STANFORD

Lecture 3

Modulation Types

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Announcements & Agenda

- Announcements
  - Homework due tomorrow
  - Solutions distributed Friday, or earlier if no late
    - This of courses causes anything after that to be a zero.
  - PS2 is due 1/24/24

- Problem Set 2 = PS2 due Tuesday Jan23 at 17:00
  1. 1.14 SQ odd-b QAM constellations
  2. 1.19 shaping gain
  3. 1.22 basic QAM design
  4. 1.34 hexagonal constellation QAM
  5. 1.35 Baseband equivalents
  6. 1.38 2-tap channel

- Today (1.3)
  - Finish Lecture 2
  - Simple AWGN Modulation and SNR
  - PAM
  - QAM
  - Filtered AWGN Channels

Finish Lecture 2

Section 1.3.3.1

Normalization to Dimensionality

- $\bar{P}_e$, $\bar{N}_e$, $\overline{\mathcal{E}_x}$, $\bar{b}$, ...

$$\bar{A} \triangleq \frac{A}{N}$$

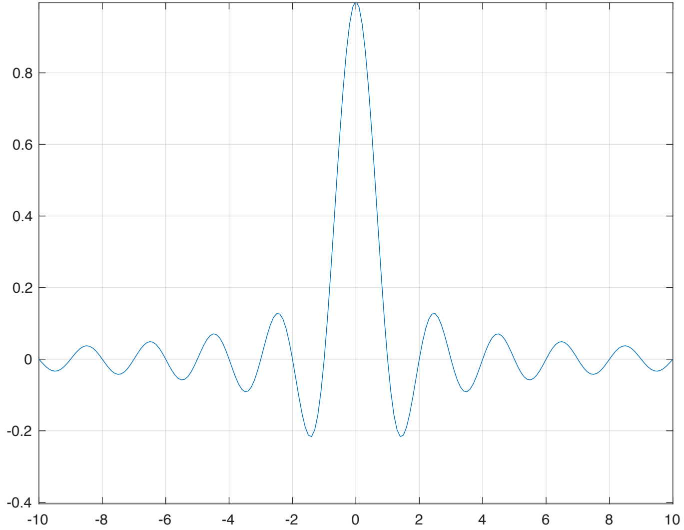
- The measure, indicated by a bar, normalizes to the “resources” - the number of dimensions.

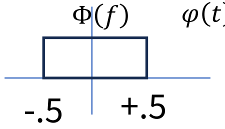
$$N = 2 \cdot W \cdot T$$

W is positive-frequency bandwidth

T is transmission length (symbol period)

$$sinc(t) \triangleq \frac{\sin(\pi \cdot t)}{t}$$





$\varphi(t) = sinc(t)$

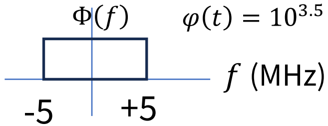
$W = .5$

Fair Comparisons

Many engineers, including some really famous ones, have erred on comparisons

- Fix 4 of these 5 – compare last
  - Data rate $R = \frac{b}{T}$
  - Power $P_x = \frac{\mathcal{E}_x}{T}$
  - System bandwidth W
  - Total transmission time or symbol period T
  - Error probability P_e
- Or fix 2 of these 3 – compare the last
  - Bits/dim $\bar{b} = \frac{b}{N}$
  - Energy/dim $\bar{\mathcal{E}}_x = \frac{\mathcal{E}_x}{N}$
  - Error Prob/dim $\bar{P}_e = \frac{P_e}{N}$

System 1: $\frac{1}{T}$=10 MHz, $N = 1, \pm 1\ (M = 2), R = 10$ Mbps

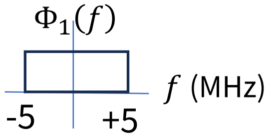


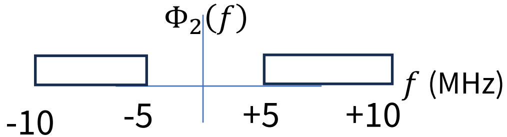
$\varphi(t) = 10^{3.5} \cdot sinc(10^7 \cdot t)$

$W = 5$ MHz

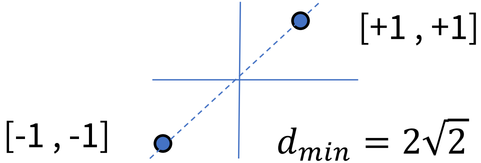
$d_{min} = 2$

System 2: $R = 20$ Mbps, $N = 2, \pm 1\ (M = 2), \frac{1}{T}$=10 MHz





$W = 10$ MHz



$d_{min} = 2\sqrt{2}$

Bigger d_{min} and twice rate ?
Not fair, both $\bar{b}$ and $\bar{P}_e$ differ
(these two are really the same)
set $\bar{b}$ =1 and becomes QPSK, smaller d_{min}

Packet Error Rate

- **Block error rate**, $P_{e,block}$, is the average probability that a packet or "block" of B bits contains at least one erred bit.

$$P_{e,block} \approx B \cdot \bar{P}_b$$

- More accurately, $P_{e,block}$ counts the ways that bit errors can occur:

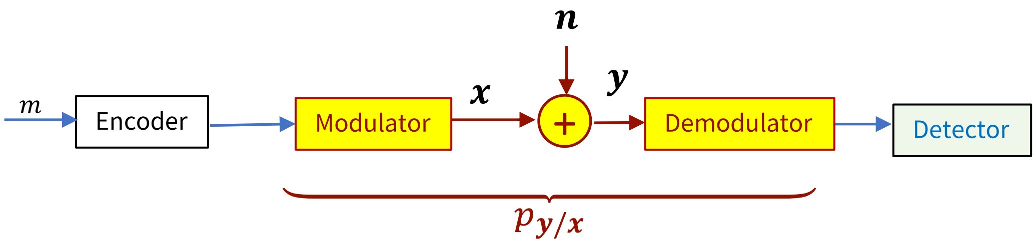
$$P_{e,block} = \sum_{i=1}^{B} \binom{B}{i} \cdot (1 - \bar{P}_b)^{B-i} \cdot \bar{P}_b^i$$

- Other examples
  - An **erred second** is second in which at least one (uncorrectable) bit error occurred.
  - **Code violations** – usually measured in 15min intervals -- and count the number of erred packets in that interval.

Simple AWGN Modulation & SNR

Section 1.3.3

AWGN Channel and SNR



$$\boxed{SNR = \frac{\bar{\mathcal{E}}_x}{\sigma^2}}$$

- The numerator and denominator must have same number of dimensions.
  - It's easy with use of barred quantities (σ^2 is the noise energy/dimension for any AWGN at MF output = power-spectral density, 2-sided).

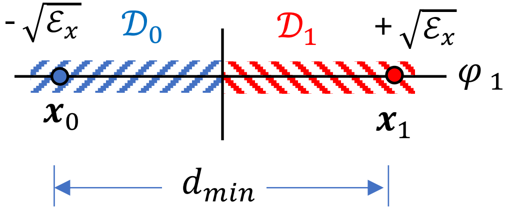
- A constellation's SNR maps to P_e (like 10^{-6}) through the (possibly scaled) Q-function argument's square root.

- **margin** = the amount of SNR in excess of that required to meet a P_e target.

- **energy/bit** $\mathcal{E}_b = {}^{\mathcal{E}_x}/_{b}$; caution, this measure confuses issues when $\bar{b} \geq 1$ (SNR works everywhere).
  - Energy quadruples with each additional bit/dim when $\bar{b} \geq 1$, an exponential growth.
    - So normalizing by linear-factor can create inconsistencies.
  - When $\bar{b} \leq 1$, the energy growth is more consistently linear, so this measure then is self-consistent.

