STANFORD

Lecture 18

Transmit Optimization and Waterfilling

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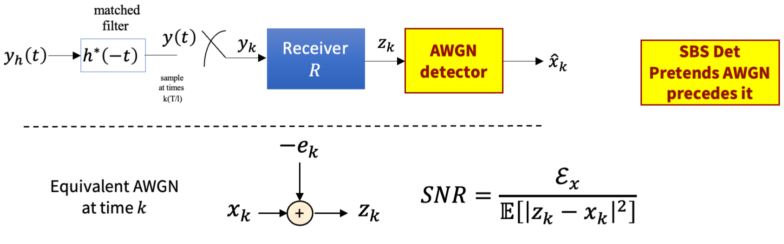
Announcements & Agenda

- Announcements
  - Optional PS8 due today - solutions early tomorrow (Wed)
    - This assignment is optional – the last two problems are from today and Thursday's lectures, respectively; Great on those looking ahead!
  - Final – end of class on Thursday – take home, 25 hours
    - Does anyone want a blue book – or prefer to use your own paper/scan/laptop-direct?

- Feedback PS7
  - 6-12 hours
  - Concept of equalizers (see side →) and L13-14
  - DFEs and root finding
    - $b^* \cdot D^{-2} + a^* \cdot D^{-1} + \tilde{q}_0 + a \cdot D + b \cdot D^2$
    - r=roots([b',a',qt0,a,b]) % for ZF-DFE, $\tilde{q}_0$=1
    - (1-r(i)· D) factors for all roots with $|r| \leq 1$. That is G(D).
    - gam0=qt(highest)/G(highest) – see L15:13,17

- Today
  - DFE RAKE aond soft equalization (carried from L17)
  - MMSE DFE Transmit Optimization
    - Water Filling
  - Suboptimal Transmitter Loss
  - MMSE-LE Transmit Optimization
    - Slush Packing

Receiver to minimize ISI impact



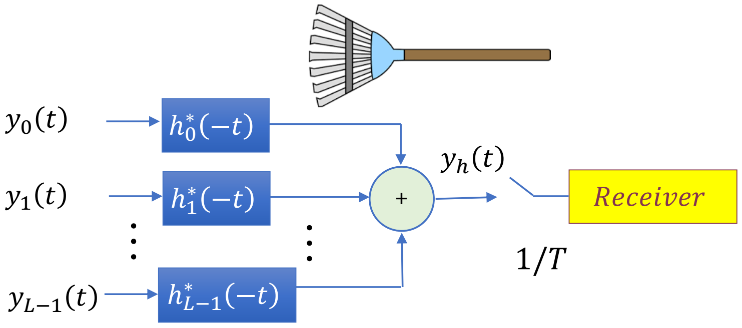
Equalizers (LE, DFE, MS,ZF) all try to create an equivalent A(W)GN channel so that uur codes apply

DFE RAKE and soft equalization

Section 3.8

DFE Rake Program

```
>> help dfeRAKE
  function [dfseSNR,W,b]=dfeRAKE(l,h,nff,nbb,delay,Ex,noise);
 DFE design program for RAKE receiver

 Inputs
 l    = oversampling factor
 L is derived as No. of fingers in RAKE (number of rows in h)
 h      = pulse response matrix, oversampled at l (size),
         each row corresponding to a diversity path
 nff   = number of feedforward taps for each RAKE finger
 nbb   = number of feedback taps
 delay = delay of system <= nff+length of p - 2 - nbb
 Ex    = average energy of signals
 noise = noise autocorrelation vector (size L x l*nff)
 NOTE: noise is assumed to be stationary, but may be spatially
correlated

 outputs:
 dfseSNR = equalizer SNR, unbiased in dB  ------------------------------
```



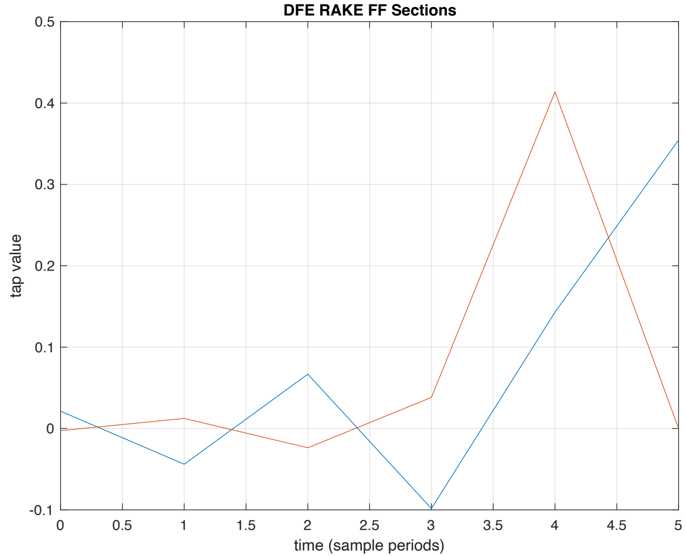

```
>> hrake=[0.9000    1.0000        0
        0    1.0500    0.8400];
>> [snr,W,b] = dfeRAKE(1,hrake,6,1,5,1,[.181 zeros(1,5) ; .181 zeros(1,5)])

snr =   11.1465 dB
W =
   0.0213   -0.0439    0.0668   -0.0984    0.1430    0.3546
  -0.0027    0.0124   -0.0237    0.0382    0.4137    0.0000
b =    0.7022
```

