STANFORD

Lecture 14

Linear Equalizers

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Announcements & Agenda

- Announcements
  - PS6 due tomorrow
  - FB PS5:
    - 10-18 hours
    - Interleavers less covered, but on homework
    - LLRs?

- Problem Set 7 = PS7 due Wednesday March 6
  1. 3.7 DFE is even better
  2. 3.8 Noise Predictive DFE
  3. 3.19 Two-Band Equalizer
  4. 3.23 Finite-Delay Tree Search
  5. 3.28 IIR channel DFE

- Today
  - Finish Zero-Forcing Equalization (3.4)
  - MMSE-LE (3.5)
  - Fractional Spacing and Passband Equalization (3.5)

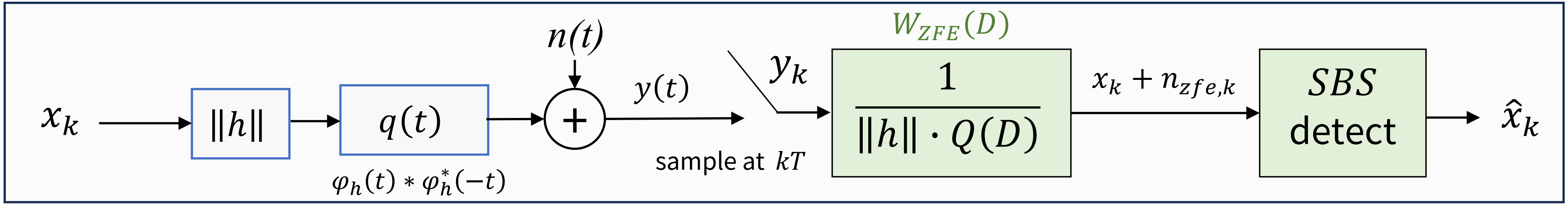
Zero-Forcing Equalization (ZFE)

Section 3.4

ZFE – Just filter with channel inverse

- The ZFE **Forces** ISI to **Zero.**

Don't confuse Q's ! (sorry)



- Calculate $\sigma_{ZFE}^2 = \frac{\mathcal{N}_0}{2} \cdot \frac{w_{ZFE,0}}{\|h\|}$; where $w_{ZFE,0}$ is the time-zero value of $w_{ZFE,k}$.

- SNR of SBS then is $SNR_{ZFE} = \frac{\bar{\mathcal{E}}_x}{\sigma_{ZFE}^2} = SNR_{MFB} \cdot \gamma_{ZFE}$.

Loss Factor γ_{ZFE}

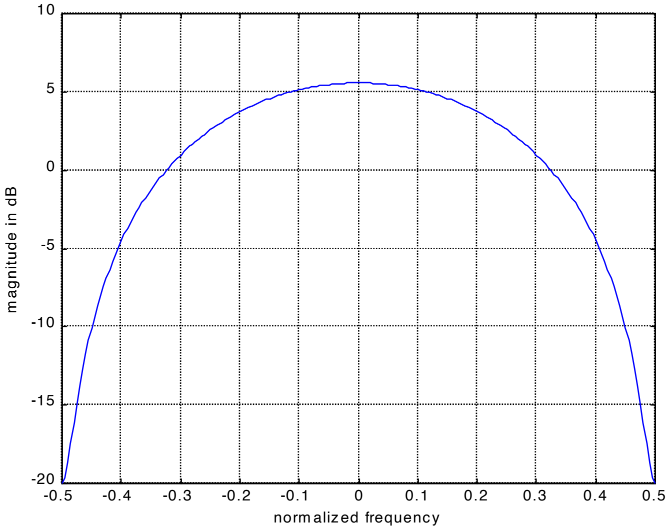
$$\gamma_{ZFE}^{-1} = \frac{T}{2\pi} \cdot \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{d\omega}{Q(e^{-j\omega T})} = [q^{-1}]_0 = w_{ZFE,0} \cdot \|h\|$$

Our Example channel: $H(D) = 1 + .9 \cdot D^{-1}$

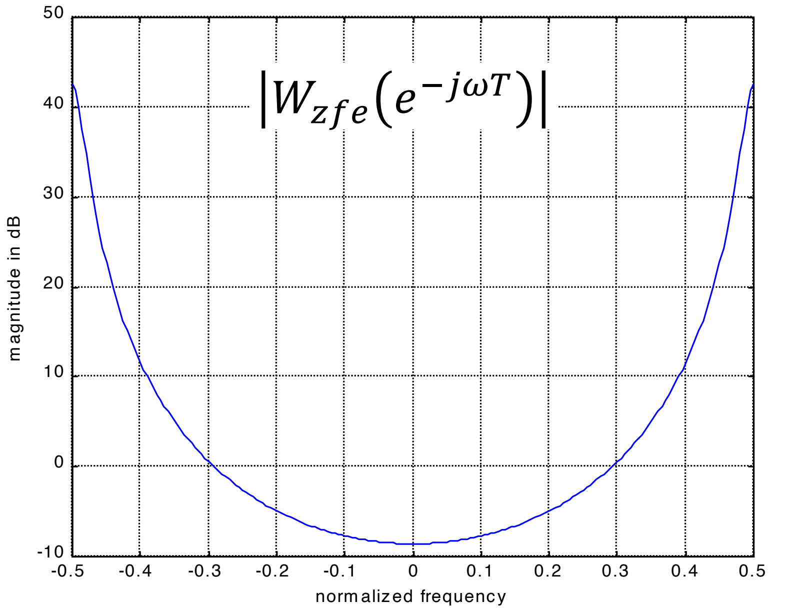
$$H(\omega) = \begin{cases} \sqrt{T} \cdot \left(1 + .9e^{\jmath\omega T}\right) & |\omega| \le \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases} \qquad \Phi_h(\omega) = \begin{cases} \sqrt{\frac{T}{1.81}} \cdot \left(1 + .9 \cdot e^{\jmath\omega T}\right) & |\omega| \le \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases}$$

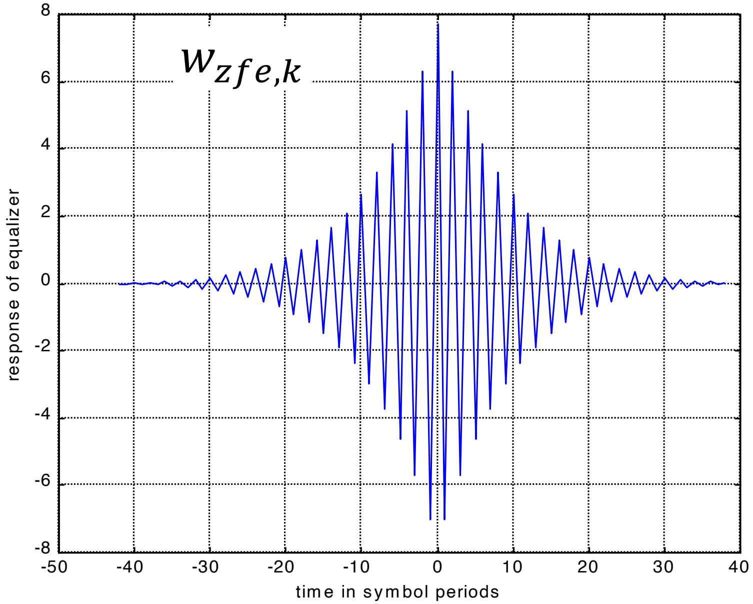
$$\|h\|^2 = 1.81 \quad \sigma^2 = 0.181 \qquad SNR_{MFB} = 10 \;\; (= 10 \text{ dB also})$$

- Real baseband, so is symmetric lowpass about DC.
- It has significant ISI (2nd large tap).
- The SNR is already not so good.
  - SNRmfb is 10 dB.
  - So, design needs a code outside the receiver!
    - One designed for AWGN.