Binary Antipodal (NRZ) – simple Binary

- 2 constellation points $\pm\sqrt{\mathcal{E}_x}$ best one-dimensional binary code

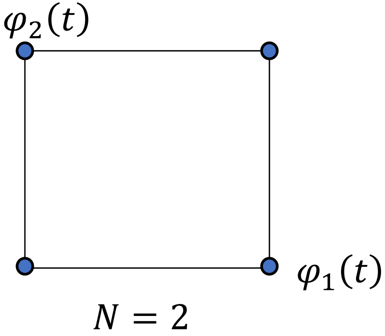


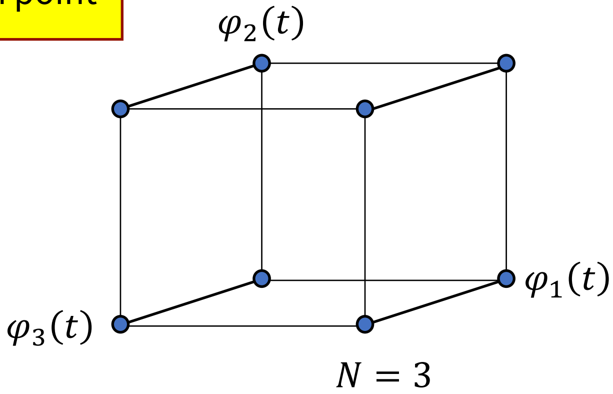
$$\sqrt{SNR} = \frac{\sqrt{\mathcal{E}_x}}{\sigma} = \frac{d_{min}}{2\sigma}$$

$$P_e = \bar{P}_e = 1 \cdot Q\left(\sqrt{SNR}\right)$$

- Extend to $N \geq 1$

● = constellation point



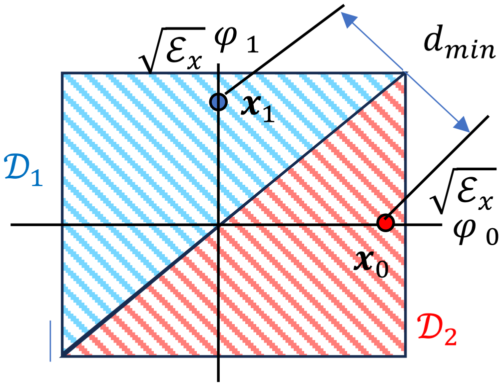


$$\bar{P}_e \leq 1 \cdot Q\left(\sqrt{SNR}\right)$$

(bound when $N \geq 2$)

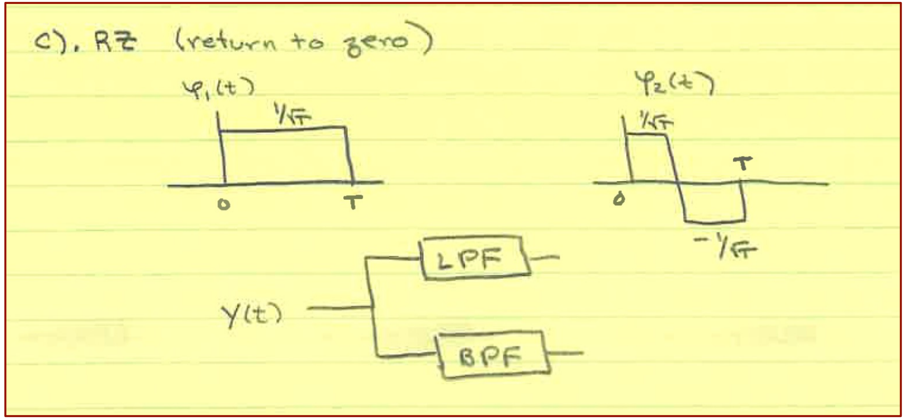
Binary-Orthogonal Constellations

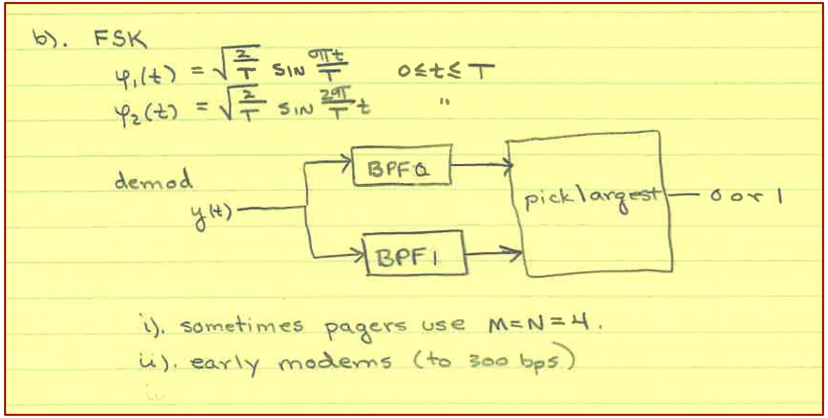
- 2 constellation points $\pm\sqrt{\mathcal{E}_x}$ (antipodal) is the best one-dimensional binary code, but how about:



$$\frac{d_{min}}{2\sigma} = \frac{\sqrt{\mathcal{E}_x}}{\sqrt{2}\sigma}$$

$$P_e = \bar{P}_e = 1 \cdot Q\left(\frac{\sqrt{\mathcal{E}_x}}{\sqrt{2}\sigma}\right)$$





Block-Orthogonal Constellations

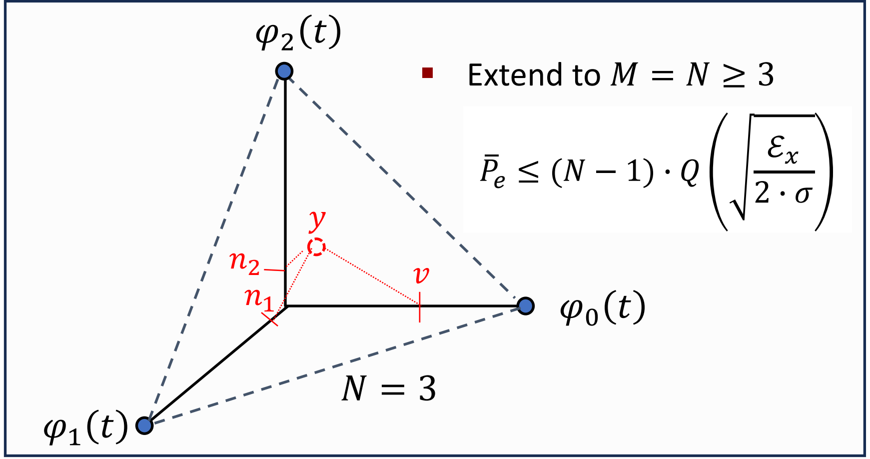
- Extending to more dimensions is wasteful of system resources (temporal in particular, time or freq).
- $x_m(t) = \sqrt{\mathcal{E}_x} \cdot \varphi_m(t)$

$$
\begin{aligned}
P_{c/0,y_0=v} &= P\{n_i \le v,\ \forall\ \ i \ne 0\} \\
&= \prod_{i=1}^{N-1} P\{n_i \le v\} \\
&= [1 - Q(v/\sigma)]^{N-1} \quad .
\end{aligned}
$$

$$\lim_{N\to\infty} \bar{\mathcal{E}}_x = 0$$

$$SNR = \frac{\mathcal{E}_x}{N \cdot \sigma^2}$$

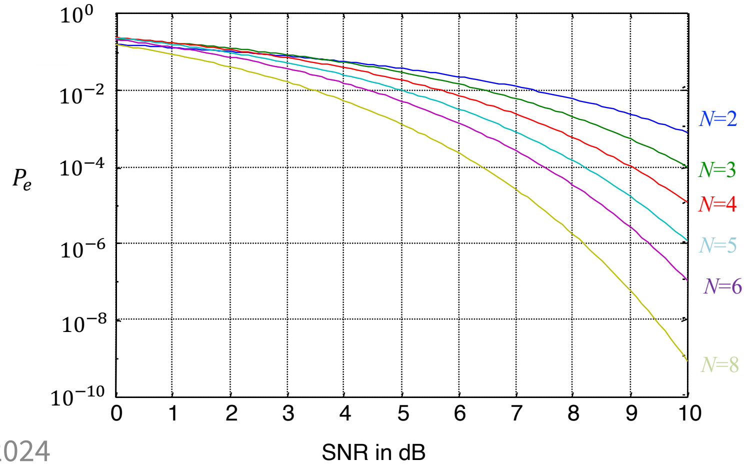
Not good,
but simple demods



- Extend to $M = N \ge 3$

$$\bar{P}_e \le (N-1) \cdot Q\left(\sqrt{\frac{\mathcal{E}_x}{2 \cdot \sigma}}\right)$$