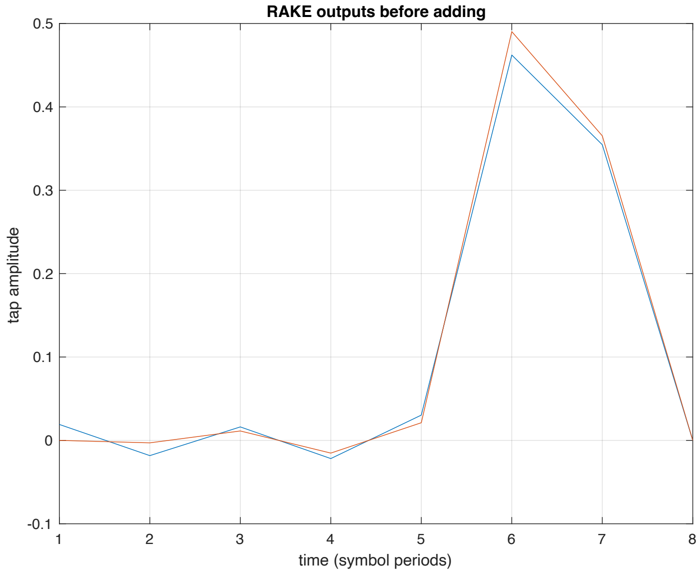
Student Project: Add the -1 = delay option to find best delay.

- Few taps, matches infinite-length result.

DFE Rake Plots

- The MS-WMF's try to align to on another as well as in time to their respective paths.





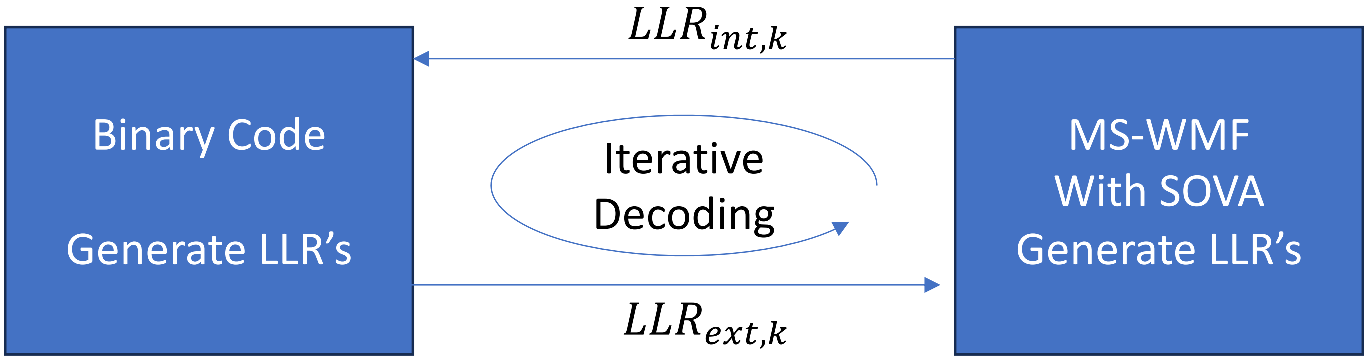

```
>> plot(conv(W(1,:),[.9 1 0]))
>> hold
Current plot held
>> plot(conv(W(2,:),(1.81/1.64)*[0 1 .8]))
```

- The equalized channel clearly looks causal in last 3 positions, and the two outputs align the large first tap.

Turbo Equalization

- These are packet adaptive equalizers where L16:26's channel identification (of H) or partial-response equalization (L17:13) is used.

- A MLSE (Viterbi Detector) for the channel ISI is used in place of the feedback section.

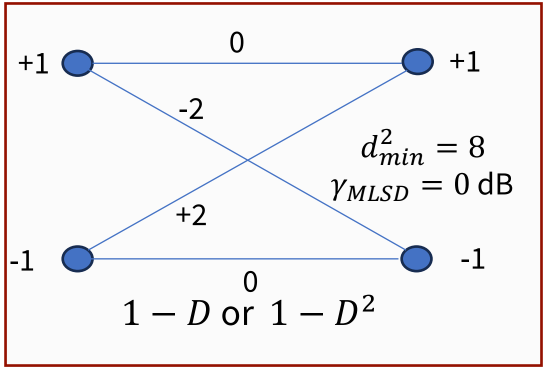


- The channel's memory is treated like a code with the SOVA generation of soft information

The intrinsic channel information

- Initially, Viterbi/SOVA produces ratios:
  - Sum of such terms if $M^{\nu} > 2$.
  - Evaluate each stage 0/1 among survivors.

$$\frac{e^{-\frac{1}{2\sigma^2}\cdot\left\|\boldsymbol{y}-\boldsymbol{H}\cdot\boldsymbol{x}_{k,0}\right\|^2}}{e^{-\frac{1}{2\sigma^2}\cdot\left\|\boldsymbol{y}-\boldsymbol{H}\cdot\boldsymbol{x}_{k,1}\right\|^2}}$$



- Later runs
  - Include the code's soft extrinsic information in the Viterbi partial-response updates.

- The MLSD on channel trellis is optimum – lower initial Pe
  - But loses advantage as number of levels increase in PAM/QAM
  - Precoder can reduce this loss, but not eliminate it.

- The code and channel may interleave order w.r.t. each other.
  - The SNRmfb attained by Viterbi does **NOT** add to coding gain.

- Turbo Equal tends to complicate/prevent transmit-filter optimization.

Much better to use Decision Feedback & Good Code

Those can achieve reliable transmission at any rate up to capacity

MMSE DFE Transmit Optimization

Section 3.12

MMSE for DFE (frac-spacing → optimize 1/T)

- Sections 3.11-12 review information-theoretic formulation, following Section 2.3.
  - That approach is further developed in EE379B next quarter.
- Only continuous-frequency theoretical optima appears in this EE379A lecture, see L15:10.

$$\sigma^2_{MMSE-DFE} = e^{-k\cdot(integral)}$$

Integral of output spectra

$$\frac{T}{2\pi}\cdot\int_{-\pi/T}^{\pi/T}\log_2\left\{\frac{\bar{\mathcal{E}}_x}{\sigma^2}\cdot\underbrace{\left|\Phi\left(e^{-j\omega T}\right)\right|^2}_{S_x(\omega)}\cdot\underbrace{\left|H\left(e^{-j\omega T}\right)\right|^2}_{S_h(\omega)}+1\right\}\cdot d\omega$$

Maximize over $S_x(\omega)$ Energy constraint

Simplify to $S_h(\omega)$

- This maximizes equivalently the $SNR(\Phi) = SNR_U(\Phi) + 1$ for all MMSE receivers.
- The side energy constraint is $\frac{T}{2\pi}\cdot\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} S_x(\omega)\cdot d\omega = \bar{\mathcal{E}}_x$.

