STANFORD

*Lecture 17*

# Precoders and Diversity

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# Announcements & Agenda

- Announcements
  - PS8 is due March 12
    - Solutions immediately available, thus no late homework
    - Final distributed March 14,end of lecture.  Due 5 pm on Friday 3/15.

- Today
  - Announcing our winner on the “chirp” sequence
  - Transmit Precoders
  - Partial-response channels
  - Diversity and Turbo Equalizers


Kevin Marx

(Normalized) DFT

$$X_n = \frac{1}{\sqrt{N}} \cdot \sum_{k=0}^{N-1} x_k \cdot e^{-j\frac{2\pi}{N}kn}$$

$$x_k = \frac{1}{\sqrt{N}} \cdot \sum_{n=0}^{N-1} X_n \cdot e^{j\frac{2\pi}{N}kn}$$

# Norm Tap Error

- Parsevals $||\boldsymbol{h}||^2 = ||\boldsymbol{H}||^2$
  - And all other frequency-domain/time-domain vectors.

$$
\begin{aligned}
\text{NTE} &\triangleq E\left\{\|\boldsymbol{h}-\hat{\boldsymbol{h}}\|^2\right\} = E\left\{\|\boldsymbol{\delta}\|^2\right\} \\
&= E\left\{\|\boldsymbol{H}-\hat{\boldsymbol{H}}\|^2\right\} = E\left\{\|\boldsymbol{\Delta}\|^2\right\} \\
&= \sum_{n=0}^{\bar{N}-1} |H_n - \hat{H}_n|^2 = \sum_{k=0}^{\bar{N}-1} |h_k - \hat{h}_k|^2
\end{aligned}
$$

$$\hat{H}_n = H_n + \sum_{l=1}^{L} \frac{U_{l,n}}{L \cdot X_{l,n}} \quad ,$$

$$\Delta_n = -\sum_{l=1}^{L} \frac{U_{l,n}}{L \cdot X_{l,n}} \quad .$$

$$
\begin{aligned}
E_n &= Y_n - \hat{H}_n \cdot X_n = \Delta_n \cdot X_n + U_n \\
&= U_n + \frac{1}{L} \cdot \sum_{l=1}^{L} U_{l,n} \cdot e^{\jmath(\theta_n - \theta_{l,n})} \quad .
\end{aligned}
$$

$$SNR_{\hat{H},n} = \frac{R_{xx,n} \cdot |H_n|^2}{R_{ee,n} - R_{uu,n}} = \frac{SNR_n}{1/L} = L \cdot SNR \;(all\ n)$$

- Same in all dimensions
  - The $L$ =40 leaves 0.1 dB gain-estimation error (1+1/40 = 0.1 dB)

- Constant magnitude in both domains
  - “White”
  - Lowest peak-to-average

Good training sequence?

$$x_k = e^{j\frac{2\pi}{N}\cdot k^2}$$

(chirp)

DFT is

$$X_n = \frac{1}{\sqrt{N}} \cdot e^{-j\frac{\pi}{4N}\cdot n^2}$$

Another chirp

# Noise Estimation

- Average the errors in frequency domain

$$\hat{\sigma_n^2} = \frac{1}{L} \cdot \sum_{l=1}^{L} |E_{l,n}|^2$$

$$\mathrm{var}\left(\hat{\sigma_n^2}\right) = \frac{1}{L^2}\left(3 \cdot L \cdot \sigma_n^4 - L \cdot (\sigma_n^2)^2\right) = \frac{2}{L}\sigma_n^4$$

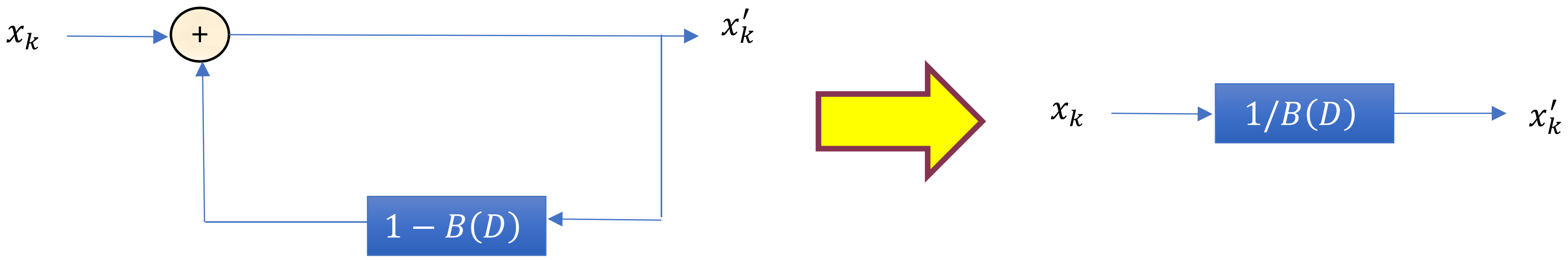
- Noise miss only reduces with sqrt(L).

$$\sqrt{2/L} \cdot \sigma_n^2$$

# Precoders

*Section 3.8*

# Channel Inversion at the transmitter?

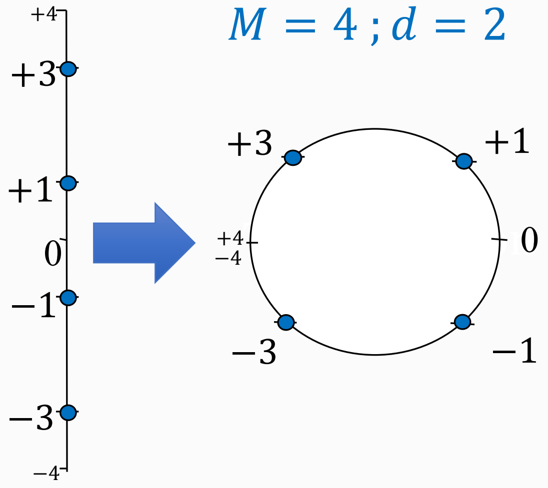



- So why not “pre-equalize” and remove ISI?  Indeed, there is no noise enhancement at transmitter, so?
- Transmit energy exceeds the limit.  Pre-inversion design reduces distance, and thus incurs a loss.
- But it still can be a good idea with a little help ….

# Modulo Arithmetic

- Mod arith wraps the real line around circle of circumference $2M$ ; $\Gamma_M(x) = x - M \cdot d \cdot \left\lfloor \frac{x + \frac{M \cdot d}{2}}{M \cdot d} \right\rfloor$,

$M = 4 ; d = 2$




| $x_k$ | $\Gamma_4(x_k)$ |
|---|---|
| +3 | +3 |
| -3 | -3 |
| -4.5 | +3.5 |
| -6 | +2 |
| +6.1 | -1.9 |

**Maps uniform Distribution $[-4i \quad 4i]$ $i \in Z$ to another uniform Distribution $[-4 \quad 4]$**

- Any two numbers with result on circle :

$$x \oplus_M y \triangleq \Gamma_M(x + y)$$
$$x \ominus_M y \triangleq \Gamma_M(x - y)$$

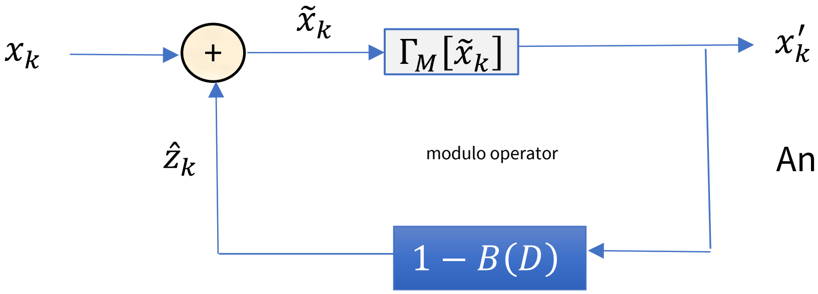
- Trivially:

$$\Gamma_M(x + y) = \Gamma_M(x) \oplus_M \Gamma_M(y)$$
$$\Gamma_M(x - y) = \Gamma_M(x) \ominus_M \Gamma_M(y)$$

# The Precoder

- Tomlinson – Harashima (and probably others …. ….) illustrates basics, not used to instructor's knowledge.