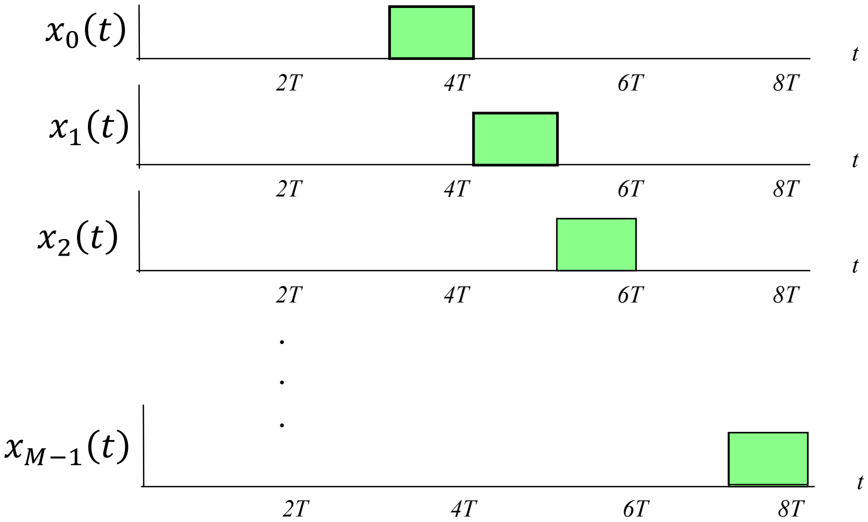
$$P_e = 1 - \int_{-\infty}^{\infty} (\sqrt{2\pi\sigma^2})^{-1} \cdot e^{-\frac{1}{2\sigma^2}(v - \sqrt{\mathcal{E}_{\boldsymbol{x}}})^2} \cdot [1 - Q(v/\sigma)]^{N-1}\, dv$$

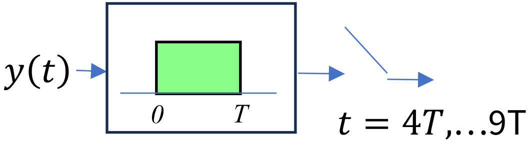


$$\frac{\mathcal{E}_x}{\sigma^2} = N \cdot \mathrm{SNR}$$

So energy increases at fixed SNR with N, while $\bar{b}$ decreases.

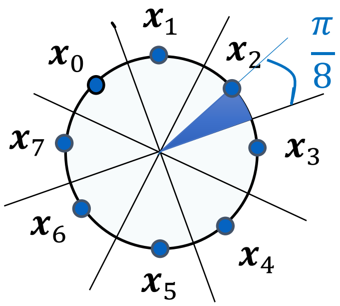
Example: Pulse Position Modulation (PPM)





- For instance, radar and lidar where the message is the delay/distance (so position).
- Visible Light Communication (VLC) systems (inside rooms, “Li-Fi”) – wide bandwidth (10’s Mbps)
  - IEEE 802.15.7.
- Similarly, pulse-duration modulation, where detector accumulates receiver input energy.

Phase-Shift Keying – Circular Constellations



$$radius = \sqrt{\mathcal{E}_x}$$

Often used in satellite transmission
e.g., LEO QPSK, 8PSK (some 16 QAM)
$^1/_T = 10 - 400$ MHz, roughly
carriers are typically above 10 GHz

Low peak-to-average can simplify design.

Performance degrades (low d_{min} for given Energy) when $M > 4$.

- PSK is shown here with $M = 8$, but generally M is any positive integer.
  - All points equal energy $\mathcal{E}_x$ – this can simplify energy driver/receiver-amplifier circuits implementation and overall energy consumption.
- Minimum distance is $d_{min} = 2 \cdot \sqrt{\mathcal{E}} \cdot \sin\frac{\pi}{M}$.
- Error Prob is $P_e < 2 \cdot \; Q\left(\sqrt{SNR} \cdot \sin\frac{\pi}{M}\right)$.

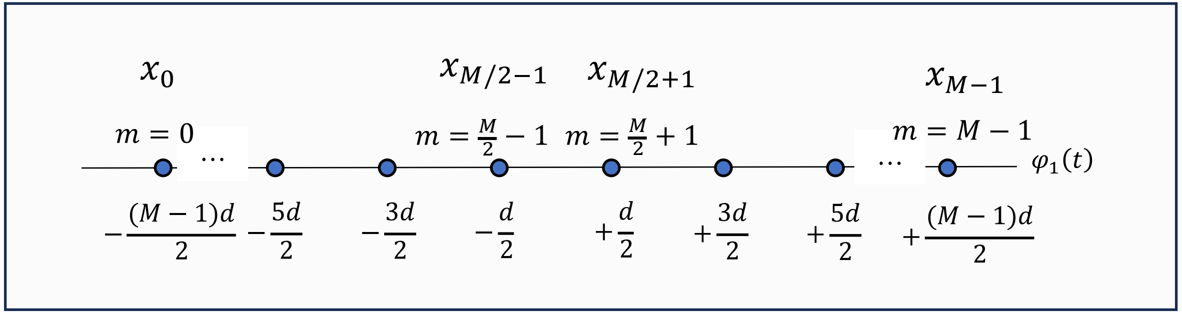
BPSK (M=2) basically wastes a dimension (although 3 dB larger distance than QPSK)

Pulse Amplitude Modulation (PAM)

Section 1.3.4.1

M'ary PAM (Sec 1.3.4.1)

- **Pulse Amplitude Modulation (PAM)** has $M = 2^b$ symbol values equally spaced in $N = 1$ dimension.



$$\varphi_1(t) = \frac{1}{\sqrt{T}} \cdot sinc\left(\frac{t}{T}\right)$$

- PAM symbol energy is $\boxed{\mathcal{E}_x = \frac{M^2-1}{12} \cdot d^2}$ $\qquad d = \sqrt{\frac{12 \cdot \mathcal{E}_x}{M^2 - 1}}$

- PAM bits/dim are $\boxed{b = \frac{1}{2} \cdot \log_2\left(1 + 12 \cdot \frac{\mathcal{E}_x}{d^2}\right)}$

- PAM error prob is $\boxed{P_e = \underbrace{2 \cdot \left(1 - \frac{1}{M}\right)}_{N_e} \cdot Q\underbrace{\left(\sqrt{\frac{3 \cdot SNR}{M^2-1}}\right)}_{d/_{2\sigma}}}$

PAM Table for Pe=1e-6

$b = \bar{b}$	M	$\frac{d_{min}}{2\sigma}$ for $\bar{P}_e = 10^{-6} \approx 2Q\left(\frac{d_{min}}{2\sigma}\right)$	SNR $= \frac{(M^2-1)\cdot 10^{1.37}}{3}$	SNR increase $= \frac{M^2-1}{(M-1)^2-1}$
1	2	13.7dB	13.7dB	—
2	4	13.7dB	20.7dB	7dB
3	8	13.7dB	27.0dB	6.3dB
4	16	13.7dB	33.0dB	6.0dB
5	32	13.7dB	39.0dB	6.0dB

Table 1.2: **PAM constellation energies.**

- **PAM** is often cited as needed **6dB/bit** (last column, large b).
- **PCIe 6.0** Uses PAM4 with $1/T \approx 28, 44, 56, 112, 228$ GHz (for 56, 88, 112, 228, and 456 Gbps)
  - Per wire (PCIe allows up to 8-16 wires in parallel).