Solution by Calculus of Variations

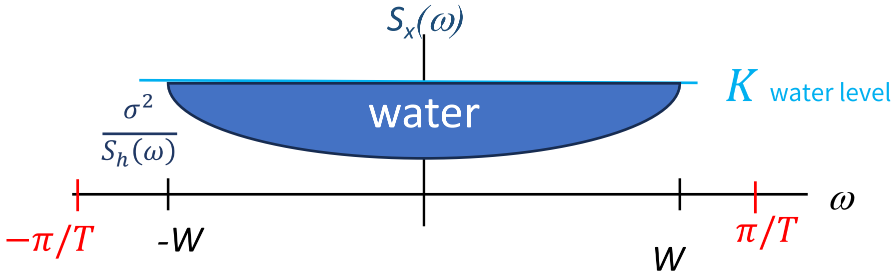
- Maximize instead

$$ln\left\{\frac{S_x(\omega)}{\sigma^2}\cdot S_h(\omega)+1\right\}+\lambda\cdot S_x(\omega)$$

$$\frac{\frac{1}{\sigma^2}\cdot S_h(\omega)}{\frac{1}{\sigma^2}\cdot S_x(\omega)\cdot S_h(\omega)+1}+\lambda=0$$

$$S_x(\omega)+\frac{\sigma^2}{S_h(\omega)}=K$$

$$S_x(\omega)\geq 0$$



- Shannon's **Waterfilling** Formula:
- Blue "water/energy" poured from above into noise-referred-to-channel-input curve.
- Waterfilling maximizes SNR (for MMSE-DFE).
- Well, almost – anyone see a problem here?
  - Uh-oh; Paley-Wiener violated.

Return to the 1+.9D^{-1} example

- **Assume $T = 1$ sec**

$$P_x = \int_{-W}^{W} \left(K - \frac{.181}{1.81 + 1.8 \cdot cos(\omega)} \right) \cdot \frac{d\omega}{2\pi}$$

$$\pi = \int_{0}^{W} \left(K - \frac{.181}{1.81 + 1.8 \cdot cos(\omega)} \right) \cdot d\omega$$

$$= K \cdot W - .181 \cdot \left\{ \frac{2}{\sqrt{1.81^2 - 1.8^2}} \right\} \cdot arctan \left[\frac{\sqrt{1.81^2 - 1.8^2}}{1.81 + 1.8} \tan\left(\frac{W}{2}\right) \right]$$

$$W = .88\pi \quad \frac{1}{T_{opt}} = .88$$

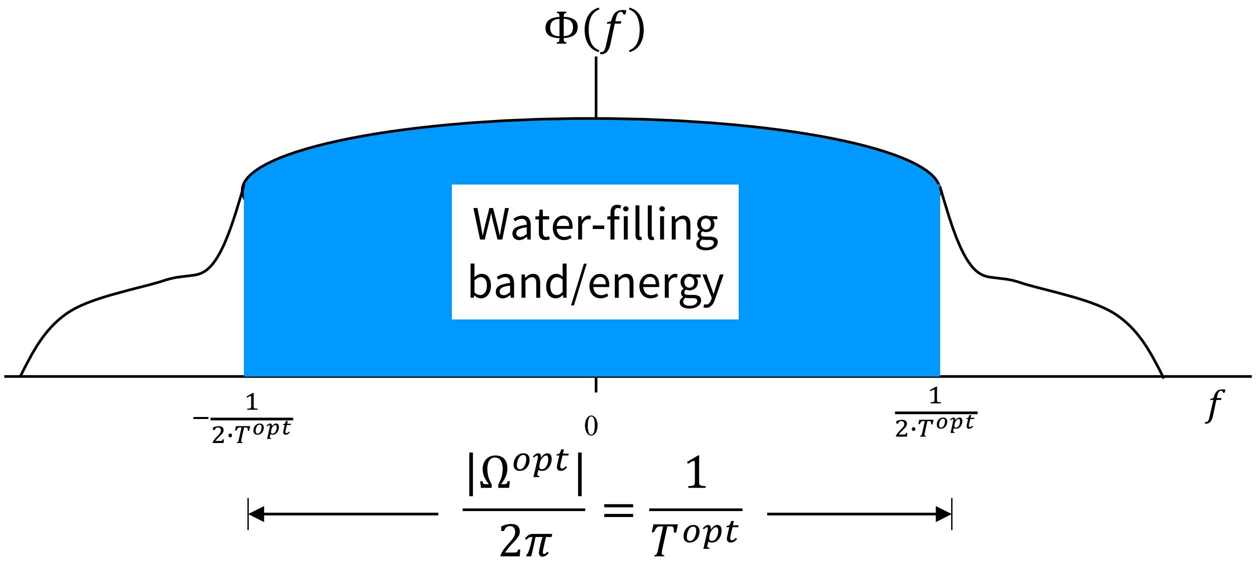
- Change symbol rate so that PWC is satisfied.

$$\begin{aligned} \mathcal{C} &= \frac{2}{2\pi} \int_0^{.88\pi} \frac{1}{2} \log_2 \left(\frac{1.33}{.181} (1.81 + 1.8 \cos \omega) \right) d\omega \\ &= \frac{1}{2\pi} \int_0^{.88\pi} \log_2 7.35 d\omega + \frac{1}{2\pi} \int_0^{.88\pi} \log_2 (1.81 + 1.8 \cos \omega) \, d\omega \\ &= 1.266 + .284 \\ &\approx 1.55 \text{bits/second} \quad . \end{aligned}$$

$$\bar{\mathcal{C}}(T^{opt}) = 1.76 \text{ bits/dimension}$$

The MMSE-DFE fix? – change the symbol rate(s)