M-ary input




Approximately uniform distribution over $[-\frac{Md}{2}, \frac{Md}{2})$

And i.i.d. ~ same as quantization noise

$$\tilde{x}_k = x_k - \sum_{i=1}^{\infty} b_i \cdot x_{k-i}'$$

$$x_k' = \Gamma_M(\tilde{x}_k) = \Gamma_M\left(x_k - \sum_{i=1}^{\infty} b_i \cdot x_{k-i}'\right)$$

Implementation is for real baseband signals.

$$z_{U,k} = \left(x_k' + \sum_{i=1}^{\infty} g_{U,i} \cdot x_{k-i}'\right) + e_{U,k}$$

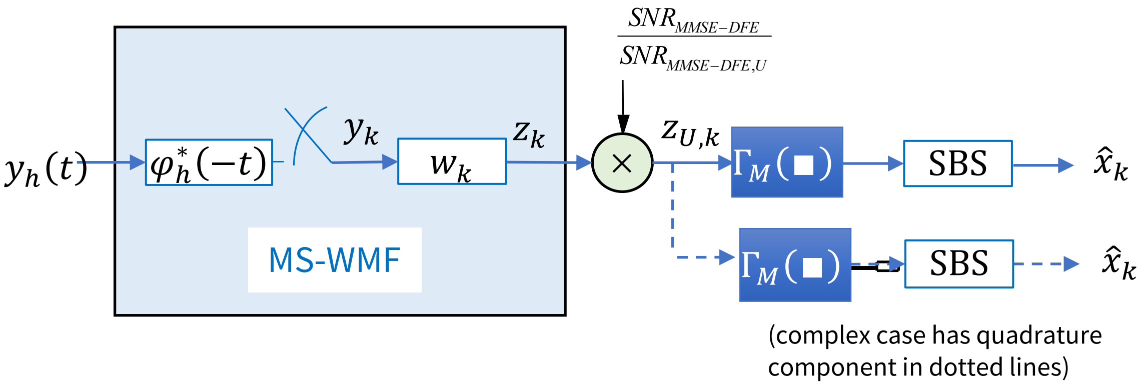
Another receiver modulo produces:

$$\Gamma_M(z_{U,k}) = x_k + e_{U,k}' \cong x_k + e_{U,k}$$

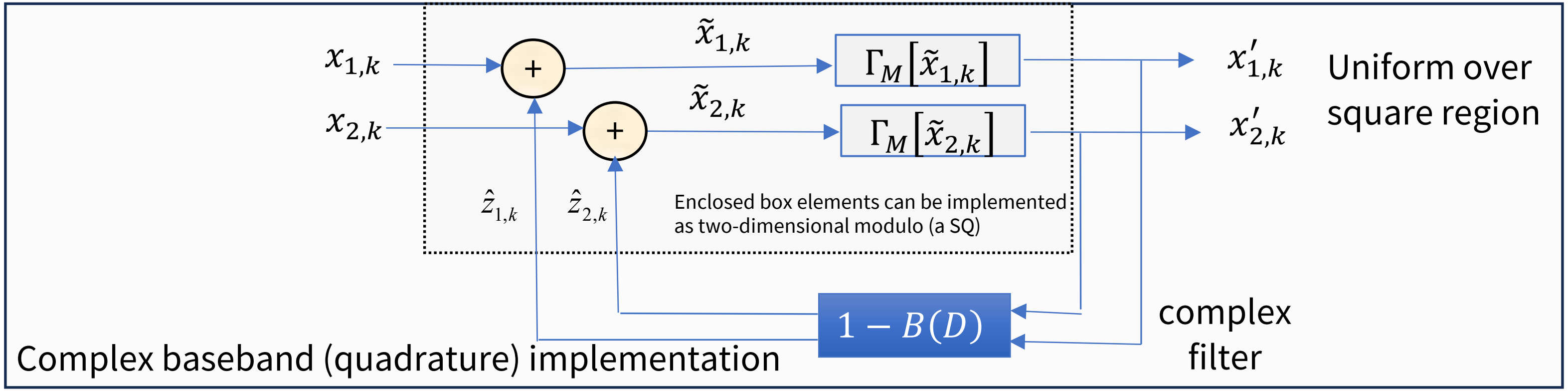
**No errors at transmitter!**

$$\begin{aligned}
\Gamma_M[z_{U,k}] &= \Gamma_M\left[\Gamma_M(x_k - \sum_{i=1}^{\infty} g_{U,i} \cdot x_{k-i}') + \sum_{i=1}^{\infty} g_{U,i} \cdot x_{k-i}' + e_{U,k}\right] \\
&= \Gamma_M\left[x_k - \sum_{i=1}^{\infty} g_{U,i} x_{k-i}' + \sum_{i=1}^{\infty} g_{U,i} \cdot x_{k-i}' + e_{U,k}\right] \\
&= \Gamma_M[x_k + e_{U,k}] \\
&= x_k \oplus_M \Gamma_M[e_{U,k}] \\
&= x_k + e_{U,k}' \quad .
\end{aligned}$$

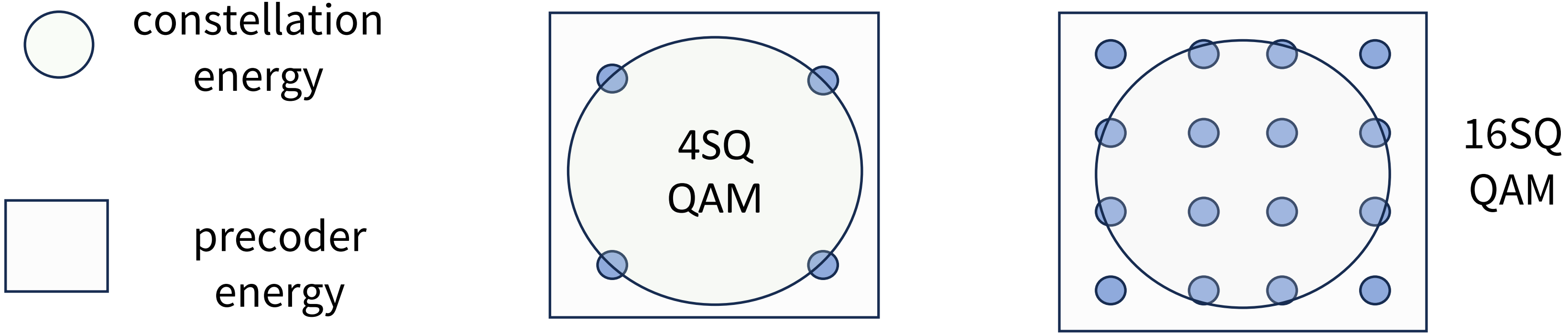
# Receiver Block Diagram (no feedback)




- Receiver only needs (unbiased) feedforward filter (feedback already presubtracted in transmitter).