Matlab Commands

- Modulator is **pammod.m**

```
>> real(pammod(0:3,4)) % =   -3    -1     1     3     - lists the 4-PAM outputs (d=2 default)
>> real(pammod(0:3,8)) % =    -7    -5    -3    -1   - lists first four 8-PAM outputs
>> real(pammod([3 0 7],8)) %  =    -1    -7     7    - some message sequence to 8-PAM
>> d=1;
>> (d/2)*real(pammod([3 0 7],8)) % =  -0.5000   -3.5000    3.5000  - change d
>> randi(4,1,5) % =     2     1     4     1     3              - random (uniform) messages
>> (d/2)*real(pammod((randi(4,1,5)-1),8))      %  Put it all together, 5 successive 8-PAM symbols
=    0.5000    1.5000   -2.5000    0.5000    1.5000
```

- ML detector is **pamdemod.m**

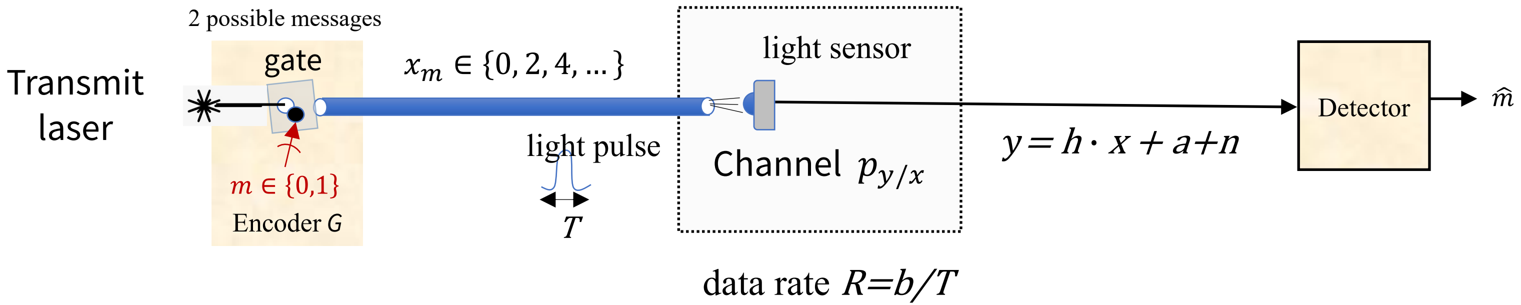
```
>> message=randi(8,1,10000)-1; %1 x 10k ∈ [1: 8]
>> x=real(pammod(message,8));
>> SNR=23;
>> Ex=63/3;
>> sig2=10^(-2.3)*Ex;
>> n=sqrt(sig2)*randn(1,10000);
>> y=x+n;
>> xhat = pamdemod(y,8);
>> sum( xhat ~= message) =    16
>> Pe=ans/10000 =   0.0016
> 1.5*qfunc(sqrt((3/63)*10^(SNR/10))) =   0.0015
```

Can run this with longer data sequences, and test several SNR.
Generate Pe versus SNR

☠ **Avoid matlab “awgn.m” program – can be confusing on “SNR”**

```
>> real(pammod([3 0 7],8,0,'gray’)) % =    -3    -7     3  - the 0 is initial phase
% gray code bit-mapping to constellation differs only by 1 bit in adjacent points, so then
Pb = Pe
>> bits= [     0     1     3     2     6     7     5     4];
%             000 001 011 010 110 111 101 100
>> real(pammod(bits,8,0,'gray’)) % =    -7    -5    -3    -1     1     3     5     7
>> pamdemod(real(pammod(bits,8,0,'gray')),8) % =   0     1     2     3     4     5     6     7
>> pamdemod(real(pammod(bits,8,0,'gray')),8,0,'gray’) % =    0     1     3     2     6     7     5     4
```

Example: Noncoherent fiber transmission



- Amplitude of light is modulated – typically all positive so constant $a = \frac{1-M}{2} d$ is added (& $x \geq 0$).
  - All positive loses 6 dB immediately to the constant, but still heavily used.
- The light wavelength is part of modulation, so not relevant to PAM (directly) – consistent with theory.
- 4PAM finds use in ethernet-fiber (IEEE 802.3xxx) standards with $^1/_T \approx 26.56$ GHz – longer fiber does introduce lowpass filtering of signals (see Chapter 3, L13-18, ISI later).
- Multiwavelength combinations (ITU standards) use $^1/_T \approx (up\ to)\ k \cdot 12.5$ GHz (ITU G.964.1) with PAM4 (contemplating PAM8 and PAM16), with $k = 1,2, \ldots, 8$ (8 probably is part of a DMT/OFDM system, see 379B).
- GPON (broadband fiber access, G.989.2 with G.sup64) – 4PAM with $^1/_T \approx 25$, 50 GHz.

Some interconnect uses (Copper or Fiber)

- **PCIe (Peripheral Component Interconnect Express)** has - $\frac{1}{T} = 16\ GHz; b = 4$ (4PAM) $R = 64$ Gbps:
  - PCIe can have up to 16 lanes (each at this speed) – 128 GBYTES/sec,
  - PCIe helps connect computer processor to peripheral components,
  - PCIe is typically (short) copper wires.

- **GDDR6 (Graphics Double Data Rate)** - $\frac{1}{T} = 6\ GHz; b = 4$ (4PAM) $R = 24$ Gbps:
  - GDDR6 is a memory interface (version 6 went to 4 PAM) that is used in gaming,
  - GDDR6 is also for copper wires.

- **Ethernet** 100 and 200 Gbps:
  - Fiber - $\frac{1}{T} = 50\ or\ 100\ GHz; b = 4$ (4PAM) $R = 100$ or 200 Gbps,
  - 200 Gbps is relatively new – just entering market.

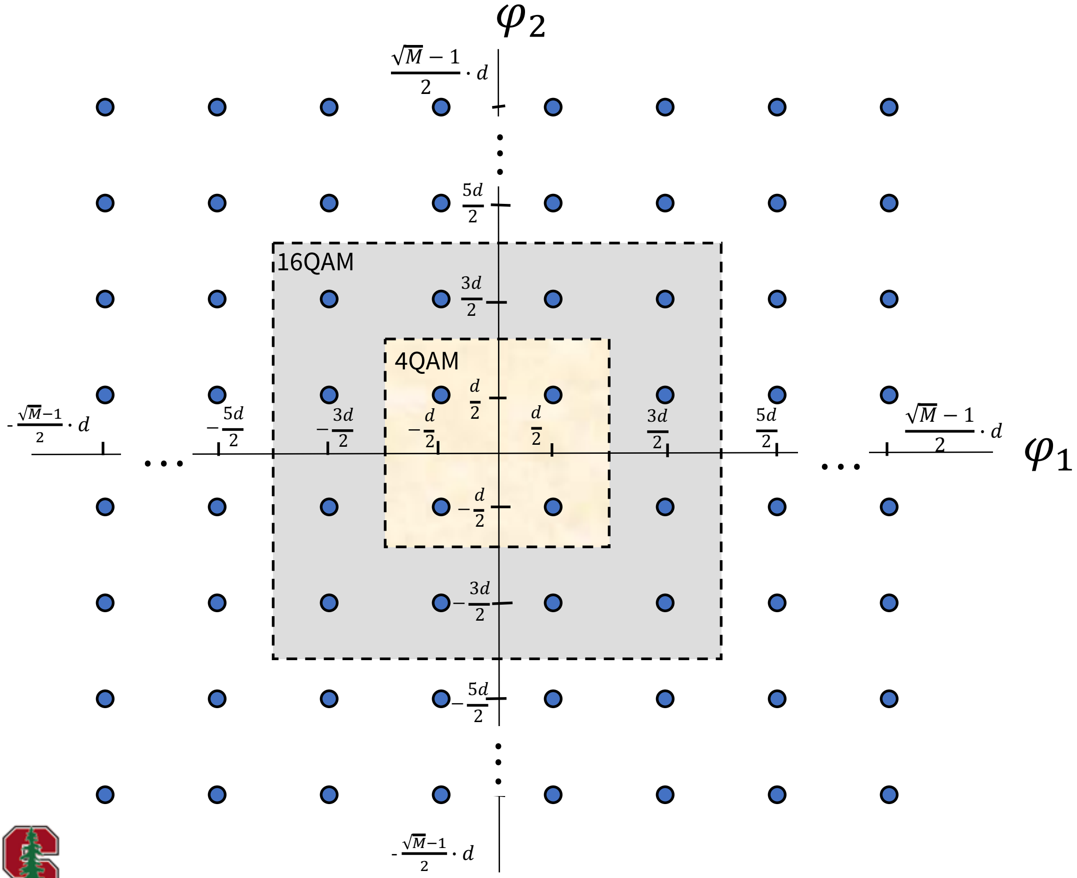
- **Coherent Fiber** – 16 QAM at $\frac{1}{T} = 130$ GHz → 400 Gbps of actual information (some code overhead).
  - Actually, use two polarizations per wavelength (so 2 16-QAMs that are spatially orthogonal, so 2x2 channel).
  - We'll see more "MIMO" instances later, so the data rate advertised is 800 Gbps (500 meters length).
  - Called Coherent DSP (two former students of this class P. Voois/N. Swenson started the company, ClariPhy that began this whole passage to QAM in coherent fiber – now part of Marvell).