$\bar{\mathcal{C}}(T < T^{opt}) = \mathcal{C} \cdot T \qquad \bar{\mathcal{C}}(T > T^{opt}) \leq \mathcal{C} \cdot T \qquad SNR_{MMSE-DFE,U}(T^{opt}) = 2^{2\mathcal{C} \cdot T^{opt}} - 1$

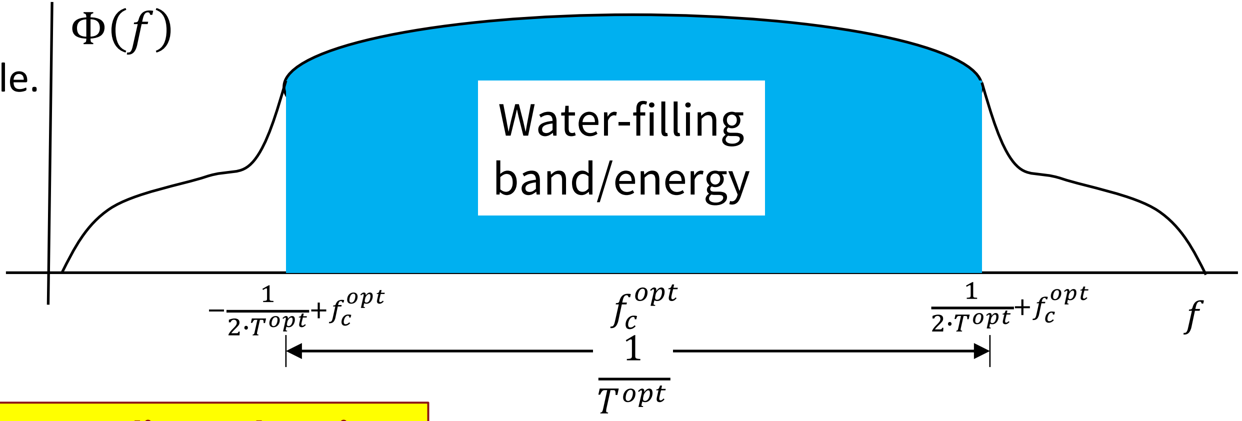


Sharp cut-off is one issue, but there are others also

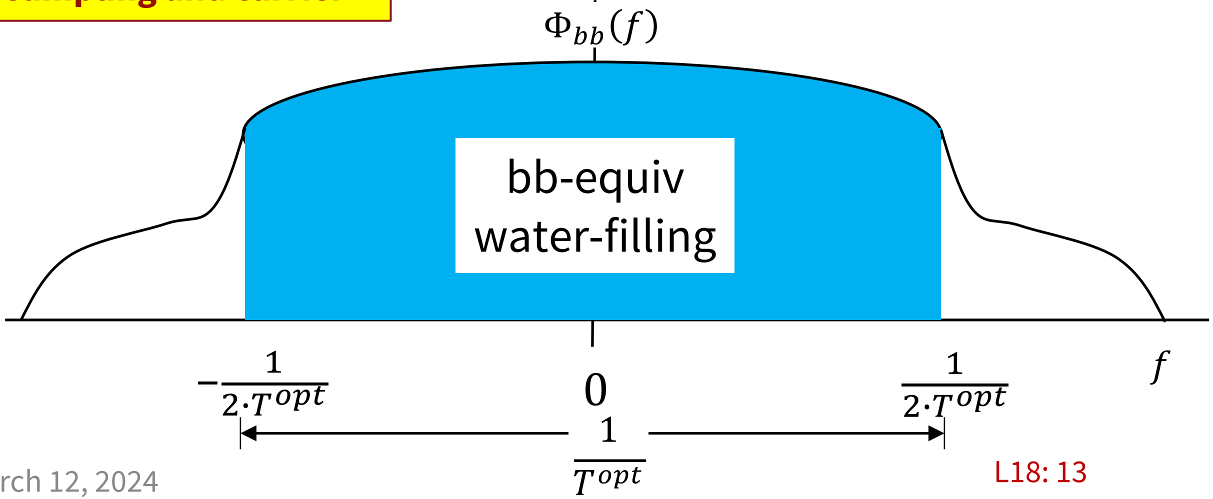
- Not so easy to do in practice (we see ways to do this digitally in 379B).

Optimum Carrier (center) Frequency

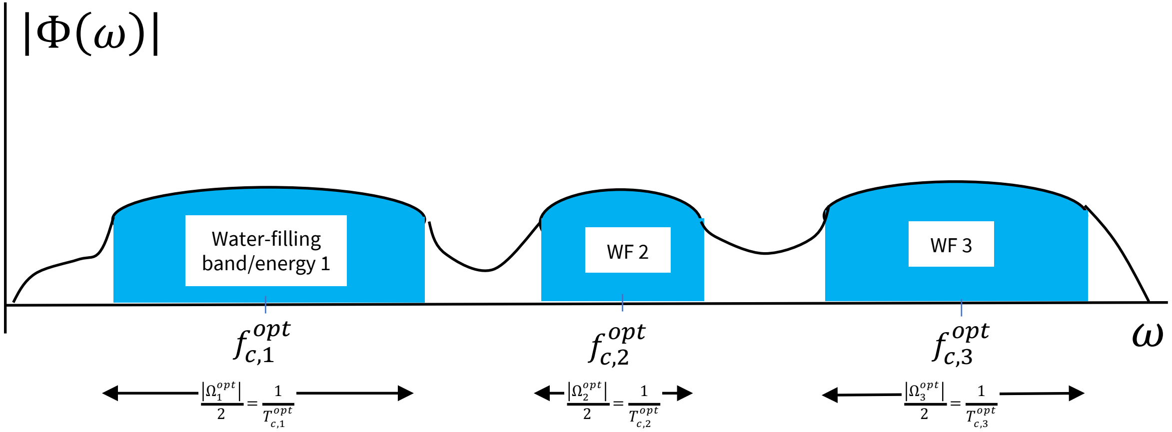
- Must be in middle.
  - Why?