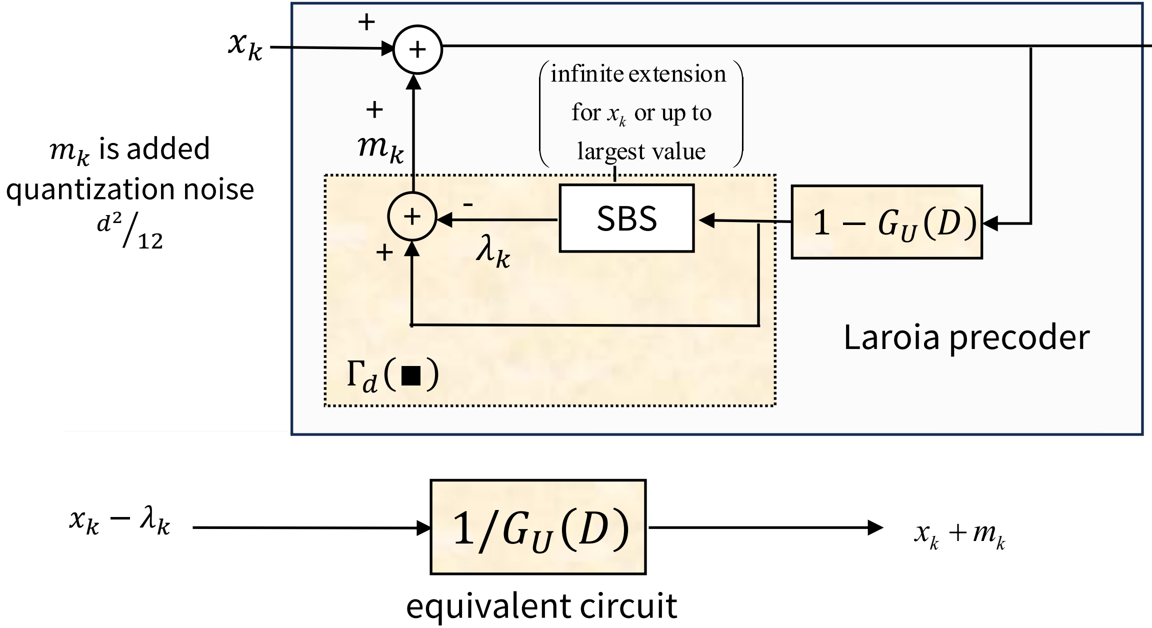
# Transmit energy increases slightly

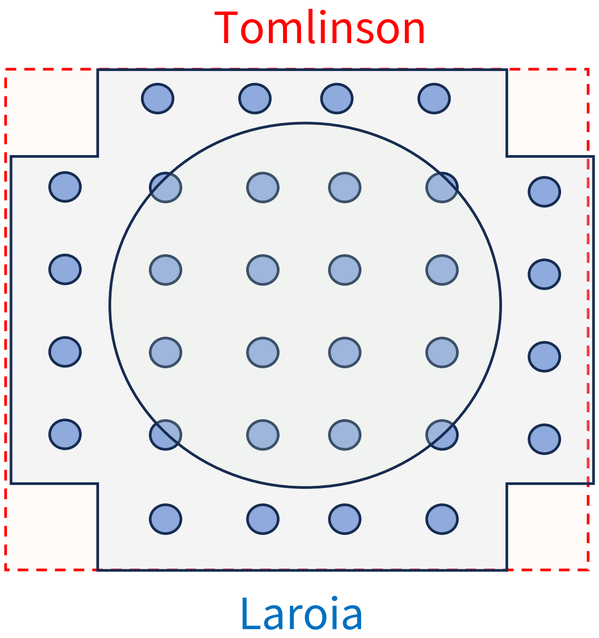



- The (Tomlinson) precoder output has (theoretically & practically) *continuous* uniform dist'n and
  - has uniform distribution over $\left[-\frac{M}{2}d \;\; , \; +\frac{M}{2}d\right]$. Mean-square (energy) $\frac{M^2 \cdot d^2}{12}$ > PAM's $\frac{(M^2-1) \cdot d^2}{12}$.
  - So there is a transmit energy effective loss of $\frac{M^2}{M^2-1}$.
  - For binary, this is 4/3 or 1.3 dB , as $M$ increases, the loss becomes negligible.
- The same applies for SQ QAM with $M^2 \rightarrow M$. Shaping is not possible with **this** precoder.

# Laroia Precoder – accommodates shaping

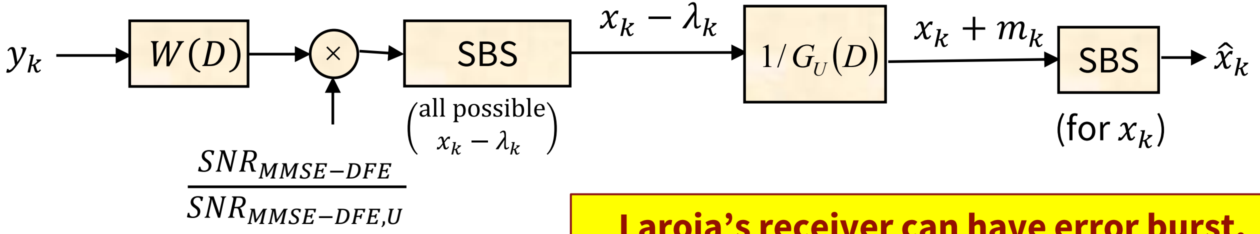
- **Laroia Precoder** inverts channel independent of constellation size (used in voiceband modem standards).




- Energy increase is $\frac{\bar{\mathcal{E}}_x + \frac{d^2}{12}}{\bar{\mathcal{E}}_x} = \frac{M^2-1}{M^2-2}$, worse than Tomlinson (3/2 for $M = 2$), preserves $\gamma_s$ at large $M$.

# Laroia's Receiver Circuit

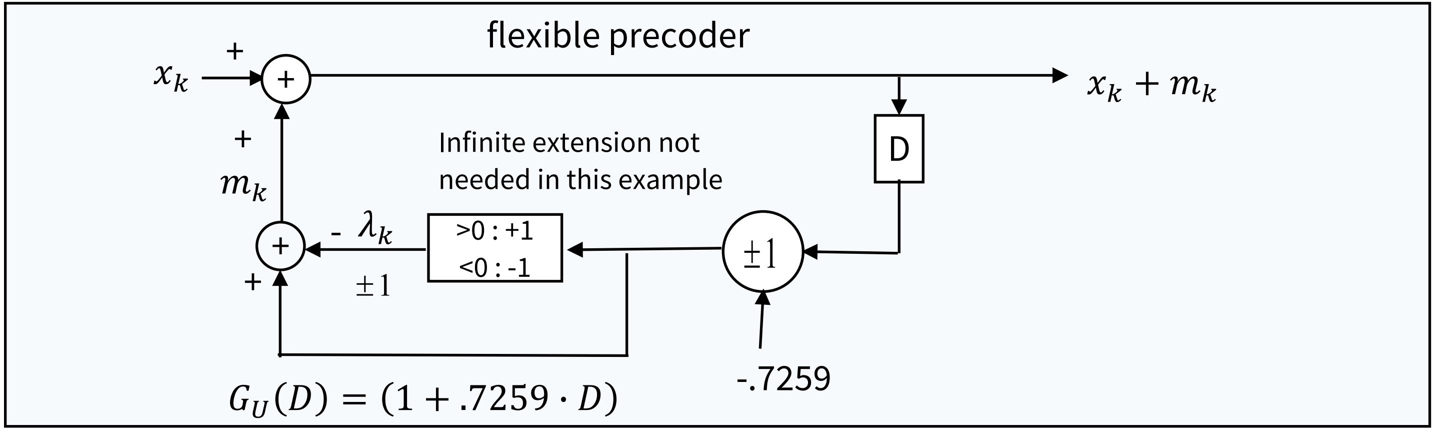


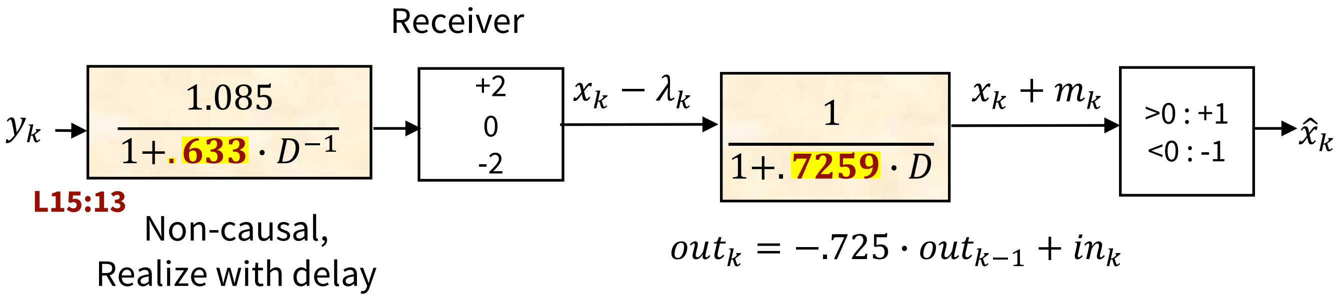

**Laroia's receiver can have error burst, but no propagation beyond. Model $G_u(D)$ as IIR, so then finite burst**

- LP was used in some wireline standards. Fortunately, there is a better solution – see EE379B (Chapter 4), which can also preserve $\gamma_s$ .
- In MIMO systems a "GDFE" is used (also see 379B, Chapter 5), and its error propagation is very finite and limited, so those GDFE systems can be used largely without error-propagation concern.

