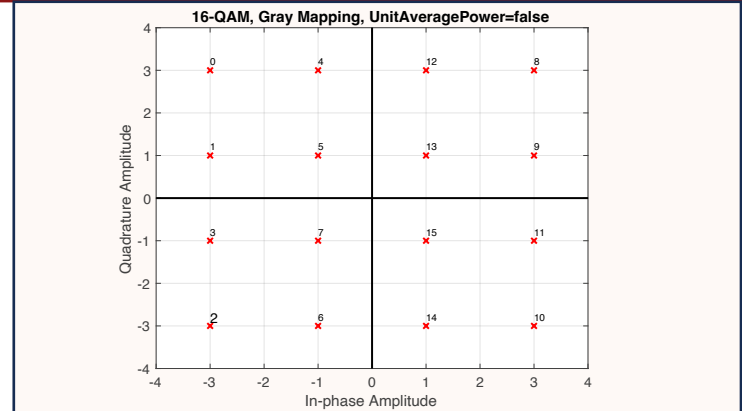
```
>> reshape(qammod(0:15,16,'plotconstellation',1),4,4) %=
-3.0000 + 3.0000i  -1.0000 + 3.0000i  3.0000 + 3.0000i  1.0000 + 3.0000i
-3.0000 + 1.0000i  -1.0000 + 1.0000i  3.0000 + 1.0000i  1.0000 + 1.0000i
-3.0000 - 3.0000i  -1.0000 - 3.0000i  3.0000 - 3.0000i  1.0000 - 3.0000i
-3.0000 - 1.0000i  -1.0000 - 1.0000i  3.0000 - 1.0000i  1.0000 - 1.0000
```

```
>>qammod(0:31, 32,'plotconstellation',1); % produces 32CR
```

Odd bits 5 or greater → cross constellation

However 8 CR is horrible and loses 0.8 dB w.r.t. 8SQ

8SQ is not well handled by matlab (**see for yourself**: matlab's constellation from qammod for M=8)



Matlab Commands

- ML detector is **qamdemod.m**

Defaults to Gray Code

```
rng(7)
message=randi(16,1,10000)-1;
x=qammod(message,16);
SNR=17;
Ex=10;
N0=10^(-1.7)*Ex;
n=sqrt(N0/2)*randn(1,10000)+i*sqrt(N0/2)*randn(1,10000);
y=x+n;
xhat = qamdemod(y,16);
sum( xhat ~= message)
= 25
ans/10000 = .0025
>> 3*qfunc(sqrt(0.2*10^(1.7)))
ans = 0.0023
```

errrate=comm.ErrorRate;

```
for indx=1:100
message=randi(16,1,10000)-1;
x=qammod(message,16);
n=sqrt(N0/2)*randn(1,10000)+i*sqrt(N0/2)*randn(1,10000);
y=x+n;
xhat = qamdemod(y,16);
errstats = errrate(xhat',message');
end
errstats(1)

ans = 0.0023 % perfect match!
```



Forney's Gap Approximation – PAM/QAM

- What Q-function argument gives $(\bar{P}_e =) 10^{-6}$? (assume $\bar{N}_e = 2$)

$$\frac{3}{2^{2\bar{b}} - 1} \cdot SNR = 10^{1.38}$$

13.8 dB ??

>> $20 \cdot \log_{10}(\text{qfuncinv}(1e-6/2)) = 13.7891$

- Solve for \bar{b} to get **Forney's Gap Formula**:

- $\bar{b} = \frac{1}{2} \cdot \log_2 \left(1 + \frac{SNR}{\Gamma} \right)$, where

- $\Gamma = 8.8$ dB (for both PAM and QAM), and
- $\Gamma = 9.5$ dB for $\bar{P}_e = 10^{-7}$.
- Gap is largely independent of $\bar{b} > 0.5$; we'll see this applies to most good codes built on PAM/QAM also.

- This looks like a very famous formula (Chapter 2, we'll see),

- where $\Gamma = 1$ (0 dB).

- That, is, the maximum reliably decodable data rate on AWGN, **Shannon's capacity**.