Must have correct sampling and carrier



Multiple Bands ?



- Need a separate optimized carrier frequency and symbol rate for each discontiguous band.

- Transmitters blasting through the zeroed bands often experience large performance reduction,
  - especially if the applied code has nonzero gap ….

Multiband DFEs equivalent rate/SNR

- Add/stack used optima bandwidths

$$\frac{1}{T_{opt}} \triangleq \sum_{i=1}^{M} \frac{1}{\bar{T}_i^{opt}}$$

$\bar{T}_i^{opt} = cb \cdot T_i^{opt}$ (1 *or* 2 for complex or real, respectively)

- Each band has a data rate.

$$R_i = \frac{1}{\bar{T}_i^{opt}} \cdot \log_2\left(1 + \frac{SNR_i\left(T_i^{opt}\right)}{\Gamma}\right)$$

- Add the bands' data rates:

$$R \triangleq \sum_{i=1}^{M} R_i$$

$$= \sum_{i=1}^{M} \frac{1}{\bar{T}_i^{opt}} \cdot \log_2\left(1 + \frac{SNR_i\left(T_i^{opt}\right)}{\Gamma}\right)$$

- Infer an average (geometric) SNR:

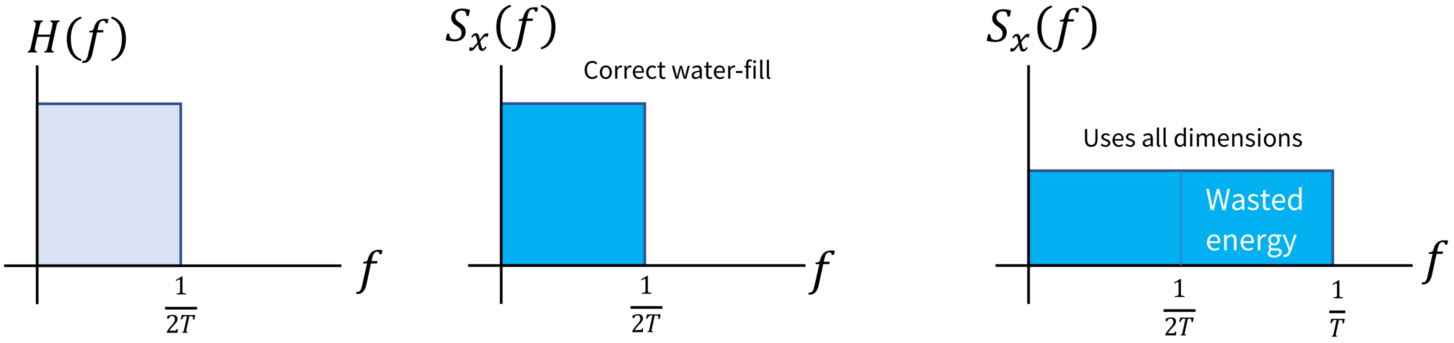
$$SNR_{MMSE-DFE,U}^{opt} \triangleq \Gamma \cdot \left\{\left[\prod_{i=1}^{M}\left(1 + \frac{SNR_i(T_i^{opt})}{\Gamma}\right)^{T^{opt}/\bar{T}_i^{opt}}\right] - 1\right\}$$

$$\bar{b}^{opt} \triangleq \frac{1}{2} \cdot \log_2\left(1 + \frac{SNR_{MMSE-DFE,U}^{opt}}{\Gamma}\right) \text{ bits/dimension} .$$