# Return to 1+.9D$^{-1}$ example

- $N_f = \infty$ ; $\bar{\mathcal{E}}_x = 1$ ; $N_b = 1$ ; $\Delta = 1$ ; $\sigma^2 = .181$



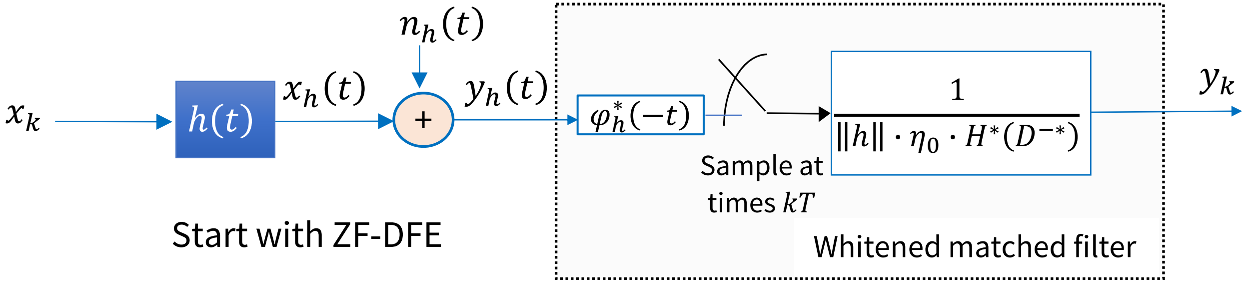



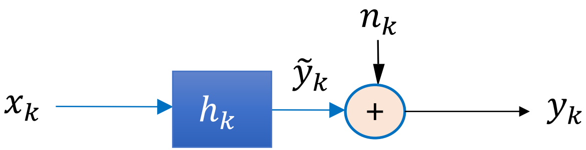
# Partial Response & Precoders

*Section 3.8.2*

Generally: “precoders” move decision feedback to the transmitter

# Model (“equalize”) channel to $h_k \in \mathbb{Z}$



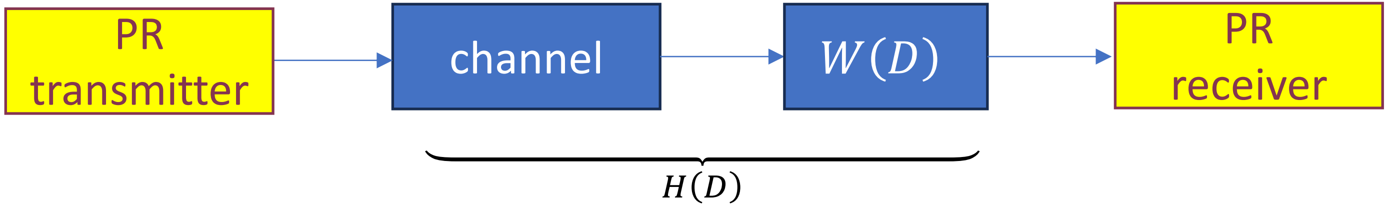




controlled ISI or **partial-response** channel, $h_{k<0} = 0$ , $h_{k>\nu} = 0$ and $h_k \in \mathbb{Z}$

- The precoder simplifies, and there is no transmit energy loss if $h_k \in \mathbb{Z}$.
- The receiver has ZF-DFE performance for this integer channel with no error propagation.
- A Viterbi Detector for ISI can be used – increases performance (often, not always) to $SNR_{MFB}$.
  - Requires more receiver complexity.
  - Often imposed at transmitter, but there may be an equalizer(with loss) to force the integer channel coefficients.
  - Any external coding gain is largely lost unless iterative decoding is used (see Turbo Equalization at end of this L17.)

# Partial-Response Channel Definition

- $H(D)$ has the **monic finite-length** form $H(D) = 1 + h_1 \cdot D + h_2 \cdot D^2 + \cdots + h_v \cdot D^\nu$, AND
- $H(D)$ is **minimum-phase** (all roots/zeros outside or on unit circle), **AND**
- **Integer coefficients** $h_k \in \mathbb{Z}$ .




- The channel may need equalization to such a response (See Section 3.13 – not taught).
  - If the $H(D)$ is pretty close to actual channel, this equalization loss can be small.

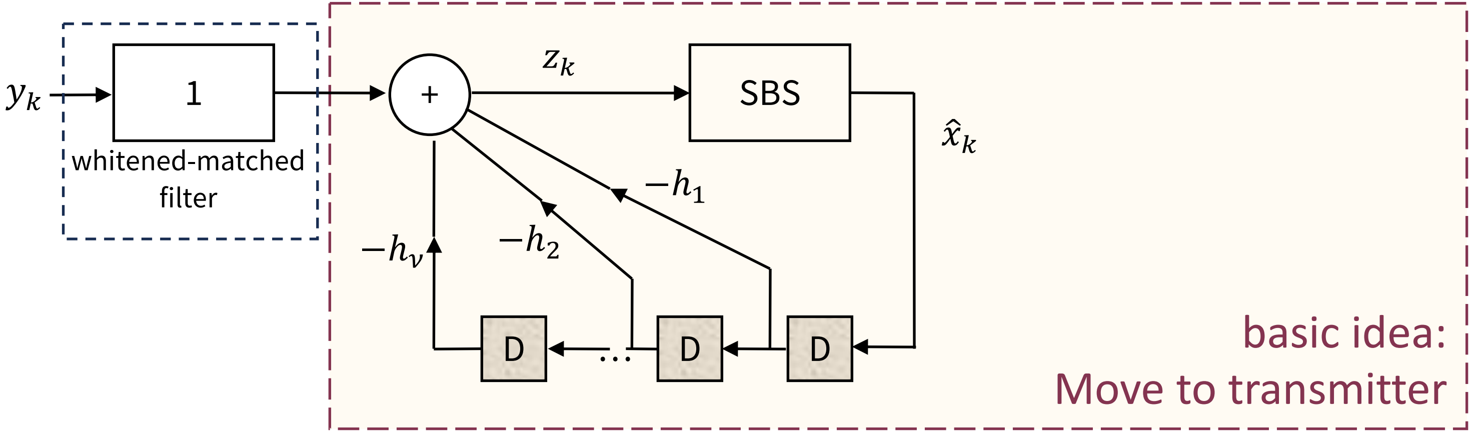
- We're going to assume here the blue part is already done.

# Precoding the PR's ZF-DFE is easy

Simple DFE operations

For $H(D) = 1 - D$ , $z_k = y_k - \hat{x}_{k-1}$

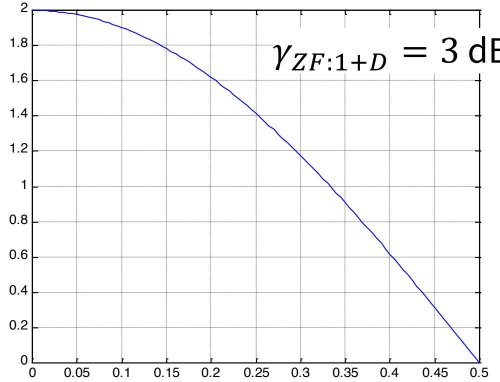
For $H(D) = 1 + D - D^2 - D^3$, $z_k = y_k - \hat{x}_{k-1} + \hat{x}_{k-2} + \hat{x}_{k-3}$

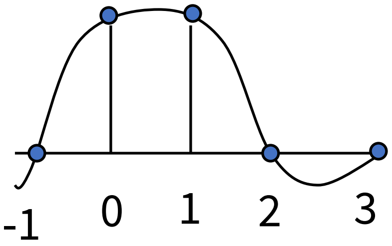



- The original channel is minimum phase, monic, and ZF-DFE becomes trivial.
- The ZF-DFE has error propagation.
- The precoder is easier, has no energy increase, and preserves the constellation.