- The gap measures SNR reduction (in dB) relative to this capacity (here for "uncoded" PAM/QAM).



Examples

2. Examples

a. SNR = 13.5 dB? $P_e = 10^{-6}$

$$\bar{b} = \frac{1}{2} \log_2 \left(1 + \frac{10^{1.35}}{10^{.88}} \right) = 1 \quad (\text{we knew this})$$

b. SNR = 24.4 dB $P_e = 10^{-7}$

$$\bar{b} = \frac{1}{2} \log_2 \left(1 + 10^{2.44-.95} \right) = 2.5$$

$$2\bar{b} = b = 5 \quad \underline{32CR QAM}$$

5 bits/Hz (4 PAM, not enough SNR for 8PAM)

c. SNR = 44.7 dB 10^{-6}

$$\bar{b} = \frac{1}{2} \log_2 \left(1 + 10^{4.47-.88} \right) = 6$$

64 PAM or 4096 SQ QAM

- Best to date? (no MIMO dimensions)
 - QAM, $b = 15$ bits/Hz.
 - Bits/Hz = $2\bar{b}$.

- Most useful codes also based on sequences of QAM/PAM symbols.
 - Their gaps are also constant for $2\bar{b} > 1$.
 - The good ones have $\Gamma \rightarrow 0$ dB (or maybe in practice more like 1 dB).

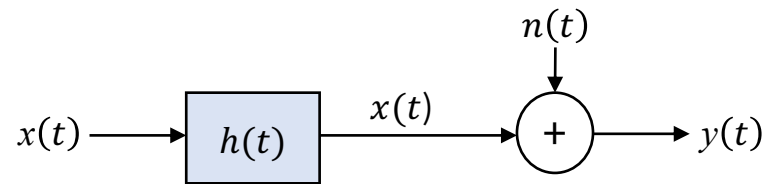


Filtered AWGN Channels

Section 1.3.7

Real channels don't pass all frequencies

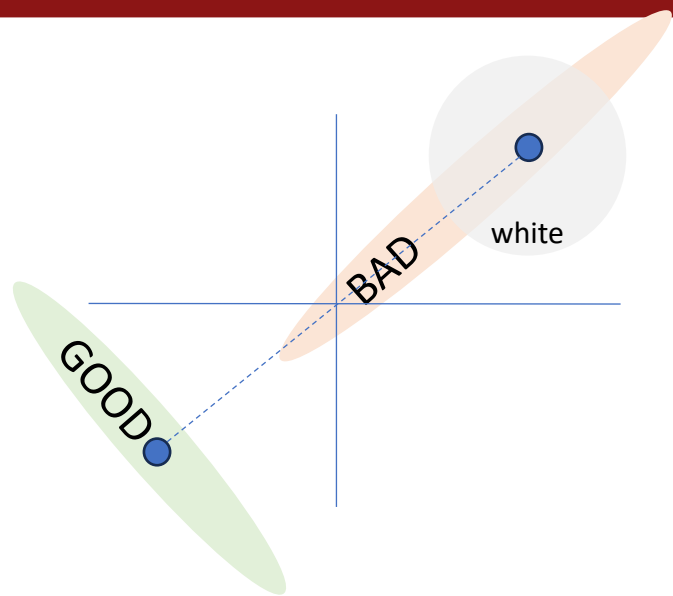
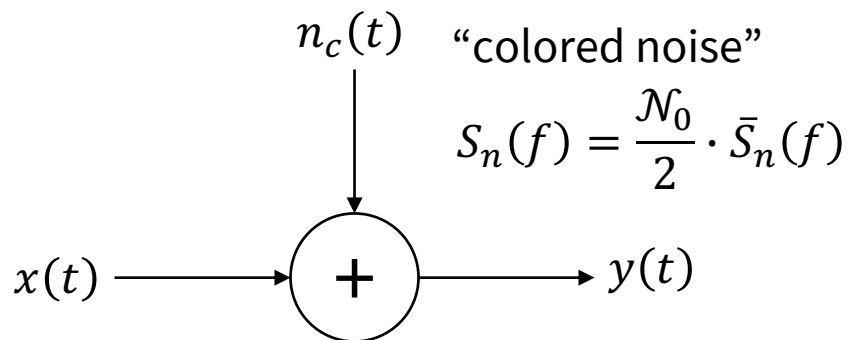
- The filtered AWGN has linear filter $h(t)$.