Quadrature Amplitude Modulation (QAM)

Section 1.3.4.2

M'ary QAM

- **Quadrature Amplitude Modulation (PAM)** has $M = 2^b$ symbol values in $N = 2$ dimension, "squares PAM"



Cartesian product of 2 PAMS

$$\varphi_1(t) = \sqrt{\frac{2}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right) \cdot \cos \omega_c t$$

$$\varphi_2(t) = -\sqrt{\frac{2}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right) \cdot \sin \omega_c t$$

- QAM symbol energy

$$\mathcal{E}_x = \frac{M-1}{6} \cdot d^2 \qquad \bar{\mathcal{E}}_{\boldsymbol{x}} = d^2 \left(\frac{M-1}{12}\right)$$

- QAM bits/sym are

$$b = \log_2\left(1 + 6 \cdot \frac{\mathcal{E}_x}{d^2}\right)$$

- QAM error prob is

$$P_e = 4 \cdot \underbrace{\left(1 - \frac{1}{\sqrt{M}}\right)}_{N_e} \cdot \underbrace{Q\left(\sqrt{\frac{3 \cdot SNR}{M-1}}\right)}_{Q(d/2\sigma)}$$

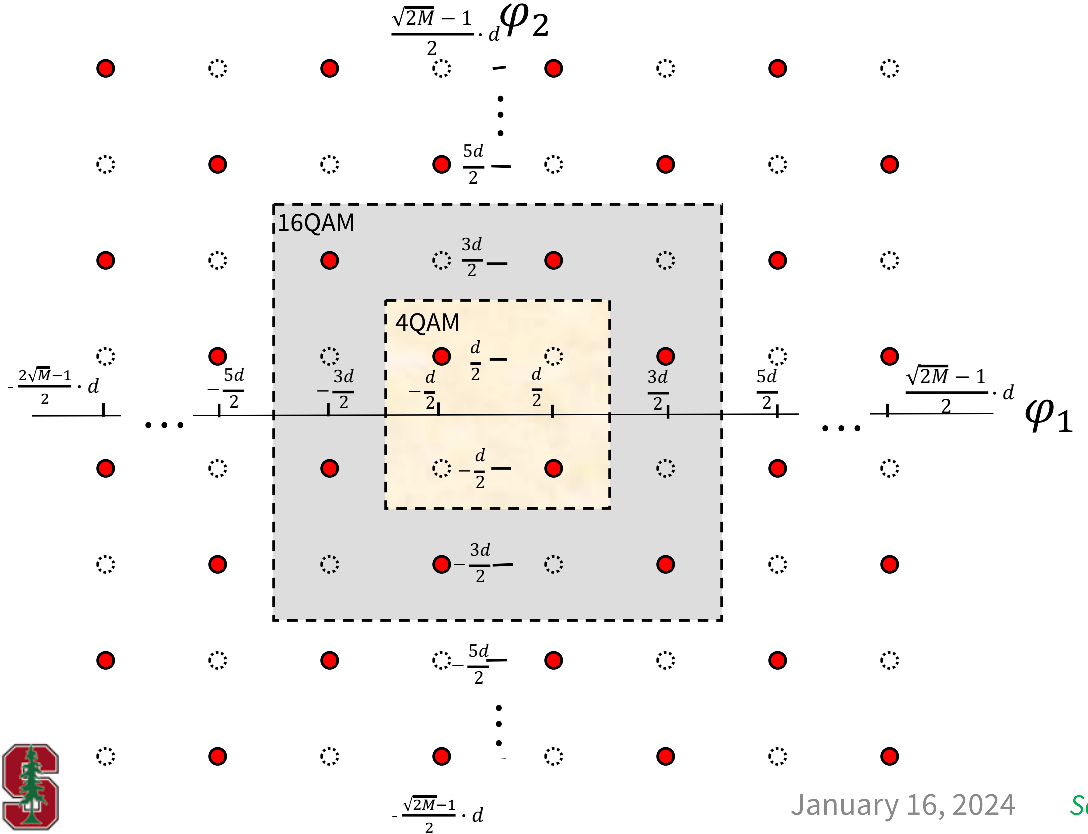
QAM Table for Pe=1e-6

$b = 2\bar{b}$	M	$\frac{d}{2\sigma}$ for $\bar{P}_e = 10^{-6} \approx 2Q\left(\frac{d_{\min}}{2\sigma}\right)$	SNR $= \frac{(M-1)\cdot 10^{1.37}}{3}$	SNR increase $= \frac{M-1}{(M-1)-1}$	dB/bit
				$M_{QAM} = M_{PAM}^2$	
2	4	13.7dB	13.7dB	—	—
4	16	13.7dB	20.7dB	7.0dB	3.5dB
6	64	13.7dB	27.0dB	6.3dB	3.15dB
8	256	13.7dB	33.0dB	6.0dB	3.0dB
10	1024	13.7dB	39.0dB	6.0dB	3.0dB
12	4096	13.7dB	45.0dB	6.0dB	3.0dB
14	16,384	13.7dB	51.0dB	6.0dB	3.0dB

- **QAM** is often cited as needed **3dB/bit** (last column, large b) – same as PAM, but over 2 dimensions
- Some wireless (satellite 16QAM, cellular 1G,2G,3G, Wi-Fi 802.11b, early US digital TV)
- Wireline – coherent fiber, ${}^{1}/_{T} \approx k \cdot 32$ GHz or ${}^{1}/_{T} \approx k \cdot 12.5$ GHz $k = 1,2,3,4$ ... 8 so far

SQ QAM for odd b ?