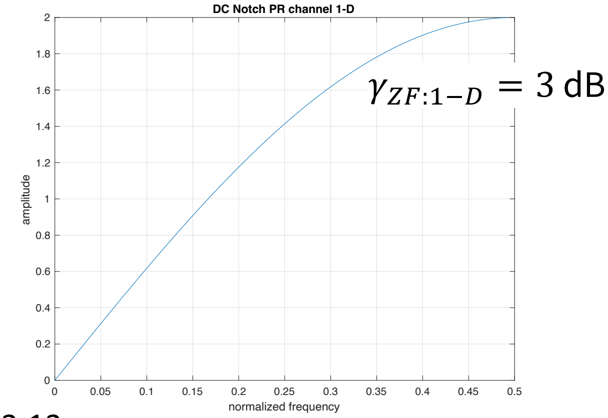
# First, some popular PR channels

- **Duobinary** $H(D) = 1 + D$ - models lowpass channel.

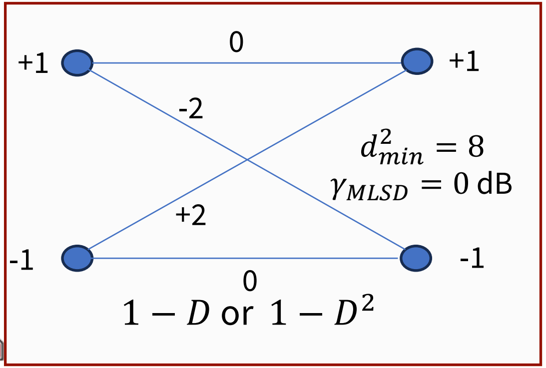


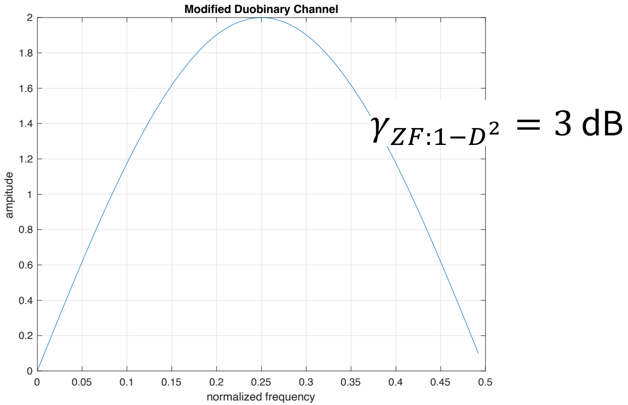




- **DC-Notch** $H(D) = 1 - D$ – models highpass



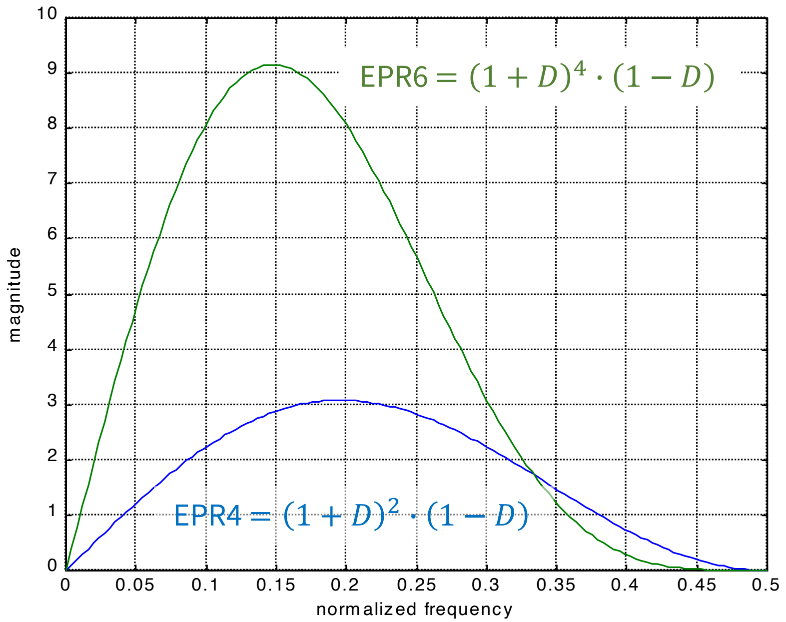

- **Modified Duobinary** $H(D) = 1 - D^2$ - lowpass with DC notch "PR4/PRML" 😀, 3.13

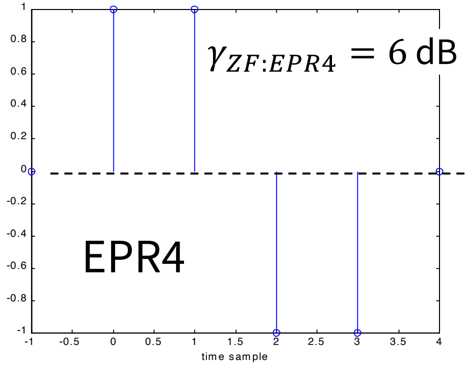


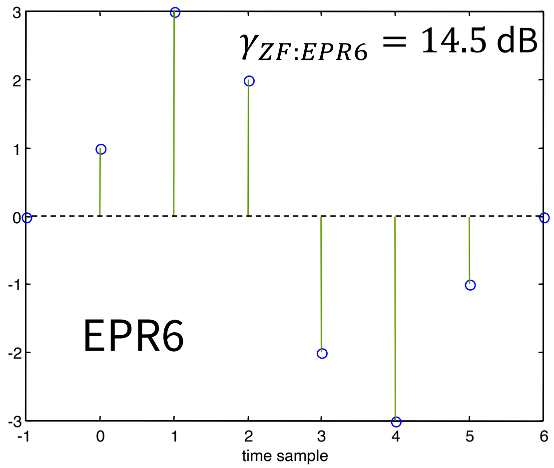




**A potential project: run Chap 7 dmin program for MLSD with Euclidean distance replace bdistance.m. All have 0 dB loss w.r.t. BINARY MFB**

# Extended Partial Response (Thapar)



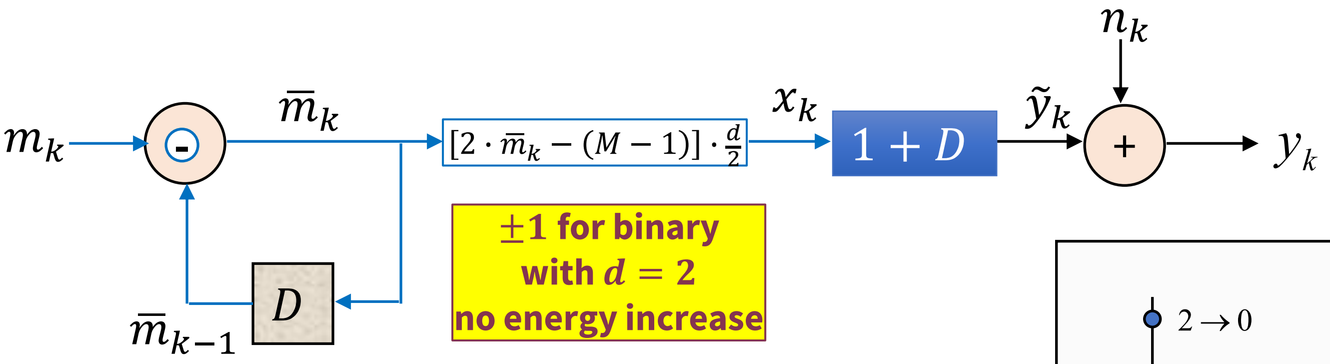




**MLSD losses are**
**EPR4: 0 dB**
**EPR6: 2.2 dB**
**...**
**EPR8: 3.7 dB**
**EPR10: 4.5 dB**

- 1+D factors increase ISI / lowpass effect.
- $H(D) = (1+D)^n \cdot (1-D)$.
- EPR is often used in “recording” channels (disk).