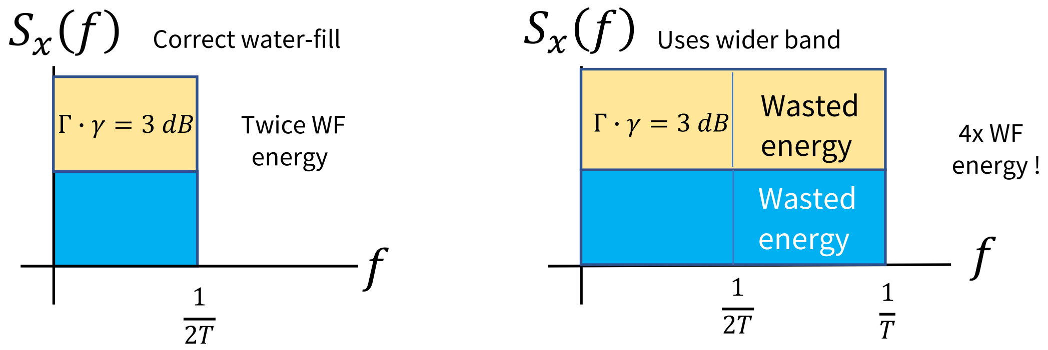
Suboptimal Transmitter Loss

Section 3.12

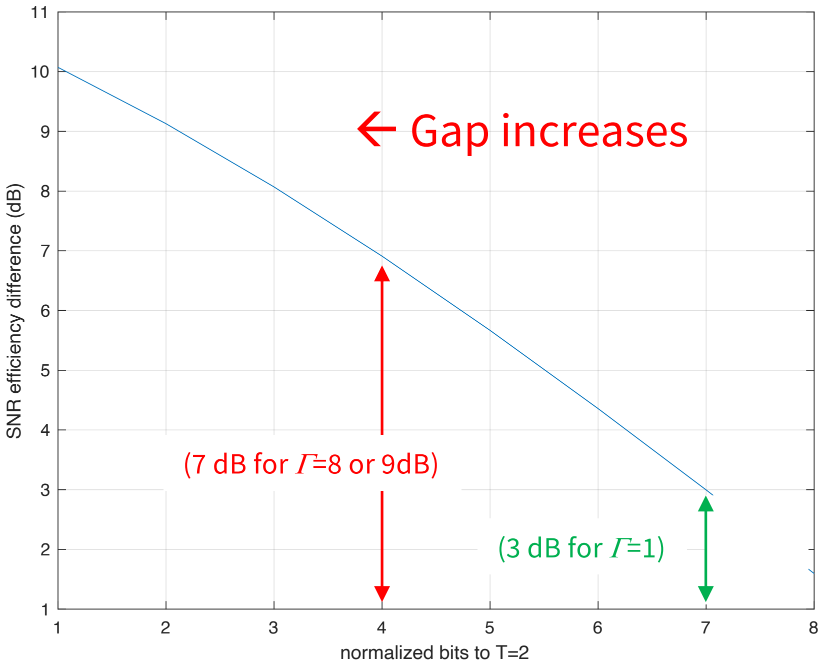
Half-Band Example



- $SNR_{MMSE-DFE,U}(T^{opt})$ is 3 dB higher than the "full" bandwidth example.
- This amount is amplified below capacity by non-unity (not 0 dB) gap-margin product.



margin difference for half-band optimum versus full band



Using wrong transmit bandwidth has performance loss, and this loss amplifies with code imperfection.

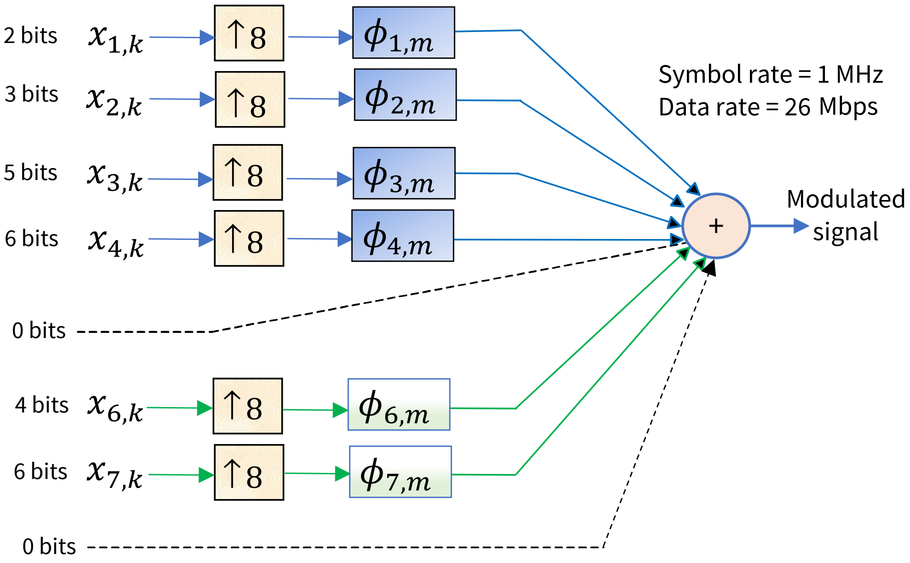
This effect can be enormous, often dwarfing code-selection as a contributor to system performance

margin difference for half-band optimum versus full band

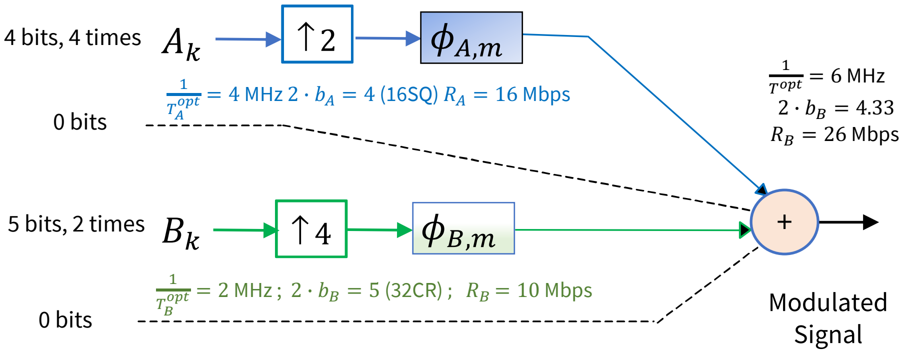
- Capacity of AWGN with WF is 8 bits/subsymbol (4 bits/dimension)
- So in addition to the 9 dB (say uncoded QAM) loss, there is another 7 dB margin loss (16 dB total loss, not 3 dB).

Dead-band DFE example – 2 Transmitters

- Use a set of (up to 8) transmitters.
  - **Waterfill** – WF integral separates into narrow bands or tones.
  - **MMSE-DFE trivializes** to simple SBS (no FF nor FB sections needed)
  - bits/subsymbol, per tone – relation to capacity still holds $SNR = 2^C - 1$
  - All have same 1/T.



- set of 2 transmitters
  - Variable 8-tone bits/dim means there is now ISI.
  - They carry the same data rate.
  - MMSE-DFE is in same relation to capacity (CDEF) holds $SNR = 2^C - 1$.

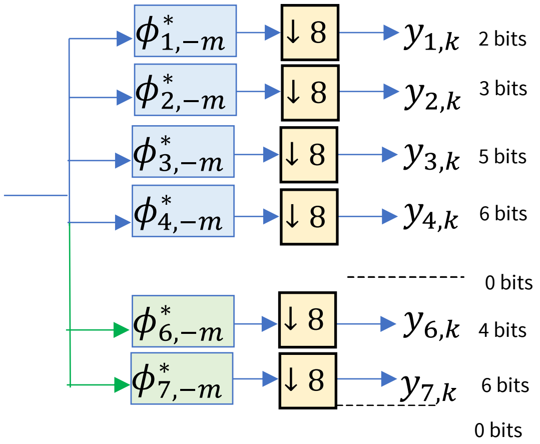


The receiver needs two DFE's for this

- EE379B examines multi-tone transmitters (set of $\phi_{n,m}$'s) that allow the water-fill-energized "tones" to stack continuously next to one another and keep simple AWGNs (no ISI) that won't need any DFE rcvrs.