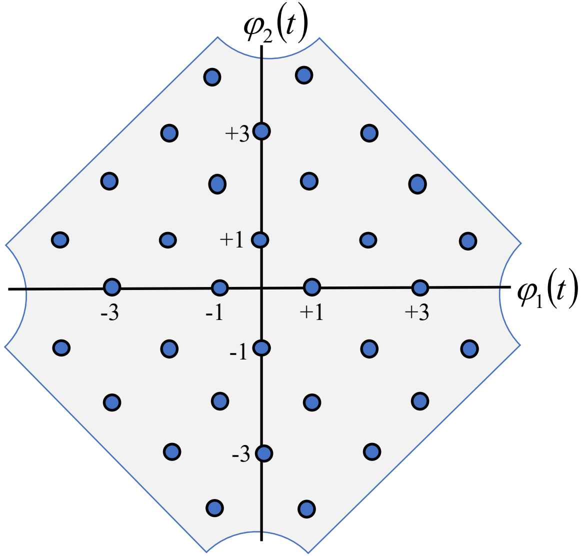
- Most typical today is "every other point" from $b+1$ size constellation
  - Use SQ QAM formulas, but increase $d_{min} \rightarrow \sqrt{2} \cdot d$ - below

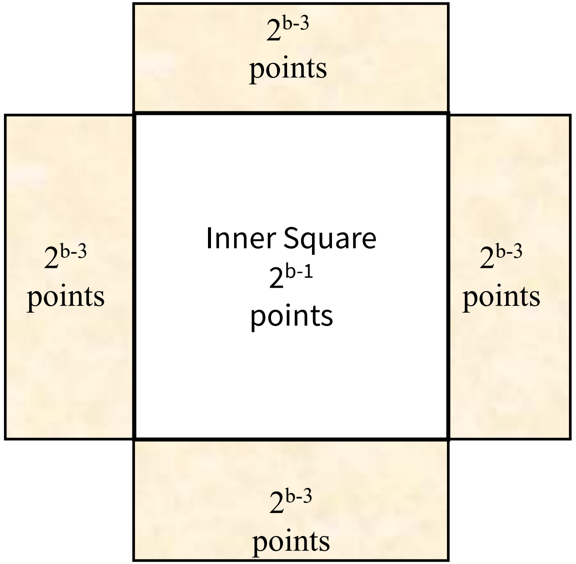


PS2.1 (1.14) – your turn to find formulas

CR constellations Slightly Better for odd b QAM

- More symmetrically removes high-energy corner points





$$\boxed{\mathcal{E}_x = \frac{d^2}{6}\cdot\left(\frac{31}{32}M - 1\right)}$$

$$\boxed{P_e = \underbrace{4\cdot\left(1-\frac{1}{\sqrt{2M}}\right)}_{N_e}\cdot Q\underbrace{\left(\sqrt{\frac{3\cdot SNR}{\frac{31}{32}M - 1}}\right)}_{d/2\sigma}}$$

- Used on some (of many carriers) in DMT systems (xDSL, G.fast, G.mgfast, etc)

Matlab Commands

- Modulator is **qammod.m**

```
>> reshape(qammod(0:3,4),2,2) %=
  -1.0000 + 1.0000i   1.0000 + 1.0000i
  -1.0000 - 1.0000i   1.0000 - 1.0000i
```

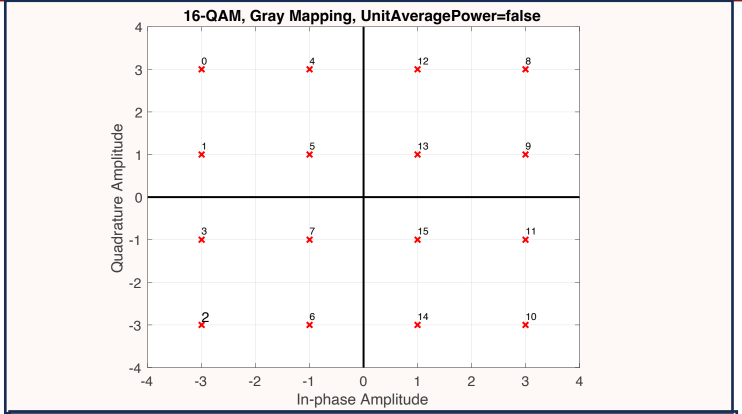
```
>> reshape(qammod(0:15,16,'plotconstellation',1),4,4) % =
  -3.0000 + 3.0000i  -1.0000 + 3.0000i   3.0000 + 3.0000i    1.0000 + 3.0000i
  -3.0000 + 1.0000i  -1.0000 + 1.0000i   3.0000 + 1.0000i    1.0000 + 1.0000i
  -3.0000 - 3.0000i  -1.0000 - 3.0000i   3.0000 - 3.0000i    1.0000 - 3.0000i
  -3.0000 - 1.0000i  -1.0000 - 1.0000i   3.0000 - 1.0000i    1.0000 - 1.0000
```

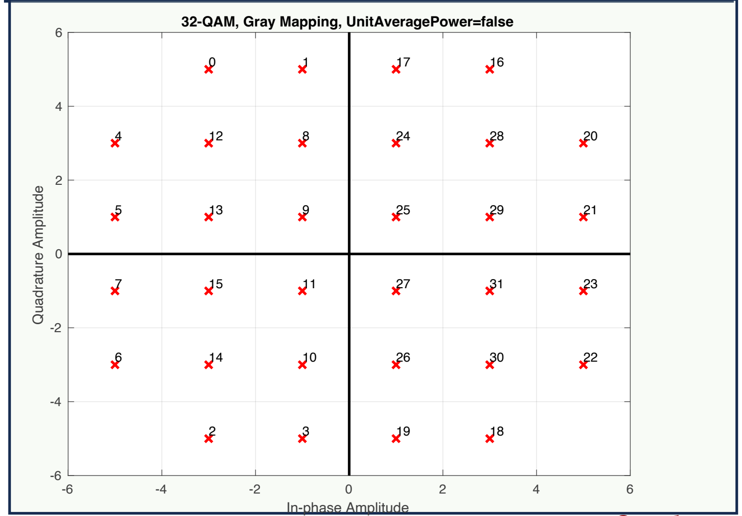
```
>>qammod(0:31, 32,'plotconstellation',1);  % produces 32CR
```

Odd bits 5 or greater → cross constellation

However 8 CR is horrible and looses 0.8 dB w.r.t. 8SQ

8SQ is not well handled by matlab (**see for yourself:** matlab''s constellation from qammod for M=8)





Matlab Commands

- ML detector is **qamdemod.m**

Defaults to Gray Code

```
rng(7)
message=randi(16,1,10000)-1;
x=qammod(message,16);
SNR=17;
Ex=10;
N0=10^(-1.7)*Ex;
n=sqrt(N0/2)*randn(1,10000)+i*sqrt(N0/2)*randn(1,10000);
y=x+n;
xhat = qamdemod(y,16);
sum( xhat ~= message)
 =   25
ans/10000 = .0025
>> 3*qfunc(sqrt(0.2*10^(1.7)))
ans =    0.0023
```

```
errrate=comm.ErrorRate;
for indx=1:100
  message=randi(16,1,10000)-1;
  x=qammod(message,16);
  n=sqrt(N0/2)*randn(1,10000)+i*sqrt(N0/2)*randn(1,10000);
  y=x+n;
  xhat = qamdemod(y,16);
  errstats = errrate(xhat',message');
end
errstats(1)

ans =    0.0023   % perfect match!
```

Forney's Gap Approximation – PAM/QAM

- What Q-function argument gives ($\bar{P}_e =$) 10^{-6} ? (assume $\bar{N}_e = 2$)

$$\frac{3}{2^{2\cdot\bar{b}} - 1} \cdot SNR = 10^{1.38}$$

13.8 dB ??
>> 20*log10(qfuncinv(1e-6/2)) = 13.7891

- Solve for $\bar{b}$ to get **Forney's Gap Formula:**
  - $\bar{b} = \frac{1}{2} \cdot \log_2\left(1 + \frac{SNR}{\Gamma}\right)$, where
    - $\Gamma = 8.8$ dB (for both PAM and QAM) , and
    - $\Gamma = 9.5$ dB for $\bar{P}_e = 10^{-7}$.
    - Gap is largely independent of $\bar{b}$>0.5 ; we'll see this applies to most good codes built on PAM/QAM also.

- This looks like a very famous formula (Chapter 2, we'll see),
  - where Γ=1 (0 dB).

- That, is, the maximum reliably decodable data rate on AWGN, the **capacity.**

- The gap measures reduction (in SNR, dB) relative to this capacity (here for "uncoded" PAM/QAM).