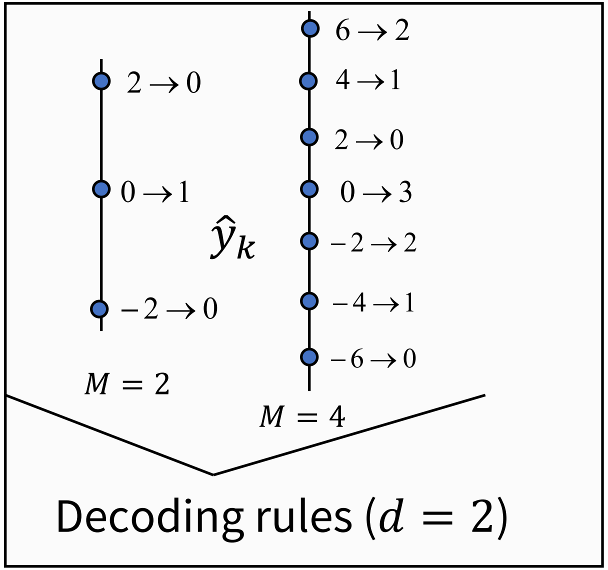
# PR Precoders




**$\pm 1$ for binary with $d = 2$ no energy increase**

- 1-D is almost the same with decoding rule flipped.
- General PR precoder is $\bar{m}_k = m_k \oplus_{i=1}^{\nu} (-h_i) \cdot \bar{m}_{k-i}$ .
- General decode:

$$\hat{m}_k = \left( \frac{\hat{y}_k}{d} + \sum_{i=0}^{\nu} h_i \cdot \left[ \frac{M-1}{2} \right] \right)_M$$

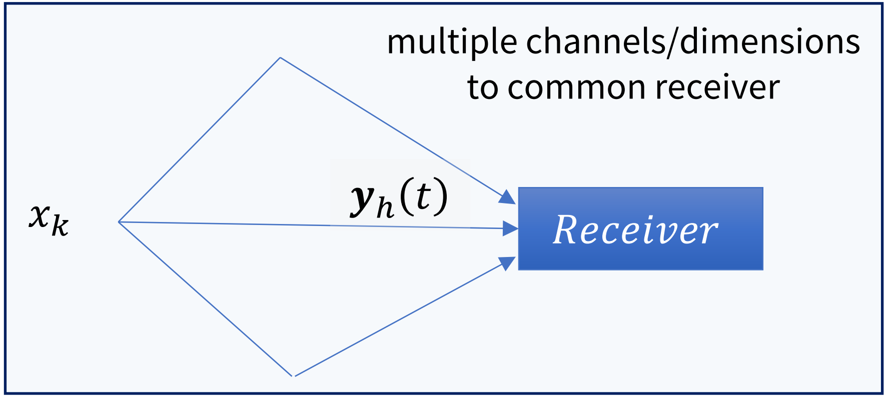


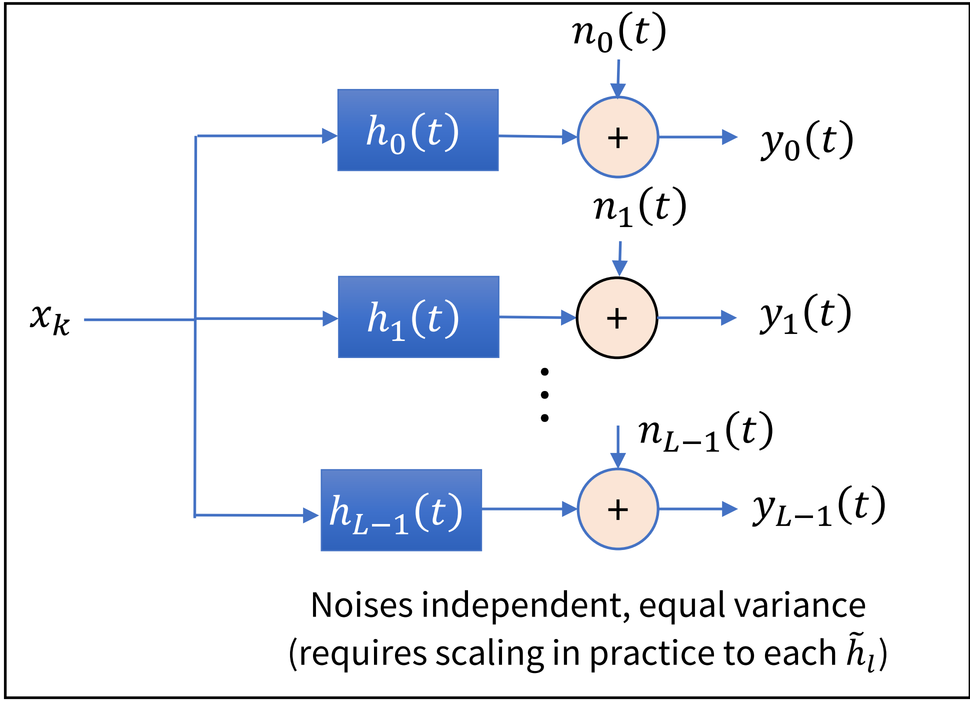

- General error prob: $P_e \leq 2 \cdot \left(1 - \frac{2}{M^{\nu+1}}\right) \cdot Q\left(\frac{d}{2\sigma}\right)$
- For quadrature versions, see 3.8.6

# Diversity Equalizers

*Section 3.8.2*

# Multiple Received Signals




$$\boldsymbol{y}_h(t) = \sum_k x_k \cdot \boldsymbol{h}(t - kT) + \boldsymbol{n}_h(t)$$

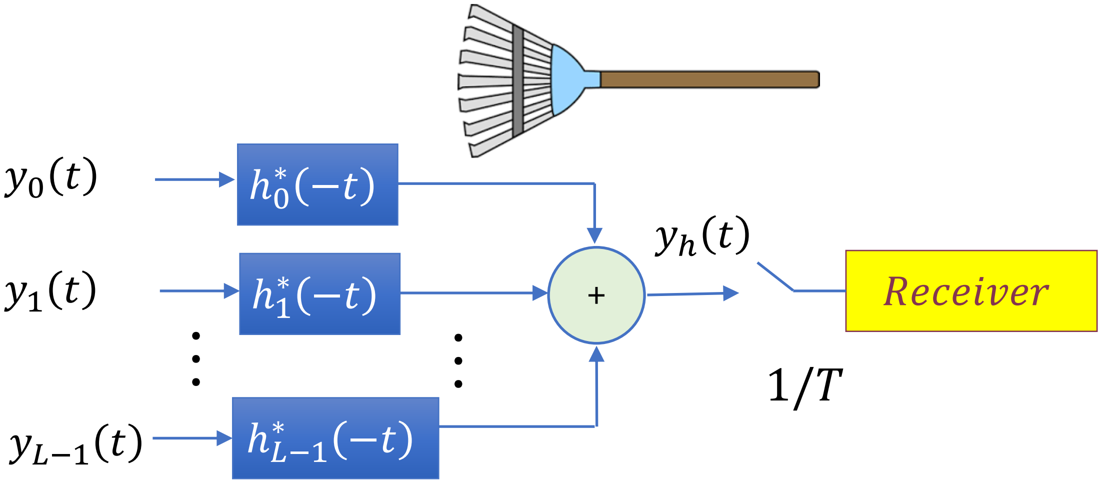
- Multiple rcvr antennas (but single xmit antenna)
- Multiple repetitions (codes are form of diversity)
- MFB? (single no-ISI transmission)

$$\boldsymbol{y}_h(t) = x_0 \cdot \boldsymbol{h}(t) + \boldsymbol{n}_h(t)$$

$$SNR_{MFB} = \frac{\bar{\mathcal{E}}_x \cdot \|\boldsymbol{h}\|^2}{\sigma^2}$$

$$\|\boldsymbol{h}\|^2 = \sum_{l=0}^{L-1} \left\|\tilde{h}_l\right\|^2$$

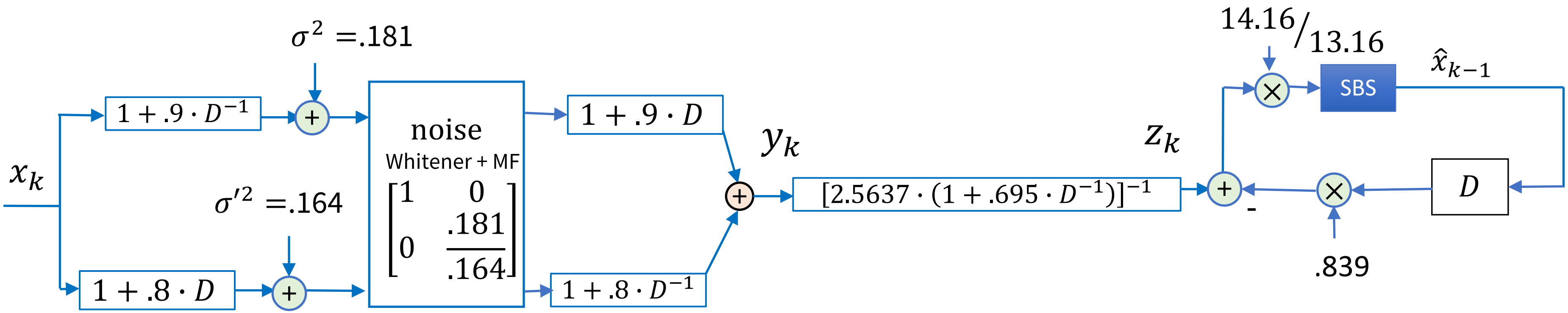
# "RAKE" Receiver (P. Green)