Corresponding ZFE Spectrum and Time Response





- Edge noise increases 50dB (with respect to center).
- Very long response (delay, or complexity if realized as FIR).
- The integral of $W_{zfe}(e^{-j\omega T})$ can be done in closed form for this example → $\sigma^2_{ZFE} = 5.26 \cdot \sigma^2$ (see Example 3.4.1)

$$\gamma_{ZFE} = 10\log_{10}(1.81 \cdot 5.26) \approx 9.8\text{dB}$$

Big Loss!

How about 1+.9D^{-1} Eye? → eye_eq.m software

- Thank former Students Dr. Jungsub Byun and Dr. Moshe Malikin (at web site).

```
>> help eye_eq

function []= eye_eq(h,Ex,noise_var,eq_type);
INPUTS
 h        = pulse response, example  [a*D^-1 + 1] ==> [a 1]
 Ex       = average energy of signals, Ex_bar
 noise_var = noise variance (sigma squared)
 eq_type   = Z => ZERO FORCING (may have numerical issues, so use M with zero
noise instead.)
           M => MMSE
           D => MMSE-DFE
 OUTPUTS, some are graphs
 outputs: pe_(zfe/mmle_dfe/mmse_le)= probability of error with Equalizer from N
input 2PAM[+1/-1] data sequences
 outputs: pe_no_eq  = probability of error without Equalizer
 outputs: dfseSNR   = receiver(equalizer)SNR, unbiased in dB
 outputs: pe_SNR = error probability estimation from receiver(equalizer)SNR, Pe =
Q function of sqrt(dfseSNR)
 this function shows Frequency response of the channel p and equalizer filter, eye
diagram, receiver SNR, and probability of error.
 N = 250; % # of input 2 PAM[+1/-1] data, you can increase the N of input data
sequences in order to calculate the Pe accurately
 created 1/06 by Jungsub Byun and M. Malkin  EE379A
 ----------------------------------------------------------------------
```

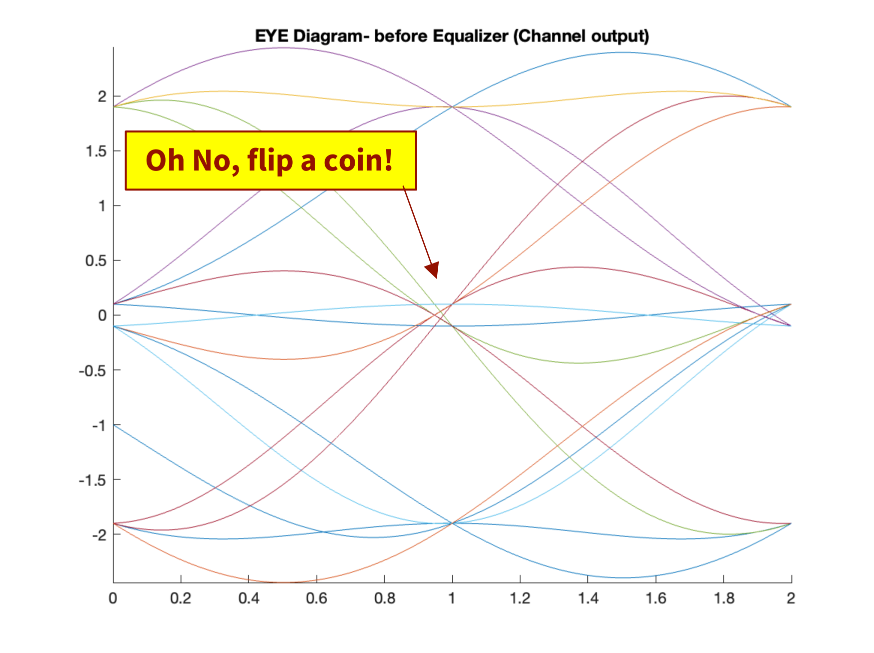
eye_eq([.9 1],1,0,'M’)

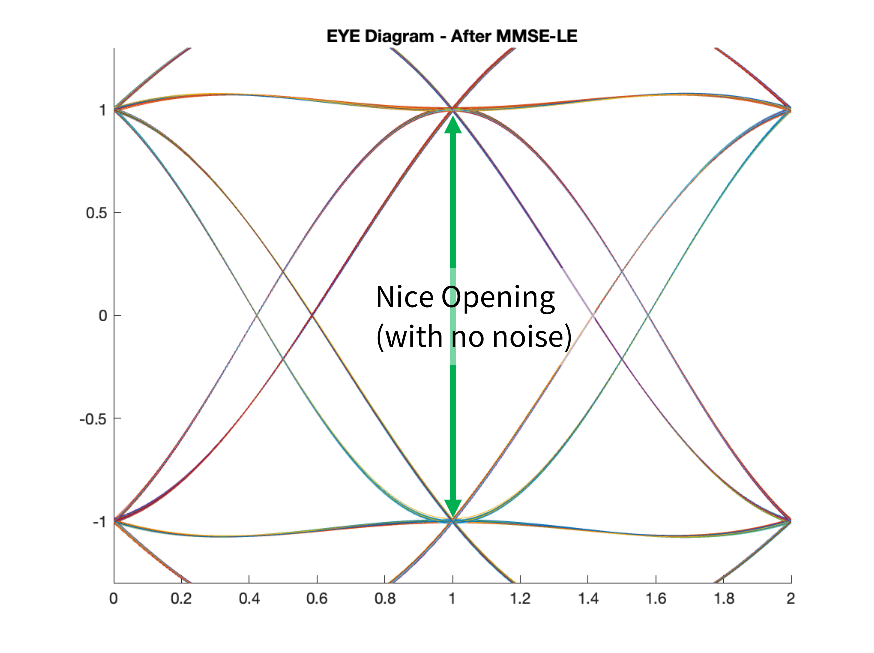
produces several plots, similar to previous slide , &

mainly makes eye diagrams easier.

Eye Diagram, before EQ and After, $1+.9D^{-1}$

\>> eye_eq([.9 1], Ex,noise,'M')





- Essentially ZFE produces ”perfect” eye opening, but of course the noise is huge (not shown).

Matlab assistance for this equalizer

- Find $Q(D)$ and invert

```
>> [r,p,k]=residue([sqrt(1.81) 0],[.9 1.81 .9])
r =
   7.8676
  -6.3728
p =
  -1.1111
  -0.9000
k =  [] % k is only relevant if num degree ≥ denom degree
```

$$W(D) = k + \sum_{m=1}^{\#\ of\ poles} \frac{r(m)}{D - p(m)}$$

$$\begin{aligned} Q(e^{-\jmath\omega T}) &= \frac{1}{T} \cdot |\Phi_h(\omega)|^2 \\ &= \frac{1}{1.81} |1 + .9 \cdot e^{\jmath\omega T}|^2 \\ &= \frac{1.81 + 1.8 \cdot \cos(\omega T)}{1.81} \\ Q(D) &= \frac{.9 \cdot D^{-1} + 1.81 + .9 \cdot D}{1.81} \ . \end{aligned}$$

$$W(D) = \frac{\sqrt{1.81} \cdot D}{.9 + 1.81 \cdot D + .9 \cdot D^2} \ .$$

- $p(m)$ and $^1/_{p^*(m)}$ are always both ZFE poles.
  - because $Q(D)$ and $W(D)$ are pos-real (autocorrelation functions).
  - Choose $|p(m)| \leq 1$ for the following, with corresponding $r(m)$,
  - So choose -.9 and -6.3728.