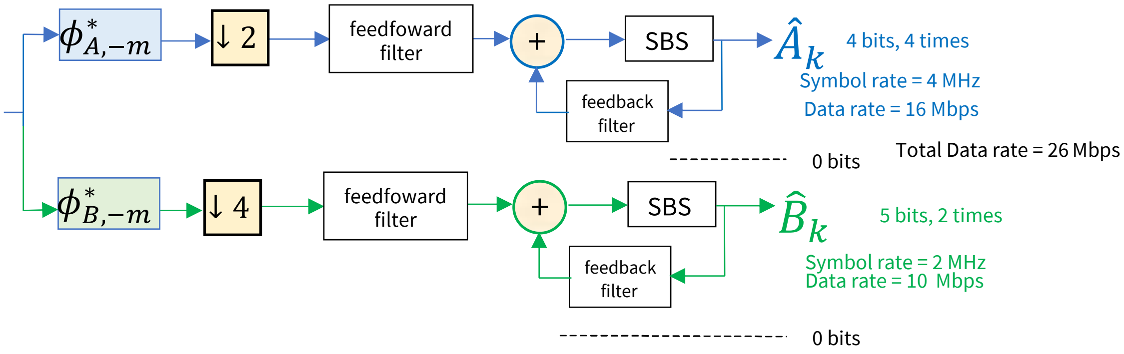
Dead-band DFE example - Receivers

Set of receivers



Symbol rate = 1 MHz
Data rate = 26 Mbps

Minimal Set of MMSE-DFEs



- Both systems have same performance (at same gap).
- Both create parallel AWGN channels with $SNR = 2^C - 1$.
- One has fewer, but more complex receivers.

More detailed dead-band analysis (L18:14,15)

n	g_n	$\mathcal{E}_n$	b_n
1	15.2	1.50	2
2	30.3	1.75	3
3	121.4	1.9375	5
4	242.7	1.97	6
5	2	0	0
6	60.7	1.875	4
7	242.7	1.97	6
8	2	0	0

$\mathcal{E}_x$=11

b=26

- Optimum symbol rate $\frac{1}{T^{opt}} = 4 + 2 = 1 + 1 + 1 + 1 + 1 + 1 = 6 \cdot 1$ MHz
- Overall data rate = 26 Mbps (=2+3+5+6 + 6+4) · 1 MHz
- Ave bits/6MHz-symbol is 26/6 = 4.33 bits/subsymbol.
- $SNR_{MMSE-DFe,U}(^1/_{T^*} = 6MHz) = 10 \cdot \log_{10}(\Gamma \cdot [2^{4.33} - 1]) = 21.6$ dB.
- Ave bits/8MHz-symbol is 26/8 – 3.25 bits/subsymbol.
- $SNR_{sum\ tones}(^1/_{T^*} = 8MHz) = 10 \cdot \log_{10}(\Gamma \cdot [2^{3.25} - 1]) = 18.1$ dB.
  - $\tilde{b}_{ave} = 3.25 \frac{bits}{tone}$ so lower corresponding ave SNR still yields $P_e = 10^{-6}$.
  - Different $^1/_{T^*}$, but same data rate $R = 26$ Mbps, same $P_e = 10^{-6}$.
  - 8 tones is simple implementation with two zeroed, the remaining DFE's trivialize.

- System A has 16 Mbps and $SNR_{MMSE-DFe,U}\left(^1/_{T_A^*} = 4MHz\right)$=20.6 dB
  - Complex MMSE-DFE
- System B has 10 Mbps and $SNR_{MMSE-DFe,U}\left(^1/_{T_B^*} = 2MHz\right)$=23.7 dB
  - Complex MMSE-DFE

A+B, or two-tone DFE, or 8-tone trivial DFE all have same performance – CDEF result
So, which is really simpler to implement? (EE379B)

Analysis of Loss

- Some designers want constant symbol rate with flat energy for each symbol (8MHz).
- Energy/1MHz is 11/8, which corresponds to:
  - 17 dB = $SNR_{MMSE-DFe,flat}\left({}^{1}/_{T} = 8MHz\right) = \Gamma \cdot \left\{ \left[\prod_{n=1}^{8} \left(1 + \frac{\frac{11}{8} \cdot g_n}{\Gamma} \right) \right]^{\frac{1}{8}} - 1 \right\}.$
- Compared to the optimum transmitter's SNR of 18.1 dB, so a 1.1 dB loss .
- Another .4 dB loss for 16 QAM precoders, then 1.5 dB loss total w.r.t. 8-tone simple dec's.
- Suppose channel change causes only lower band to be passable (set B is zeroed)?
- Best places all 11 energy units in set A, increasing by 11/(1.5+1.75+1.9375+1.97) =1.9 dB
- So previous band A of 20.6+1.9=22.5dB, or 1 dB margin for 16 QAM
- A single 1/T=8 MHz flat transmit energy of 11/8 yields SNR=12.8 dB, which only would do 4QAM, or is roughly 8 dB worse, including 1.3 dB (4/3) precoder loss.