$$r(t) = \sum_{l=0}^{L-1} r_l(t) = \sum_{l=0}^{L-1} \tilde{h}_l(t) * \tilde{h}_l^*(-t) \qquad r(t) = \|\boldsymbol{h}\|^2 \cdot q(t)$$

- Infinite-length theory applies to the scalar $Y(D) = X(D) \cdot \|\boldsymbol{h}\|^2 \cdot Q(D) + N(D)$.
- Receiver can be LE, DFE, MMSE or ZF …. (MLSD, etc).
- Can be implemented digitally with sufficiently high sampling rate preceding the matched filters.

# 2 Parallel Channels Example




$$\|\boldsymbol{h}\|^2 = 1.81 + \left(\frac{.181}{.164}\right) \cdot 1.64 = 2 \cdot 1.81 = 3.62$$

$$SNR_{MFB} = \frac{1 \cdot 3.62}{.181} = 20 \text{ (13) dB}$$

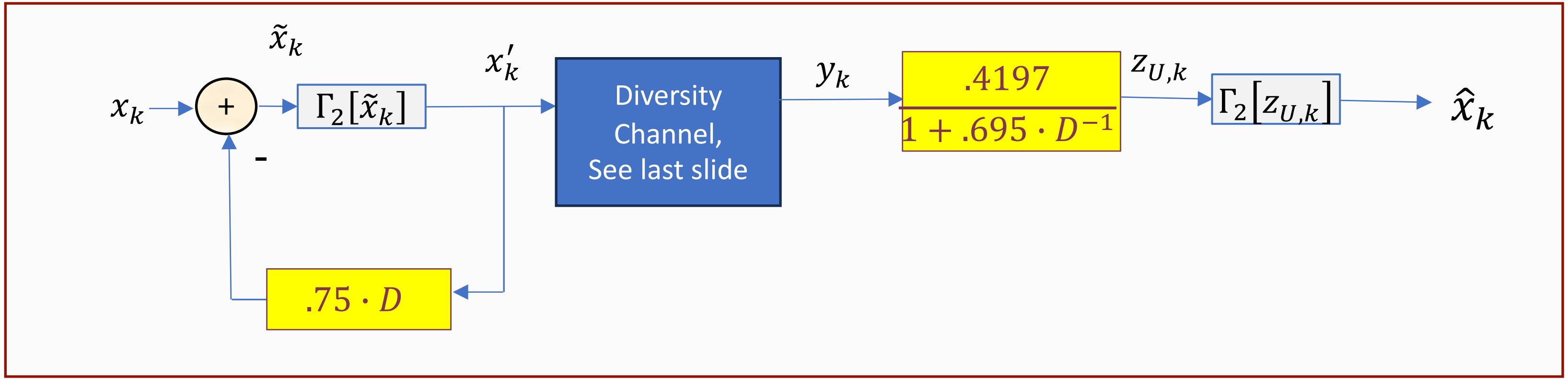
$$SNR_{MMSE-DFE,U} = .7082 \cdot 20 - 1 = 13.16 \text{ (11.15 dB)}$$

$$\begin{aligned} Q(D) &= \frac{1}{2(1.81)} \left[ (1+.9D^{-1})(1+.9D) + \frac{.181}{.164}(1+.8D)(1+.8D^{-1}) \right] \\ &= .492D + 1 + .492D^{-1} \\ &= .589 \cdot (1+.835D) \cdot (1+.835D^{-1}) \\ \tilde{Q}(D) &= .492D + (1 + 1/20) + .492D^{-1} \\ &= .7082 \cdot (1+.695D) \cdot (1+.695D^{-1}) \end{aligned}$$

- Once equivalent sum-channel is found, subsequent receiver analysis is the same as earlier (LE, DFE, even MLSD)

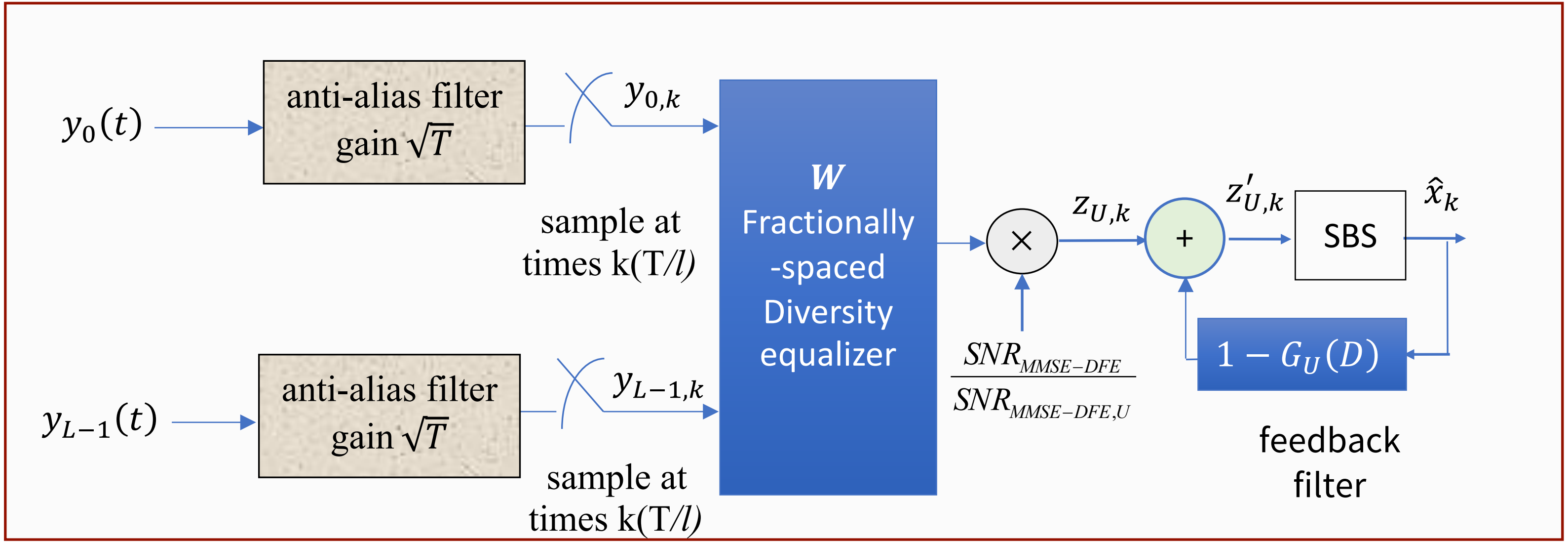
# Combine with Precoder?

- Need $G_U(D) = 1 + \left(\frac{14.16}{13.16}\right) \cdot D = 1 + .75 \cdot D$ ; similarly $W_U(D) = \left(\frac{14.16}{13.16 \cdot 2.5637}\right) \cdot \frac{1}{1+695 \cdot D^{-1}} \cong \frac{.8}{1+.7 \cdot D^{-1}}$




- Precoder loss (binary) is 4/3 or 1.3 dB, so SNR = 11.2 dB – 1.3 = 9.9 dB.
- Note that multiple paths, with appropriate receiver design, always improves the performance w.r.t. either path individually

# Multidimensional FIR Equalizer/DFE




- The design can again follow the single-channel case theory
- or use DFERake.m .
- "**Diversity Receiver**" is the modern name for it.