- Write for each pair m as
- $W(D) = k + \frac{r(m)/p(m)}{1 - p(m) \cdot D} + \frac{r(m) \cdot D^{-1}}{1 - p^*(m) \cdot D^{-1}} \rightarrow k \cdot \delta_k - r(m) \cdot [p(m)]^{k-1} \cdot u_k - r(m) \cdot [p^*(m)]^{-k-1} \cdot u_{1-k}$

$$W(D) = \underbrace{\frac{7.0809}{1 + .9 \cdot D}}_{causal,\ k=0 \ldots \infty} - \underbrace{\frac{6.3728 \cdot D^{-1}}{1 + .9 \cdot D^{-1}}}_{anticausal,\ k=-1 \ldots -\infty}$$

$w_0 = 7.0809$

See text for written details

>> 10*log10(7.0809*sqrt(1.81)) = 9.8 dB

Complex Baseband ISI Example

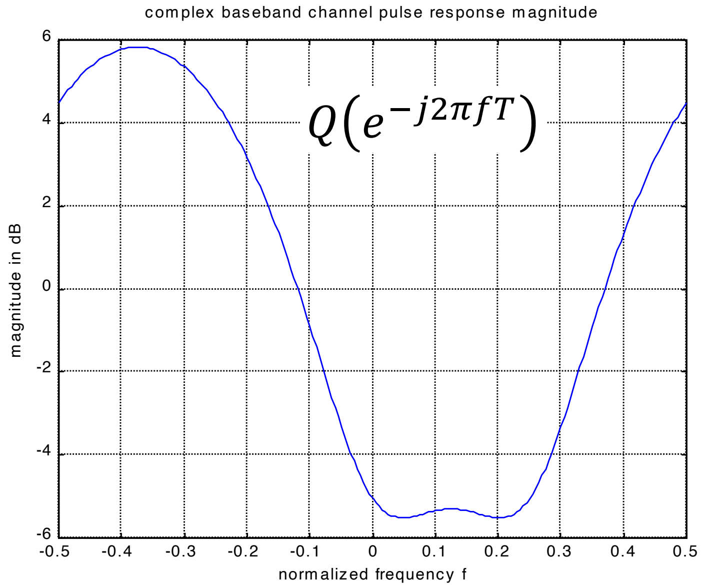
$$h(t) = \frac{1}{\sqrt{T}} \left\{ -\frac{1}{2} \cdot \text{sinc}\left(\frac{t+T}{T}\right) + \left(1+\frac{\jmath}{4}\right) \cdot \text{sinc}\left(\frac{t}{T}\right) - \frac{\jmath}{2} \cdot \text{sinc}\left(\frac{t-T}{T}\right) \right\}$$

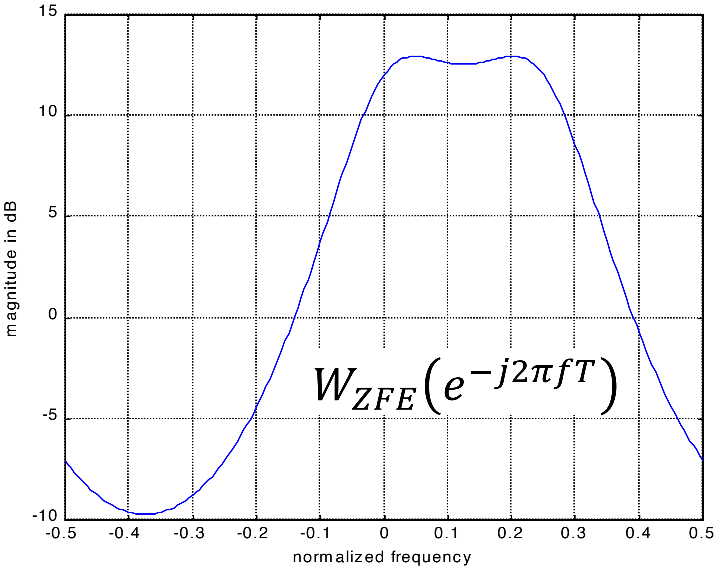
$$\sigma^2 = 0.15625$$

$$SNR_{MFB} = 10 \ \ (= 10 \text{ dB also})$$

```
>> h=[ -1/2 (1+i/4) -i/2];
>> conv(h,h(end:-1:1)) =   0.0000 + 0.2500i  -0.3750 - 0.6250i
0.9375 + 0.5000i  -0.3750 - 0.6250i   0.0000 + 0.2500i
>> norm(h)^2 =    1.5625
```

$$W_{ZFE}(D) = \frac{\sqrt{1.5625}}{(-.25\jmath \cdot D^{-2} + .625(-1+\jmath) \cdot D^{-1} + 1.5625 - .625(1+\jmath) \cdot D + .25\jmath \cdot D^2)}$$

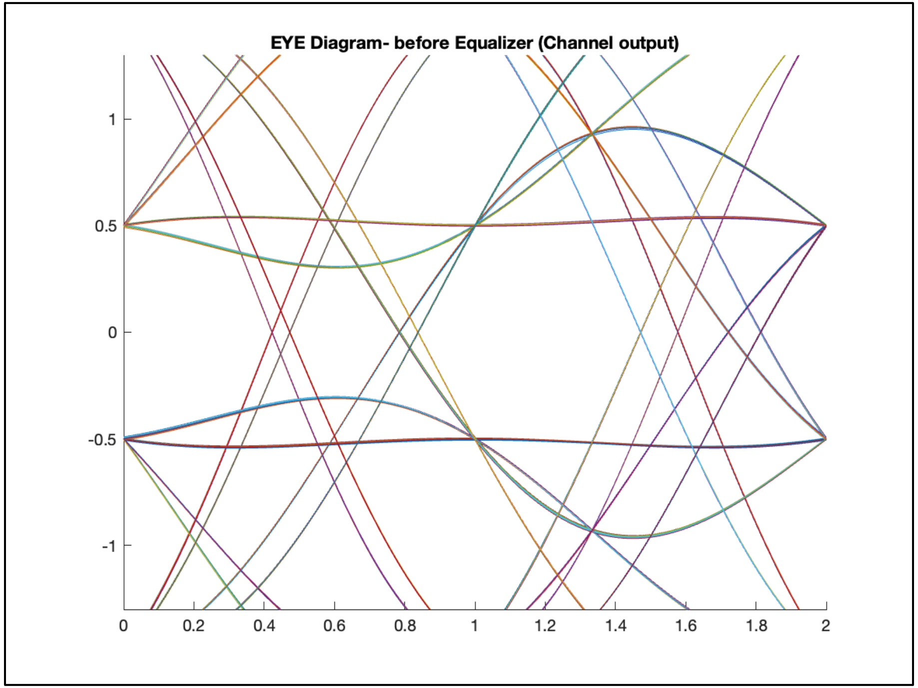


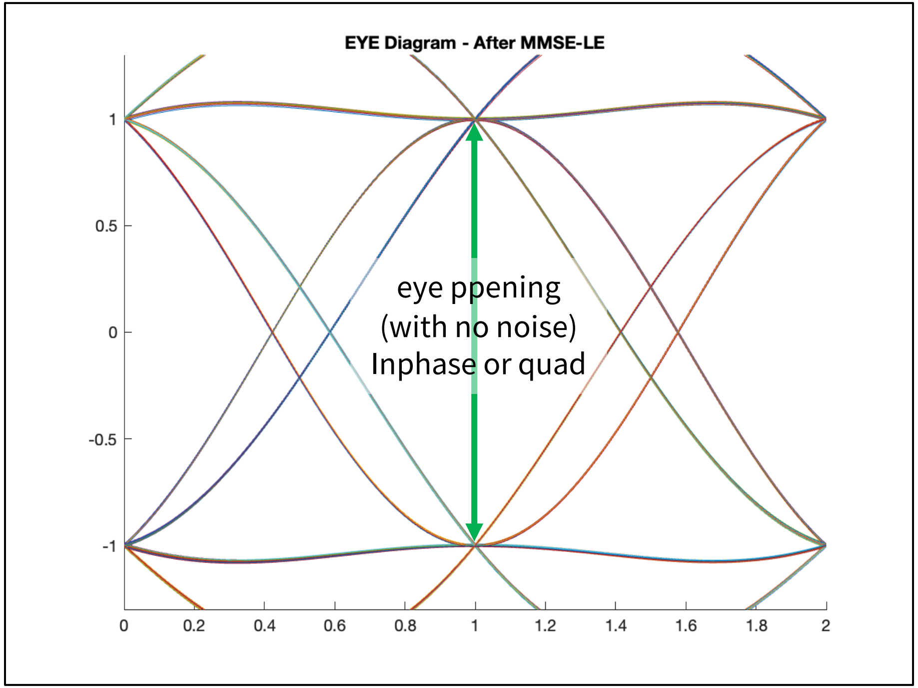


- Magnitude shows channel inversion

Eye Diagram, before EQ and After, complex

eye_eq(h,1,0,'M')





- However, as with real case, there is enhanced noise.