- Successive symbols get stretched and may overlap (intersymbol interference).
 - Can't go too fast ...
- Correlated (“colored”) noise is equivalent to filtered AWGN next slides



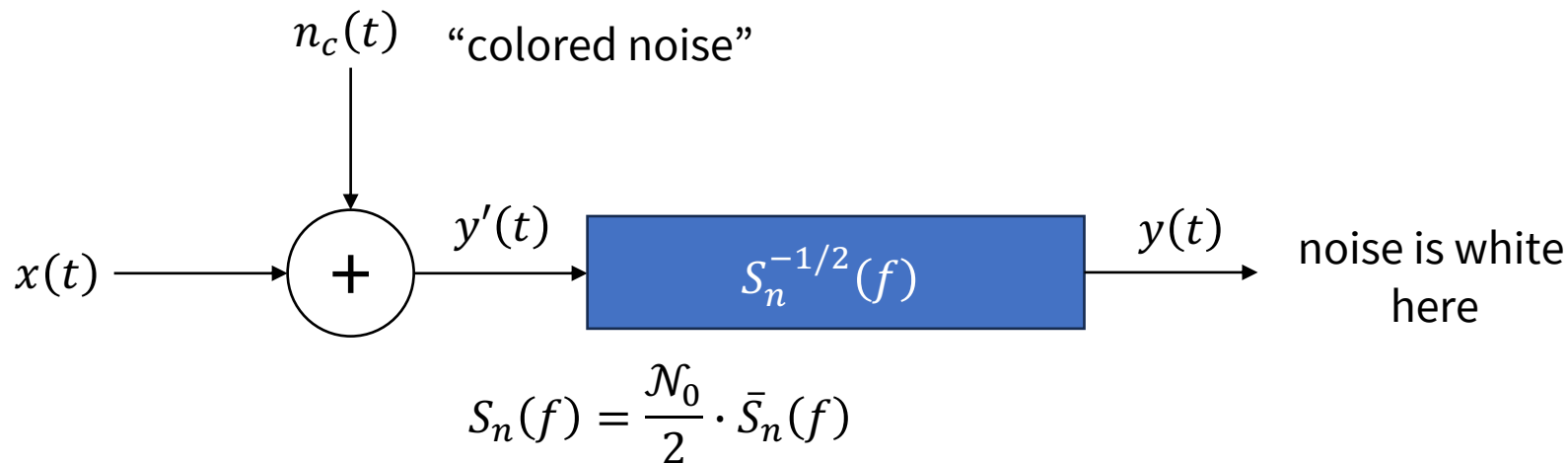
Colored noise



- Noise is not “white” (not flat PSD) – power spectral density $\frac{\mathcal{N}_0}{2} \cdot \bar{S}_n(f)$
- What is problem with this?
 - The MAP/ML detector no longer “picks the closest point”
- See Examples in Section 1.3.7.2 / 3



Noise Whitening



- 1-to-1 reversible transformation that whitens noise.
 - Loses nothing by reversibility theorem.
- And thus creates a filtered AWGN $H(f) = \bar{S}_n^{-1/2}(f)$.



But WAIT!

- This filter only exists, and is 1-to-1 causal and causally invertible, **IF**

Theorem 1.3.6 [Paley-Wiener Criterion] *If*

$$\int_{-\infty}^{\infty} \frac{|\ln \mathcal{S}_n(f)|}{1+f^2} df < \infty \quad , \quad (1.434)$$

then there exists a $G(f)$ satisfying below with a realizable inverse. (Thus the filter $g(t)$ is a 1-to-1 mapping).

$$[\bar{\mathcal{S}}_n(f)]^{-1} = |G(f)|^2$$

- See Appendix D on canonical factorization of autocorrelation/power spectra:
 - Such a filter exists for any noise typically found in practice.
 - Notice this says “noise” – does not necessarily apply to systems that optimize transmit power spectra.





End Lecture 3