# DFE Rake Program

```
>> help dfeRAKE
  function [dfseSNR,W,b]=dfeRAKE(l,h,nff,nbb,delay,Ex,noise);
  DFE design program for RAKE receiver

  Inputs
  l    = oversampling factor
  L is derived as No. of fingers in RAKE (number of rows in h)
  h    = pulse response matrix, oversampled at l (size),
         each row corresponding to a diversity path
  nff  = number of feedforward taps for each RAKE finger
  nbb  = number of feedback taps
  delay = delay of system <= nff+length of p - 2 - nbb
  Ex   = average energy of signals
  noise = noise autocorrelation vector (size L x l*nff)
  NOTE: noise is assumed to be stationary, but may be spatially
correlated

  outputs:
  dfseSNR = equalizer SNR, unbiased in dB  ------------------------------
```

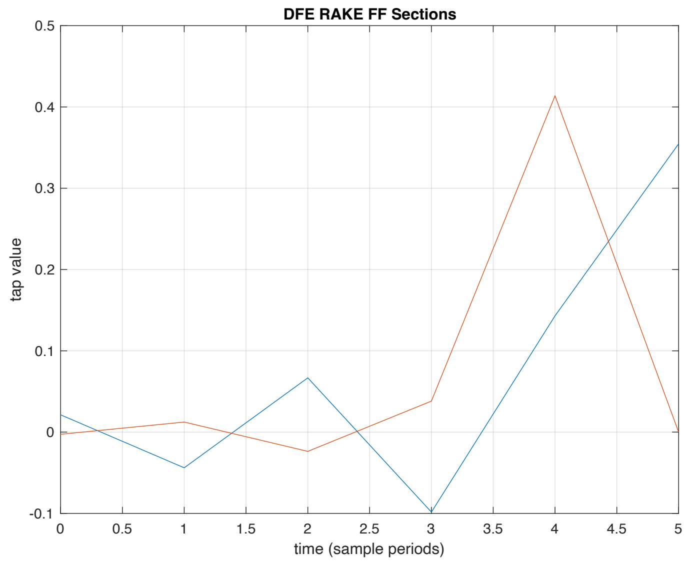
```
>> hrake=[0.9000   1.0000      0
      0   1.0500   0.8400];
>> [snr,W,b] = dfeRAKE(1,hrake,6,1,5,1,[.181 zeros(1,5) ; .181 zeros(1,5)])

snr =   11.1465 dB
W =
   0.0213  -0.0439   0.0668  -0.0984   0.1430   0.3546
  -0.0027   0.0124  -0.0237   0.0382   0.4137   0.0000
b =   0.7022
```

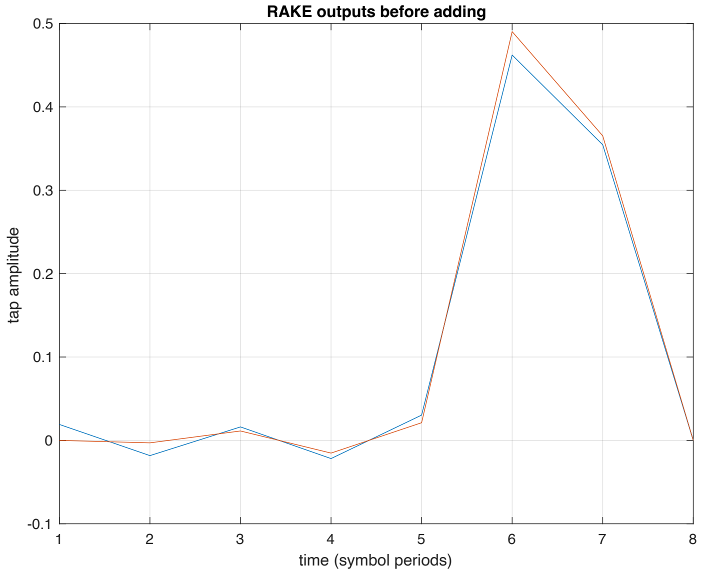
**Student Project: Add the -1 = delay option to find best delay.**

- Few taps, matches infinite-length result.

# DFE Rake Plots

- The MS-WMF's try to align to on another as well as in time to their respective paths.



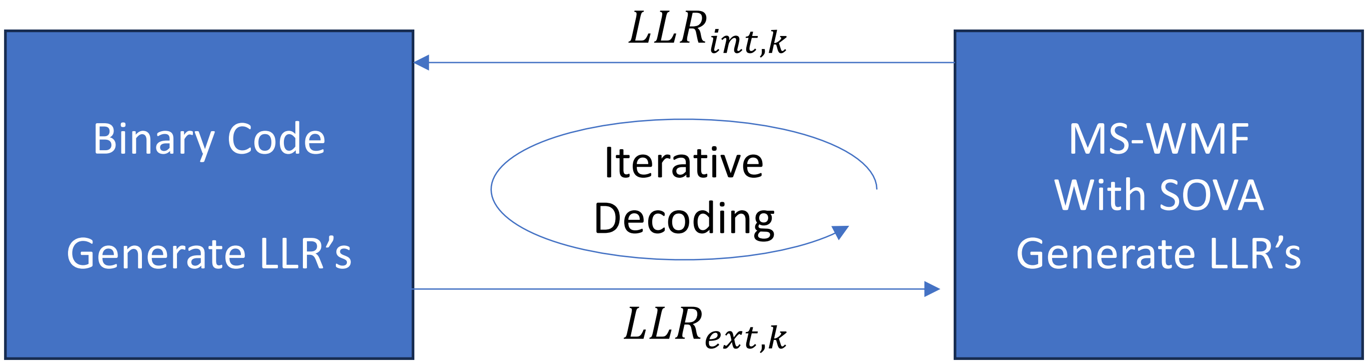

```
>> plot(conv(W(1,:),[.9 1 0]))
>> hold
Current plot held
>> plot(conv(W(2,:),(1.81/1.64)*[0 1 .8]))
```

- The equalized channel clearly looks causal in last 3 positions, and the two outputs align the large first tap.

# Turbo Equalization

- These are packet adaptive equalizers where L16:26's channel identification (of H) or partial-response equalization (L17:13) is used.

- A MLSE (Viterbi Detector) for the channel ISI is used in stead of the feedback section.

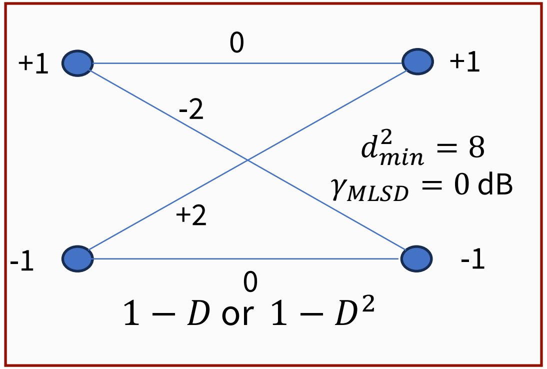



- The channel's memory is treated like a code with the SOVA generation of soft information

# The intrinsic channel information

- Initially, Viterbi/SOVA produces ratios:
  - Sum of such terms if $M^{\nu} > 2$.
  - Evaluate each stage 0/1 among survivors.

$$\frac{e^{-\frac{1}{2\sigma^2}\cdot\left\|\boldsymbol{y}-\boldsymbol{H}\cdot\boldsymbol{x}_{k,0}\right\|^2}}{e^{-\frac{1}{2\sigma^2}\cdot\left\|\boldsymbol{y}-\boldsymbol{H}\cdot\boldsymbol{x}_{k,1}\right\|^2}}$$




- Later runs
  - Include the code's soft extrinsic information in the Viterbi partial-response updates.

- The MLSD on channel trellis is optimum – lower initial Pe
  - But loses advantage as number of levels increase in PAM/QAM
  - Precoder can reduce this loss, but not eliminate it.

- The code and channel may interleave order w.r.t. each other.
  - The SNRmfb attained by Viterbi does not

- Tends to prevent transmit-filter optimization.

**Much better to use Decision Feedback & Good Code**

**Those can achieve reliable transmission at any rate up to capacity**

# End Lecture 17