Examine this complex-BB equalizer

- $W_{zfe}(D)$ directly as $^1/_{\|h\|} \cdot Q(D)$

$$W_{ZFE}(D) = \frac{\sqrt{1.5625}}{(-.25\jmath \cdot D^{-2} + .625(-1+\jmath)\cdot D^{-1} + 1.5625 - .625(1+\jmath)\cdot D + .25\jmath \cdot D^2)}$$

```
>> [r,p,k]=residue([sqrt(1.5625) 0 0],[-i/4  (5/8)*(-1+i) 1.5625 -5/8*(1+j) i/4]);
>> transpose(r)  =  -1.9608 + 1.1765i   1.1765 - 1.9608i   0.2941 + 0.4902i   0.4902 + 0.2941i
>> transpose(p) =   2.0000 - 0.0000i  -0.0000 + 2.0000i  -0.0000 + 0.5000i   0.5000 + 0.0000i
>> k =     []
>> r(3)/p(3) =   0.9804 - 0.5882i
>> r(4)/p(4) =   0.9804 + 0.5882i
```

- $W(D) = k + \frac{r(m)/p(m)}{1-p(m)\cdot D} + \frac{r(m)\cdot D^{-1}}{1-p^*(m)\cdot D^{-1}} \rightarrow r(m)\cdot[p(m)]^{k-1}\cdot u_k - r(m)\cdot[p^*(m)]^{-k-1}\cdot u_{1-k}$ (k>0 terms)

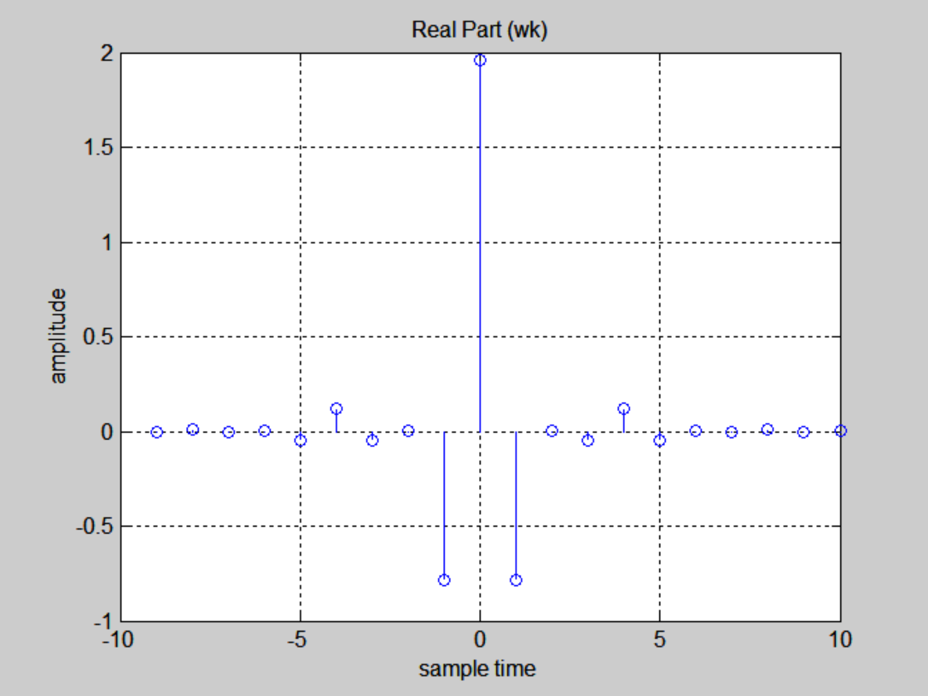
$$W(D) = \underbrace{\frac{.9804 - .5882j}{1-(\frac{j}{2})\cdot D} + \frac{.9804 + .5882j}{1-(\frac{1}{2})\cdot D}}_{causal,\ k=0\ldots\infty} - \underbrace{don't\ care\ for\ w_0}_{anticausal,\ k=-1\ldots-\infty}$$

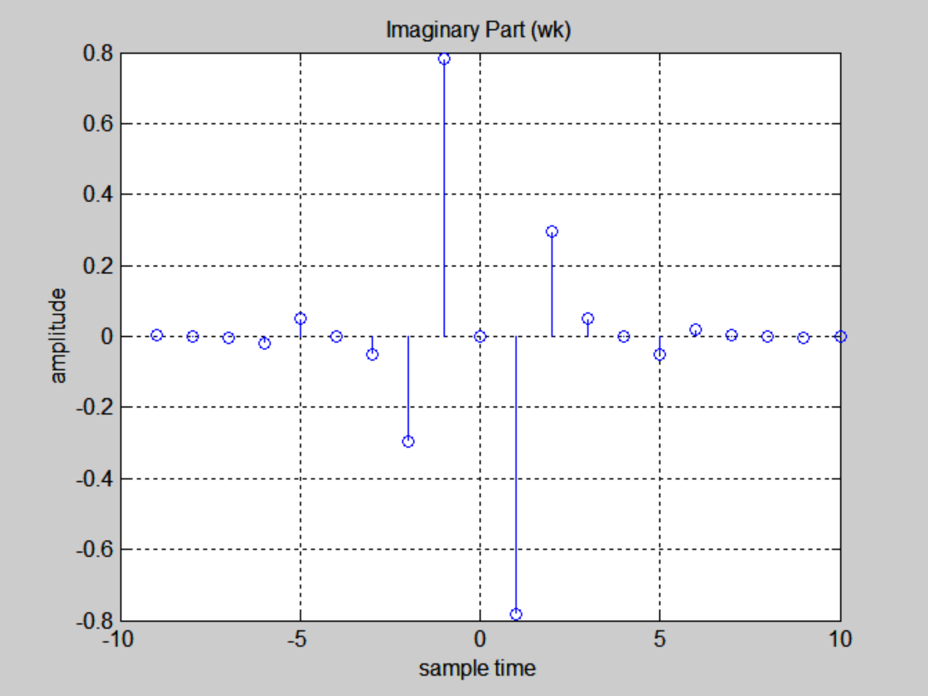
$$w_0 = 1.9608$$

See text for written details

$$\gamma_{ZFE} = 10 \cdot \log_{10}(w_0 \cdot \|h\|) = 3.9\ \text{dB}$$

Time Domain Equalizer



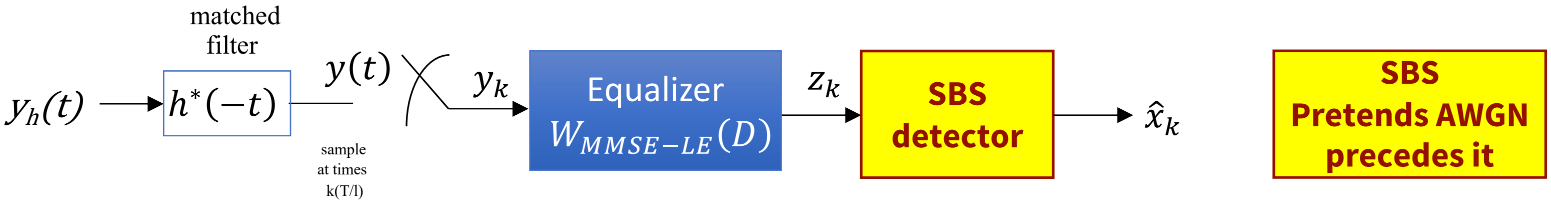


- Little easier channel (smaller loss w.r.t. MFB) – shorter equalizer.