Examples

2. Examples

a. SNR = 13.5 dB? $P_e = 10^{-6}$

$$\bar{b} = \frac{1}{2}\log_2\left(1 + \frac{10^{1.35}}{10^{.88}}\right) = 1 \quad \text{(we knew this)}$$

$\boxed{\begin{array}{c}\text{2 PAM} \\ \text{4 QAM}\end{array}}$

b. SNR = 24.4 dB $P_e = 10^{-7}$

$$\bar{b} = \frac{1}{2}\log_2\left(1 + 10^{2.44-.95}\right) = 2.5$$

$2\bar{b} = b = 5$ 32CR QAM

5 bits/Hz (4 PAM, not enough SNR for 8PAM)

c. SNR = 44.7 dB 10^{-6}

$$\bar{b} = \frac{1}{2}\log_2\left(1 + 10^{4.47-.88}\right) = 6$$

64 PAM or 4096 SQ QAM

- Best to date? (no MIMO dimensions)
  - QAM, $b = 15$ bits/Hz.
  - Bits/Hz = $2\bar{b}$.

- Most useful codes also based on sequences of QAM/PAM symbols.
  - Their gaps are also constant for $2\bar{b} > 1$.
  - The good ones have $\Gamma \to 0$ dB (or maybe in practice more like 1 dB).

End Lecture 3

Lattices and Codes (AWGN)

- **Lattice** $\Lambda = \{\boldsymbol{\lambda}_0 , \boldsymbol{\lambda}_1 , \cdots\}$ that is closed under an operation "addition" (usually normal addition, but can also be over a finite field when $|\Lambda| < \infty$. (Appendix B)

- Examples include:
  - $\mathbb{Z}$ - the integers (think PAM),
  - $\mathbb{Z}^2$- 2D integer vectors (think QAM), and
  - $\mathbb{Z}^N$ - think codewords built from PAM/QAM.

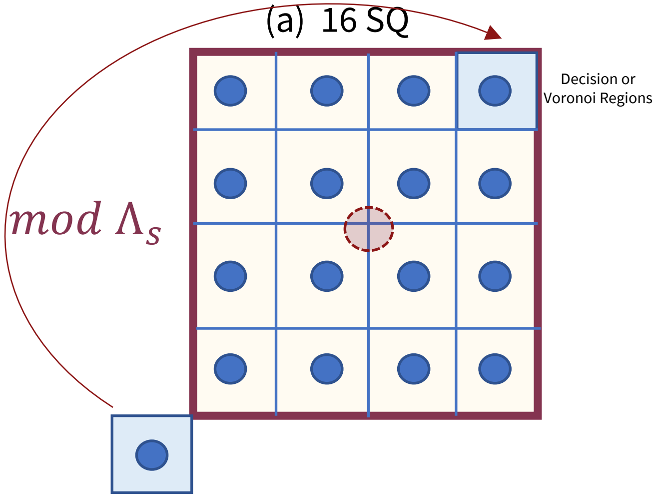
$$D_2 = 2Z^2 + \{0,1\} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

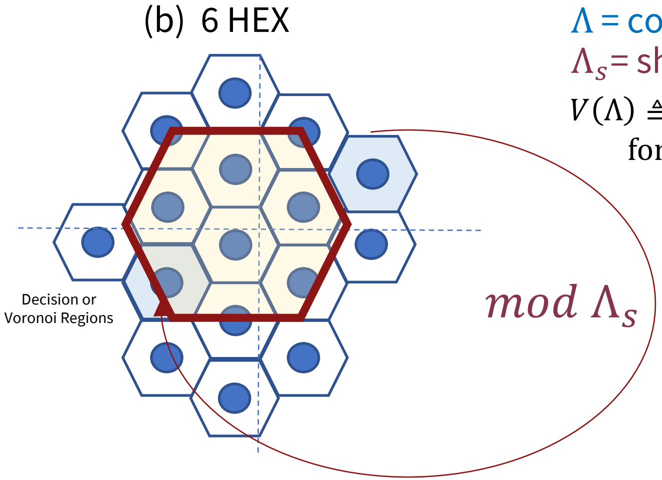
- **Coset** $\Lambda + \boldsymbol{a} = \{\boldsymbol{\lambda}_0 + \boldsymbol{a} , \boldsymbol{\lambda}_1 + \boldsymbol{a} , \cdots\}$ basically maintains all the lattice properties, but need to add ***a*** or remove it appropriately.

- Most constellations C are subsets of lattices Λ (or their cosets).
  - Designs choose M symbols from Λ , and subtract mean so that they have minimum average energy.

- Lattices are a nice way for designers to pack points evenly into given volume or energy.

Coding Gain and Constellation/Code





Λ = coding lattice for d_{min}

Λ_s = shaping lattice for $\mathcal{E}_x$

$V(\Lambda) \triangleq$ decision − region volume for (any) lattice point in Λ

$$\gamma \triangleq \frac{\left(d^2_{\min}(\boldsymbol{x})/\bar{\mathcal{E}}_{\boldsymbol{x}}\right)}{\left(d^2_{\min}(\breve{\boldsymbol{x}})/\bar{\mathcal{E}}_{\breve{\boldsymbol{x}}}\right)} = \underbrace{\frac{\left(\frac{d^2_{\min}(\boldsymbol{x})}{V^{2/N}(\Lambda)}\right)}{\left(\frac{d^2_{\min}(\breve{\boldsymbol{x}})}{V^{2/N}(\breve{\Lambda})}\right)}}_{\gamma_f} \cdot \underbrace{\frac{\left(\frac{V^{2/N}(\Lambda)}{\bar{\mathcal{E}}_{\boldsymbol{x}}}\right)}{\left(\frac{V^{2/N}(\breve{\Lambda})}{\bar{\mathcal{E}}_{\breve{\boldsymbol{x}}}}\right)}}_{\gamma_s}$$

coding gain

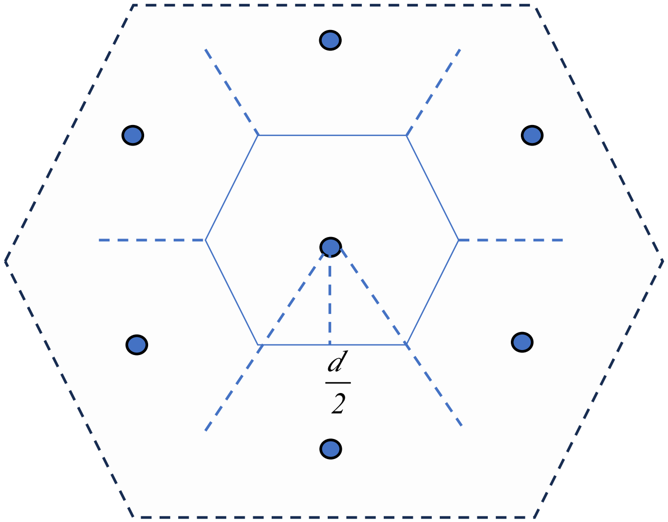
fundamental gain

shaping gain

Basic principle extends $\bar{N} \rightarrow \infty$.
Hexagon→ hypersphere (Gaussian marginals).

Good codes can follow from $\Lambda_s/\Lambda = |C|$.

Hexagon Constellation, fund gain



$\Lambda = A_2$, the **"hexagonal" lattice.**

$$V(A_2) = \underbrace{6}_{of\ \Delta's} \cdot \frac{1}{2} \cdot \underbrace{\frac{d}{\sqrt{3}}}_{altitude} \cdot \underbrace{\frac{d}{2}}_{base} = \frac{\sqrt{3}}{2} \cdot d^2 \ .$$

$$\boxed{\gamma_f = \frac{d^2}{\frac{\sqrt{3}d^2}{2}} = \frac{2}{\sqrt{3}} = .625 \text{ dB}}$$

- A_2 is (up to) .625 dB better than $\mathbb{Z}^2$ in fundamental gain.
- It's also better as a shaping lattice (see previous page).

Maximum Shaping Gain

- Let $N \rightarrow \infty$, then **best shape** is a **hypersphere**.