Olympics Results

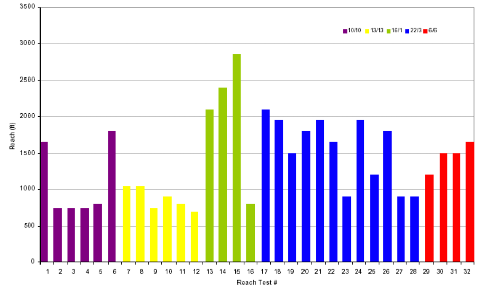
- This CDEF result has some confused predecessors
  - Price MIT
  - Zervos Bell Labs
- These ignore the “+1” term, which is equivalent to assuming infinite energy available to water fill
  - And that full flat energy is optimum

    At any 1/T ??
  - Their erroneous conclusion – “just use a ZFDFE on anything and its optimum.”
- Lead to two “Bellcore” DSL Olympics
  - 1993 ADSL – 11 dB to 30 dB margin differences across many channels
  - 2003 VDSL – see lengths for 25 Mbps at right
- After this, use of water-filling (DMT at right) became common in wired and wireless
  - See Chapter 4 or 379B

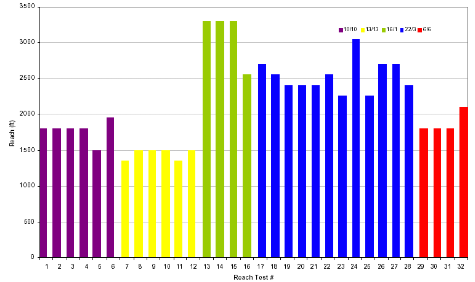
1993 ADSL Olympics – Bellcore
Margin differences at 1.6 Mbps, 4 miles, 11+dB
DMT 4x faster (6 Mbps) at 2 miles

2003 VDSL Olympics - Bellcore

Variable f_c and 1/T single-carrier QAM results



DMT* results – exact same channels as QAM



MMSE-LE Transmit Optimization

Section 3.13

Calculus of Variations

- Minimize the MMSE for LE:

$$\sigma^2_{MMSE-LE} = \frac{T}{2\pi} \cdot \int_{-\pi/T}^{\pi/T} \frac{\sigma^2 \cdot d\omega}{\|h\|^2 \cdot \left[|\Phi(e^{-j\omega T})|^2 \cdot |H(e^{-j\omega T})|^2 + 1/SNR_{MFB} \right]}$$

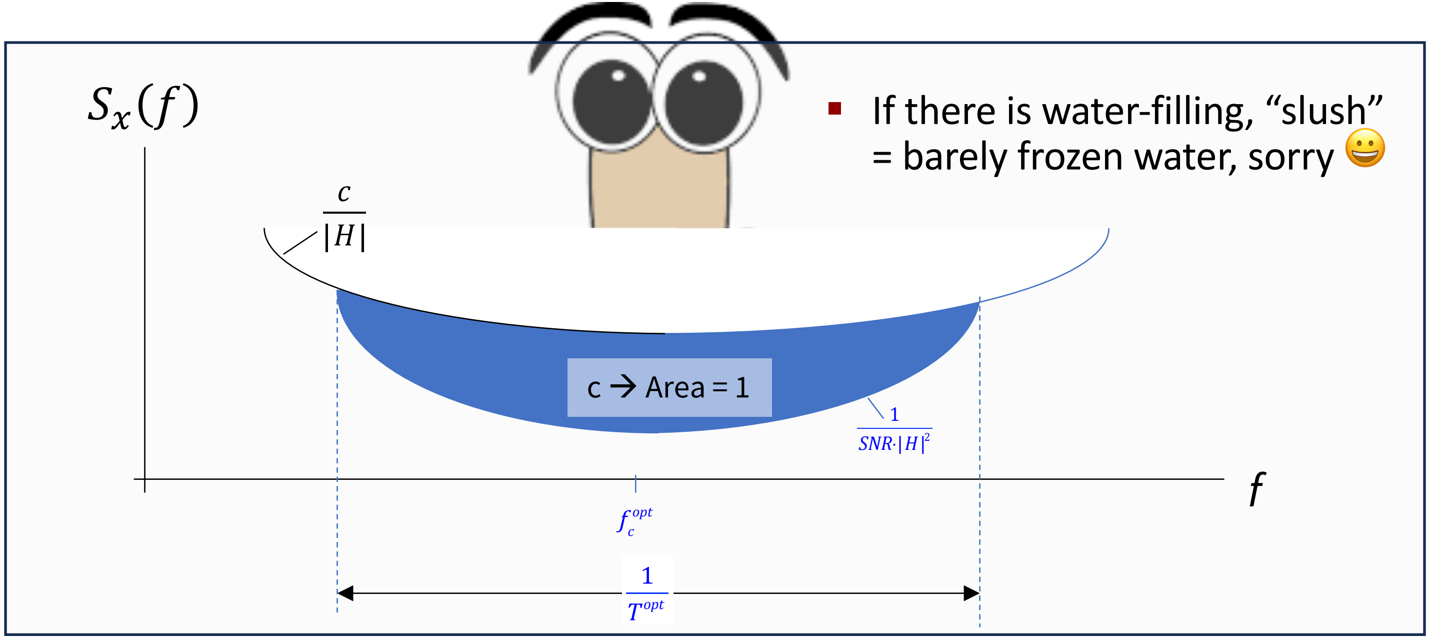
- yields

$$\left|\Phi\left(e^{-j\omega T}\right)\right|^2 = c \cdot \left|H\left(e^{-j\omega T}\right)\right| - \frac{1}{SNR \cdot |H(e^{-j\omega T})|^2}$$

$$c = \left[\frac{\mathcal{N}_0}{2} \cdot \frac{T^{opt}}{2\pi} \int_{-\frac{2\pi}{T^{opt}}}^{\frac{2\pi}{T^{opt}}} |H(\omega)| \cdot d\omega \right]^{-1} \left[1 + \frac{T^{opt}}{2\pi \cdot SNR} \int_{-\frac{2\pi}{T^{opt}}}^{\frac{2\pi}{T^{opt}}} |H(\omega)|^2 d\omega \right]$$

Slush Packing – may need iterative solution

- Solution iterates between constant c and $1/T^{opt}$.



- If there is water-filling, "slush" = barely frozen water, sorry 😀

- If linear is desirable, use many tones and no equalizer, see Chapter 4/379B – not aware of any uses of slush packing.

End Lecture 18