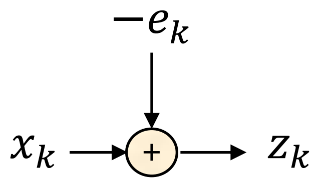
MMSE-Linear Equalizer

Section 3.5

Linear Equalizer to minimize ISI/noise impact



Equivalent AWGN at time k



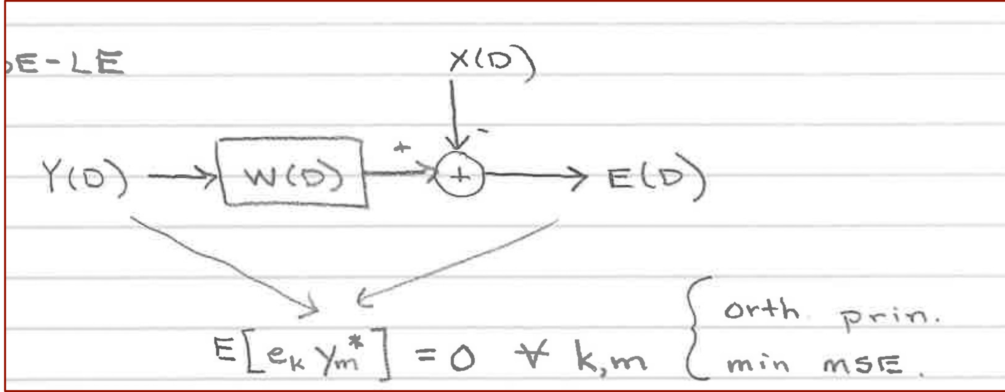
$$SNR = \frac{\mathcal{E}_x}{\mathbb{E}[|z_k - x_k|^2]}$$

- Like ZFE, SBS = Symbol By Symbol (detector) – not ML, not optimum, but maybe much cheaper.
  - All the QAM/PAM, coding formulas apply to this new channel $\bar{b} \cong \frac{1}{2} \cdot \log_2(1+SNR)$, so receiver converts to AWGN (or tries to).
- Linear filter that minimizes Mean-Square Error (maximizes SNR).
- Clearly MMSE-LE can't do any worse than ZFE, and often should do better with nonzero noise.

MMSE Criterion

- The $SNR = \frac{\mathcal{E}_x}{\mathbb{E}[|z_k - x_k|^2]}$ is maximum.
  - Output is biased when $W(D)$ is selected to minimize MSE.
- Appendix D's **orthogonality principle**, error is orthogonal to input, find $W(D)$.

$$\mathbb{E}[E(D) \cdot Y^*(D^{-*})] = 0$$



- Solution (see Sec 3.5) is

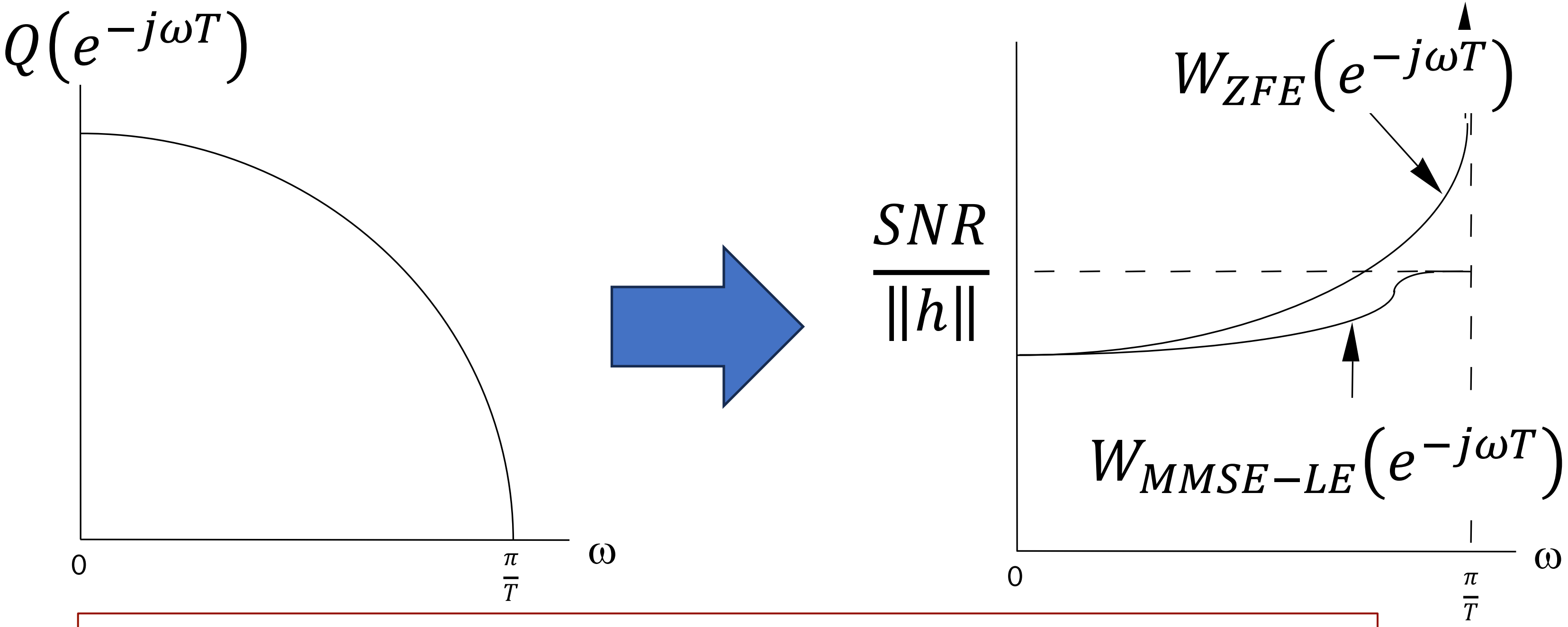
$$W_{MMSE-LE}(D) = \frac{\bar{R}_{xy}(D)}{\bar{R}_{yy}(D)} = \frac{1}{\|h\| \cdot \left(Q(D) + {}^{1}/_{SNR_{MFB}}\right)}$$

$$R_{xy}(D) = \mathbb{E}[X(D) \cdot Y^*(D^{-*})]$$

$$R_{yy}(D) = \mathbb{E}[Y(D) \cdot Y^*(D^{-*})]$$

- Differs from ZFE only in the extra ${}^{1}/_{SNR_{MFB}}$ denominator term, which conditions/controls noise increase.
  - Can't divide by zero or very small number