- For hypersphere: $$\frac{\bar{\mathcal{E}}_x}{V^{2/N}} = \underbrace{\frac{r^2}{N+2}}_{2nd\ moment} \cdot \underbrace{\frac{\left(\frac{N}{2}!\right)^{2/N}}{\pi \cdot r^2}}_{1/area} = \frac{\left(\frac{N}{2}!\right)^{2/N}}{\pi \cdot (N+2)}$$

$r^2/_4$ $\quad 1/_{\pi \cdot r^2}$

Problem 1.19

- Limit, relative to $\mathbb{Z}^N$ is $\frac{\pi e}{6}$ = 1.53 dB .
  - Proof in text.

BEST SHAPING GAIN IS 1.53 dB

- Fundamental gain can be infinite – see Chapter 2.

Peak-to-Average Ratio (PAR)

- Can be important for amplifiers (see PSK discussion)

Definition 1.3.23 [Discrete Peak Energy] *A constellation's N-dimensional* **discrete peak energy** *is* $\mathcal{E}_{peak}$.

$$\mathcal{E}_{peak} \stackrel{\Delta}{=} \max_i \sum_{n=1}^{N} x_{in}^2 \quad . \tag{1.328}$$

A modulated signal's **continuous-time peak energy** *is*

$$\mathcal{E}_{cont} \stackrel{\Delta}{=} max_{i,t} |x_i(t)|^2 \geq \mathcal{E}_{peak} \quad . \tag{1.329}$$

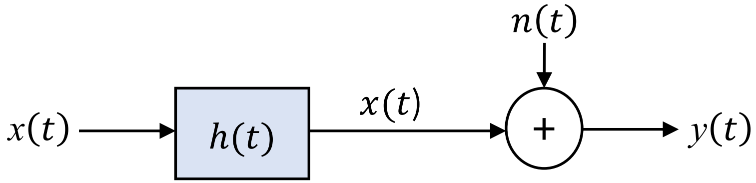
- PARs could be measured
  - At symbol instants: $PAR = {}^{\mathcal{E}_{peak}}/_{\mathcal{E}_x}$
  - In continuous time for an overall $PAR = {}^{\mathcal{E}_{cont}}/_{\mathcal{E}_x}$ - this one is always at least as large.
  - Example – simple sinusoid symbol-rate sampled at peaks has symbol-rate PAR =1 while any continuous sinusoid has PAR 3dB.

Filtered AWGN Channels

Section 1.3.7

Real channels don't pass all frequencies

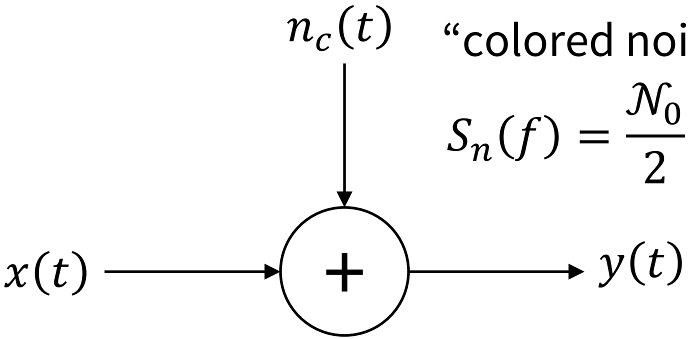
- The filtered AWGN has linear filter $h(t)$.



ISI (L12-17)

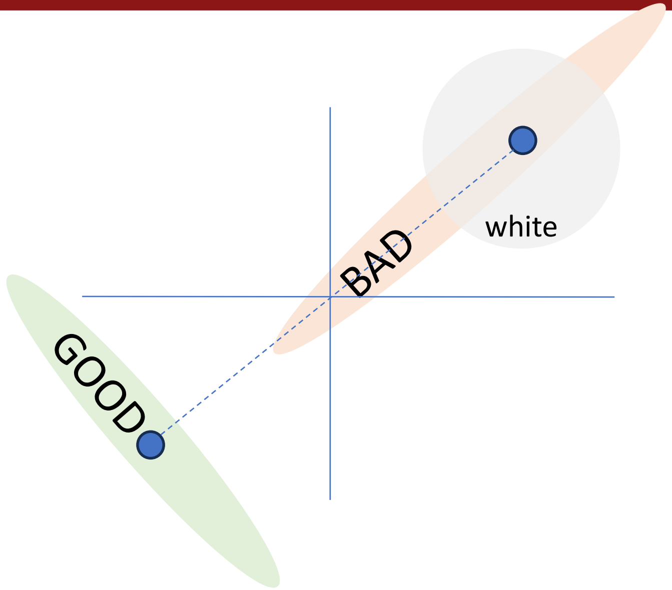
- Successive symbols get stretched and may overlap (intersymbol interference).
  - ➢ Can't go too fast …
- Correlated ("colored") noise is equivalent to filtered AWGN …. next slides

Colored noise



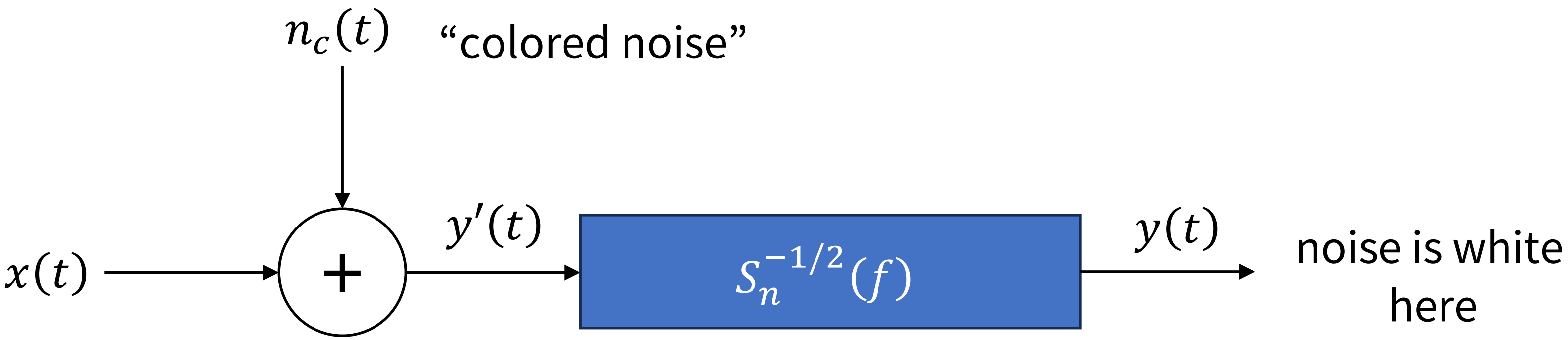
"colored noise"

$$S_n(f) = \frac{\mathcal{N}_0}{2} \cdot \bar{S}_n(f)$$



- Noise is not "white" (not flat PSD) – power spectral density $\frac{\mathcal{N}_0}{2} \cdot \bar{S}_n(f)$
- What is problem with this?
  - The MAP/ML detector is no longer "pick the closest point"
- See Examples in Section 1.3.7.2 / 3

Noise Whitening



$$S_n(f) = \frac{\mathcal{N}_0}{2} \cdot \bar{S}_n(f)$$

- 1-to-1 reversible transformation that whitens noise.
  - Loses nothing by reversibility theorem.
- And thus creates a filtered AWGN $H(f) = \bar{S}_n^{-1/2}(f)$.

But WAIT!

- This filter only exists, and is 1-to-1 causal and causally invertible, **IF**

Theorem 1.3.6 **[Paley-Wiener Criterion]** *If*

$$\int_{-\infty}^{\infty} \frac{|\ln \mathcal{S}_n(f)|}{1+f^2} df < \infty \quad , \tag{1.434}$$

then there exists a $G(f)$ satisfying below *with a realizable inverse. (Thus the filter $g(t)$ is a 1-to-1 mapping).*

$$\left[\bar{\mathcal{S}}_n(f)\right]^{-1} = |G(f)|^2$$

- See Appendix D on canonical factorization of autocorrelation/power spectra:
  - Such a filter exists for any noise typically found in practice.
  - Notice this says "noise" – does not necessarily apply to systems that optimize transmit power spectra.