Limits noise enhancement

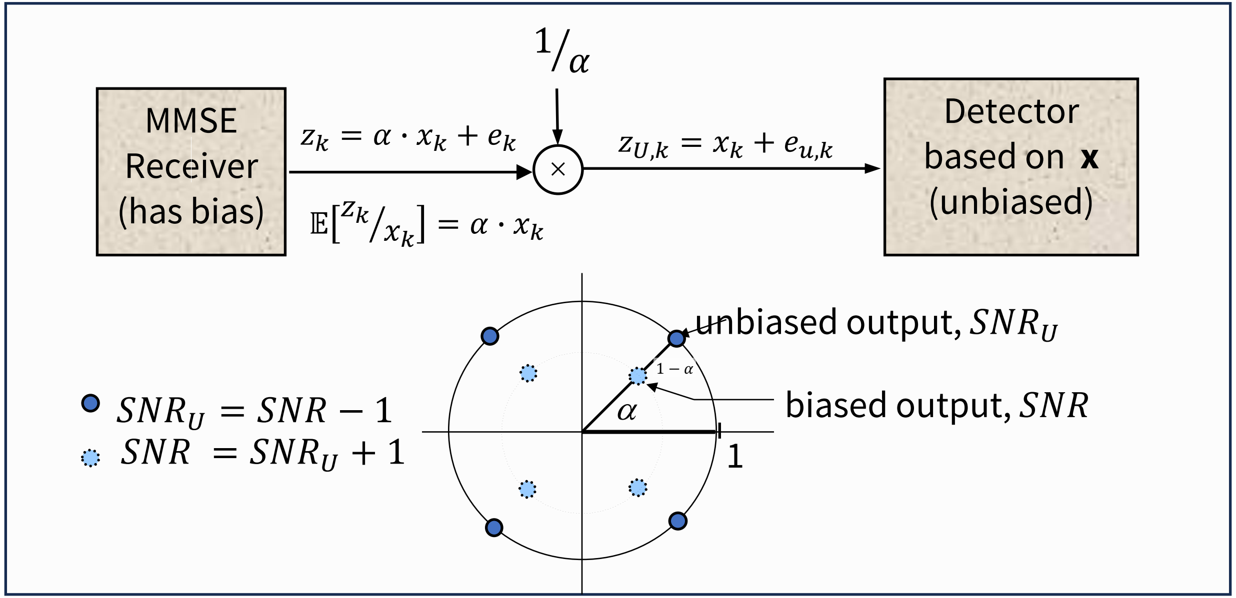


$$\sigma^2_{MMSE-LE} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \bar{R}_{ee}(e^{-\jmath\omega T})\cdot d\omega = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{\frac{\mathcal{N}_0}{2}\cdot d\omega}{\|h\|^2\cdot\left(Q(e^{-\jmath\omega T}) + 1/SNR_{MFB}\right)}$$

- Any very low energy region is offset by the 1/SNRmfb term.
- If SNR=∞, then ZF and MMSE-LE are the same.

Receiver Removal of Bias

- Reminder



- Recall the best value of α , when the receiver minimizes MSE, is $\frac{SNR}{SNR-1}$.
- This boosts desired signal as well as distortion/noise, but in a way so that receiver's decision regions based on p_{z_k/x_k} are correct.
- The correct new SNR is always (with any MMSE receiver bias removal) $SNR_U = SNR - 1$.
- If no noise, bias removal is $\alpha = 1$.

MMSE Conclusion / Use

- So ok to use MMSE, just remove the bias before entering the SBS detector, and now we have P_e

$$P_e \approx N_e \cdot Q\left(\sqrt{\kappa \cdot SNR_{MMSE-LE,U}}\right) \qquad \kappa = \frac{3}{2^{2\bar{b}}-1} \text{ for SQ QAM/PAM}$$

- Maximum value would occur when there is no ISI so that all signal energy at receiver MF output sampler is

$$SNR_{MMSE,U} \leq \frac{\bar{\mathcal{E}}_x \cdot \|h\|^2}{\sigma^2} \triangleq SNR_{MFB}$$

Matched-Filter Bound SNR

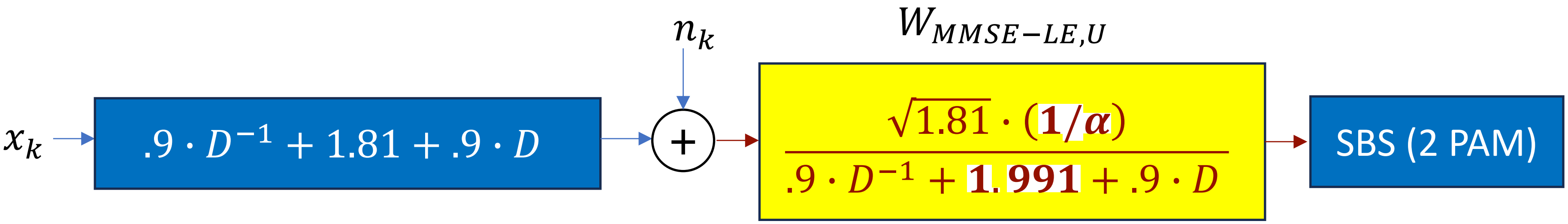
- Bound attained with no ISI, so $Q(D)$=1 in

$$Y(D) = \|h\| \cdot Q(D) \cdot X(D) + N(D)$$

Two Major Repeated Concepts for the Designer

1. $P_e \approx N_e \cdot Q\left(\sqrt{\kappa \cdot SNR_{MMSE-LE,U} \cdot \gamma}\right)$ where γ is any coding gain.
2. Minimize MMSE, orthogonality, error and channel output

Return to 1+.9D^{-1} ISI Example



- Reproduce ZFE commands for MMSE-LE

```
>> [r,p,k]=residue([sqrt(1.81) 0],[.9 1.991 .9]);
>> transpose(r ) =    2.4962   -1.0014
>> transpose(p) =   -1.5788   -0.6334
>> k =    []
>> r(2)/p(2) =   1.5811
```

```
>> w0=r(2)/p(2) %=   1.5811
>> SNRLE=1/(w0*.181/sqrt(1.81)) %=   4.7013
>> SNRLEU=SNRLE-1 %=   3.7013
>> 10*log10(SNRLEU)  5 %=   5.6835 dB
>> WU0=(SNRLE/SNRLEU)*w0 %=   2.0082
```

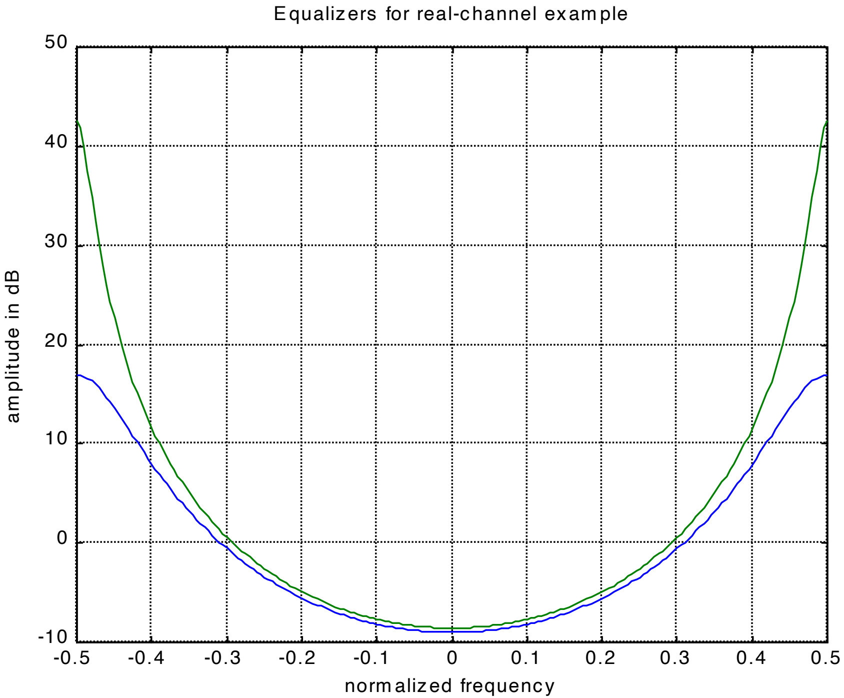
$$W(D) = \underbrace{\frac{1.5811}{1+.6334 \cdot D}}_{causal,\ k=0 \ldots \infty} - \underbrace{\frac{1.0014 \cdot D^{-1}}{1+.6334 \cdot D^{-1}}}_{anticausal,\ k=-1 \ldots -\infty}$$

$$w_{U,0} \cong 2$$

gamma= 10-5.7= 4.3dB << 9.8 dB !!

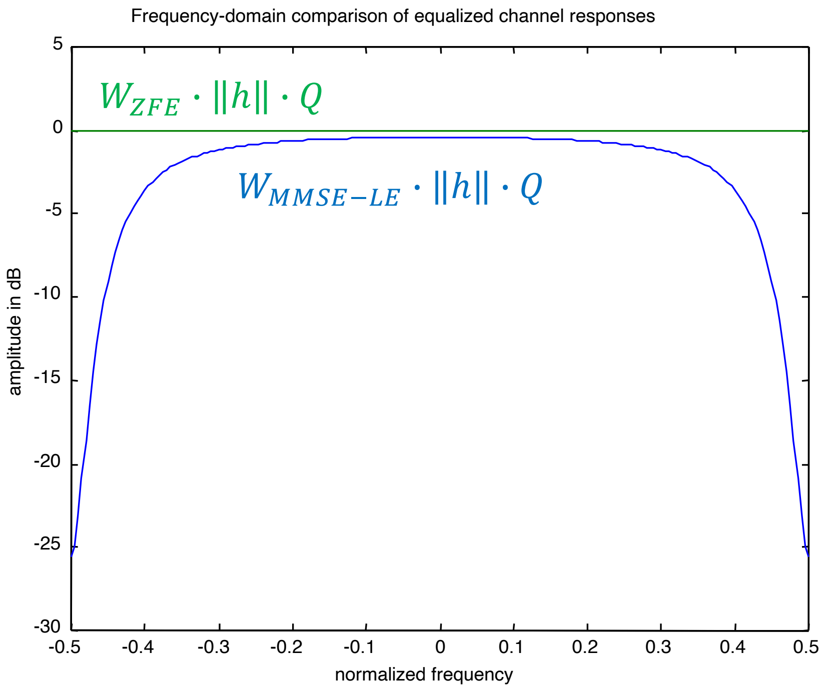
5.5 dB!: MMSE effect can be large
(~ turbo/ldpc/GRAND fight for .2 dB)

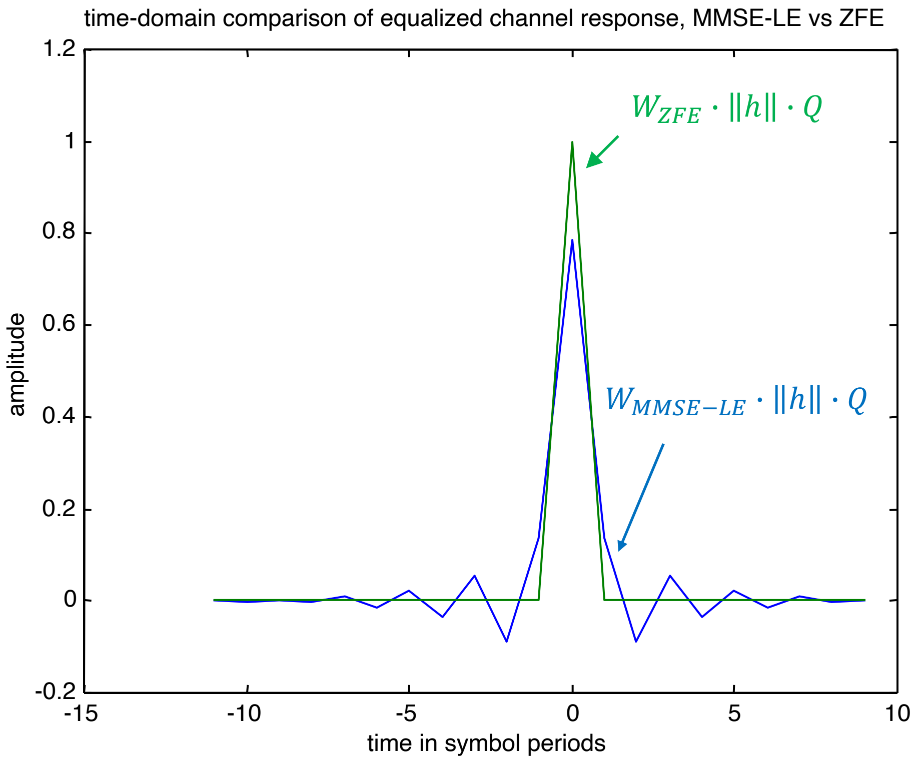
Equalizer Comparison



- Less enhancement visible, and some ISI visible in equalized response.

Equalizer Outputs





- Less enhancement is visible, but some ISI is visible in equalized response.

Complex Example

$$W(D) = \frac{\sqrt{1.5625}}{-.25\jmath \cdot D^{-2} + .625(-1+\jmath) \cdot D^{-1} + 1.5625(1+.1) - .625(1+\jmath) \cdot D^{1} + .25\jmath \cdot D^{2}}$$

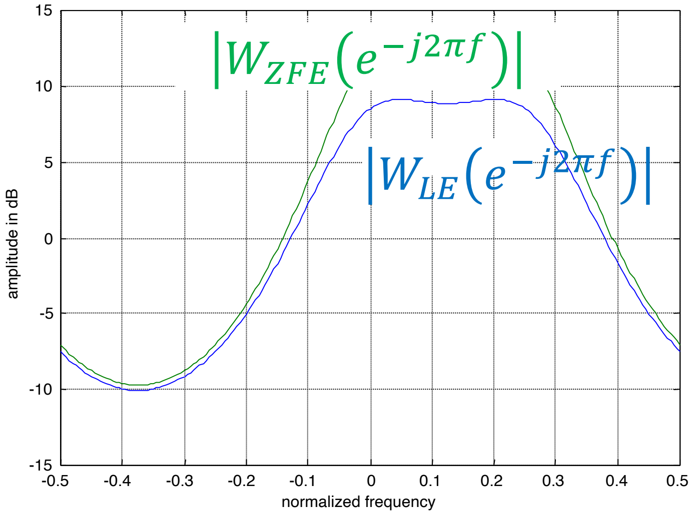
```
> [r,p,k]=residue([sqrt(1.5625) 0 0],[-i/4  (5/8)*(-1+i) 1.5625*(1+.1) -5/8*(1+j) i/4]);
>> transpose(r) =   1.0079 - 1.5025i  -1.5025 + 1.0079i   0.1662 + 0.3284i   0.3284 + 0.1662i

>> transpose(p) =  -0.1356 + 2.2130i   2.2130 - 0.1356i  -0.0276 + 0.4502i   0.4502 - 0.0276i
>> k =     []
>> A=r(3)/p(3) =   0.7042 - 0.4123i
>> B=r(4)/p(4) =   0.7042 + 0.4123i
>> w0=A+B =   1.4084 + 0.0000i
SNRLE=1/(w0*.181/sqrt(1.81)) = 5.2776
SNRLEU=SNRLE-1 = 4.2776
10*log10(SNRLEU)=6.312 dB
```

- Loss is 3.7 dB < ZFE's 3.9dB
- Why so small?

ISI is less

- See the vertical scale for ZFE
  - MMSE is better when an equalizer is really needed

MMSE-LE for Complex Channel

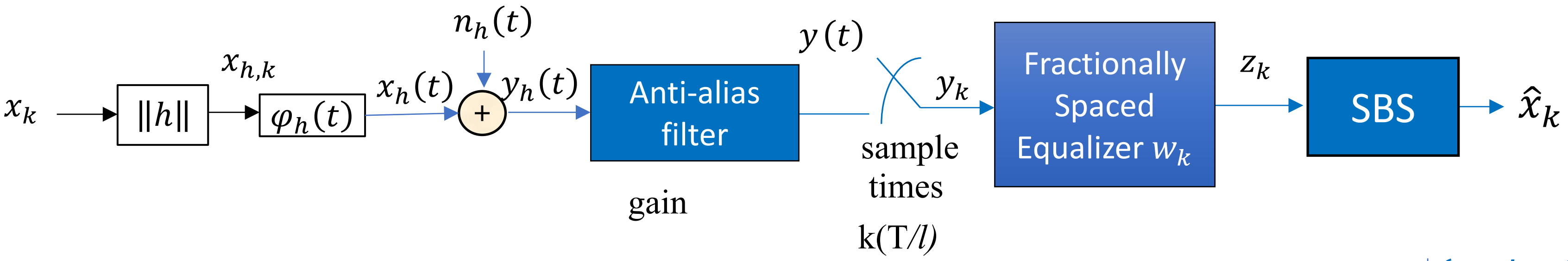


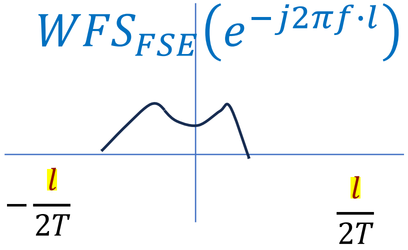
Fractionally Spaced Equalizers

Section 3.5.4

Fractional Spacing (integer multiple l)

- There is some channel passband energy above the (designer selected) Nyquist frequency.
- Then simple sampling (without matched filter) at symbol rate is not sufficient.
- Designer wants the equalizer to absorb the matched filter.





equivalent (aliased) frequency response target is "flat" , but use MMSE

L phases

- There are l phases of input samples for each symbol-sample-time output ("polyphase system")

$$y_i(kT) = y(kT - iT/l) \;\; , i = 0, \ldots, l-1 \; .$$

- Each phase has a D-Transform $Y_i(D) = H_i(D) * X(D) + N_i(D)$
  - This is a form of what is called "diversity" where several channels carry the same input to different outputs.

$$\underbrace{\boldsymbol{Y}(D)}_{l\times 1} = \underbrace{\boldsymbol{H}(D)}_{l\times 1} \cdot X(D) + \underbrace{\boldsymbol{N}(D)}_{l\times 1}$$

- There is an $l\times 1$ channel-output vector for each symbol period, creating an FSE with l times more coefficients:

$$\mathbb{E}\left[E(D) \cdot \boldsymbol{Y}^*(D^{-*})\right] = \boldsymbol{R}_{x\boldsymbol{Y}}(D) - \boldsymbol{W}(D) \cdot \boldsymbol{R}_{\boldsymbol{Y}\boldsymbol{Y}}(D) = 0 \quad ,$$

$$\boldsymbol{R}_{x\boldsymbol{Y}}(D) \;\triangleq\; \mathbb{E}\left[X(D) \cdot \boldsymbol{Y}^*(D^{-*})\right] = \bar{\mathcal{E}}_{\boldsymbol{x}} \cdot \boldsymbol{H}^*(D^{-*})$$

$$\boldsymbol{R}_{\boldsymbol{Y}\boldsymbol{Y}}(D) \;\triangleq\; \mathbb{E}\left[\boldsymbol{Y}(D) \cdot \boldsymbol{Y}^*(D^{-*})\right] = \bar{\mathcal{E}}_{\boldsymbol{x}} \cdot \boldsymbol{H}(D) \cdot \boldsymbol{H}^*(D^{-*}) + \ell \cdot \frac{\mathcal{N}_0}{2} \cdot I$$

MMSE-FSE otherwise follows same format

$$W(D) = \boldsymbol{R}_{x\boldsymbol{Y}}(D) \cdot \boldsymbol{R}^{-1}_{\boldsymbol{YY}}(D) = \boldsymbol{H}^*(D^{-*}) \cdot \left[\boldsymbol{H}(D) \cdot \boldsymbol{H}^*(D^{-*}) + \ell/SNR\right]^{-1}$$

The corresponding error sequence has autocorrelation function

$$\bar{R}_{ee}(D) = \bar{\mathcal{E}}_{\boldsymbol{x}} - \boldsymbol{R}_{x\boldsymbol{Y}}(D) \cdot \boldsymbol{R}^{-1}_{\boldsymbol{YY}}(D)\boldsymbol{R}_{\boldsymbol{Y}x}(D) = \frac{\ell \cdot \frac{\mathcal{N}_0}{2}}{\boldsymbol{H}^*(D^{-*}) \cdot \boldsymbol{H}(D) + \ell/SNR}$$

The MMSE is then computed as

Time zero value is found from integration of Fourier Transform

$$\text{MMSE}_{MMSE-FSE} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{\ell \cdot \frac{\mathcal{N}_0}{2} d\omega}{\|\boldsymbol{H}(e^{-\jmath\omega T})\|^2 + \ell/SNR} = \text{MMSE}_{MMSE-LE}$$

- FSE only computes $1/l$ FSE outputs (so the complexity only grows by l).
- Analysis no longer needs the $Q(D)$ explicitly because sampling rate is twice highest frequency.
- Everything else follows like earlier symbol-spaced, but matched filter is implemented within the FSE.

Timing Phase compensation

- The FSE needs to know only the symbol rate (i.e, the frequency $\frac{1}{T}$), $kT + t_0$.
- The FSE's sampling device can use any time sampling time t_0 – any phase is ok.

$$y(kT + t_0) = \sum_m x_m \cdot \|h\| \cdot q(kT - mT + t_0) + n_k \quad , \quad (3.2$$

which corresponds to $q(t) \to q(t + t_0)$ or $Q(\omega) \to Q(\omega)e^{-\jmath\omega t_0}$. For the system with sampling offset,

$$Q(e^{-\jmath\omega T}) \to \frac{1}{T} \cdot \sum_n Q(\omega - \frac{2\pi n}{T}) \cdot e^{-\jmath(\omega - \frac{2\pi n}{T})t_0} \quad , \quad (3.2$$

- A $t_0 \neq 0$ simply changes the matched-filter/sampler.
- This is tacitly included (optimized) in the FSE.

So, don't need to know matched filter – just do MMSE-FSE – must know sampling/symbol frequency.

Rational fraction versions

- Suppose $l = 2$ is too fast (think optical/fiber transmission)?

- Can we just do $(l'/l) \cdot T$ where $l' < l$ and both are integers (like 3T/4)?

- Yes, there will be l (4) input phases per every l' (3) output phases

- This creates 12 (more generally least-common multiple of l and l') different equalizers over 3 symbol periods of 4 samples (interpolated, ADC does not run this fast usually) samples each. Each equalizer is different (just as the l phases of T/l were.
  - Fractions closer to 1 create yet more different equalizers for the different phases.

- The FSE becomes cyclostationary and the equalizers' performance needs to be averaged
  - This simplified when $l' = 1$.

$$\|\boldsymbol{H}\|^2 \rightarrow {}^{1}/_{l'} \cdot \|\boldsymbol{H}\|^2 \text{ in the integrals and formulas}$$

Passband Equalization "Direct Conversion"

- As long as we're absorbing things into high sampling rate filters that decimate to symbol rate ….
  - Essentially the equalizer for Chapter 1's carrierless amplitude-phase modulation.

- Why not get the Hilbert Transform / Phase splitter in there also (if carrier is synchronized as rational fraction to symbol clock)? (See Section 1.3.6.2.)

$$
\begin{aligned}
x_A(t) &= \sum_k x_k \cdot \varphi(t-kT) \cdot e^{\jmath\omega_c t} \\
&= \sum_k x_k \cdot \varphi(t-kT) \cdot e^{\jmath\omega_c t} \cdot e^{-\jmath\omega_c kT} \cdot e^{+\jmath\omega_c kT} \\
&= \sum_k (x_k \cdot e^{+\jmath\omega_c kT}) \cdot \varphi(t-kT) \cdot e^{\jmath\omega_c (t-kT)} \\
&= \sum_k \breve{x}_k \cdot \varphi_A(t-kT)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_A(t) &= \varphi(t) \cdot e^{\jmath\omega_c t} \text{ and} \\
\breve{x}_k &= x_k \cdot e^{+\jmath\omega_c kT}
\end{aligned}
$$

- $y_A(t) = \sum_k \breve{x}_k \cdot h_A(t-kT) + n_A(t)$
- This estimates (using fractional spacing at sufficiently high sampling rate to be twice analytic signal's bandwidth) $\breve{x}_k$ with MMSE-FSE directly. Then rotate by $e^{-j\omega_c kT}$ to get $\hat{x}_k$ (1-to-1 reversibility).
- Modern wireless systems often do this below 6 GHz.

End Lecture 14