STANFORD

Lecture 13

Intersymbol Interference, MMSE, & SNR

February 22, 2024

JOHN M. CIOFFI

Hitachi Professor Emeritus (recalled) of Engineering

Instructor EE379A – Winter 2024

Announcements & Agenda

- Announcements
  - PS5 due today
  - PS6 next Wednesday
  - PS8 remains optional – can substitute if better grade for an earlier
  - Final is take-home March 14-15

- Today
  - Intersymbol Interference (3.1)
  - Receiver SNR (3.2)
  - Nyquist Criterion and pulse shaping (3.3)
  - Zero-Focing Equalization (3.4)

Now with good codes – we can use them with (little) AWGN channels
Keeping in mind that bandwidth/dimensions may be limited.

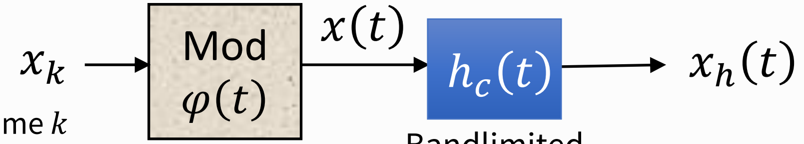
Intersymbol Interference and Equalization

13	2/22	Intersymbol Interference, MMSE, & SNR	3.1-3,3	-/-
14	2/27	Linear Equalizers	3.4-3.5	7/6 (2/23)
15	2/29	Decision Feedback Equalizers	3.6	-/-
16	3/5	FIR Equalizer Design & Software	3.7	8/7 (3/6)
17	3/7	Precoders and Diversity	3.8-9	-/-
18	3/12	Transmit Optimization and Waterfilling	3.11-12	-/8

Intersymbol Interference

Section 3.1

Pulse Response and Intersymbol Interference

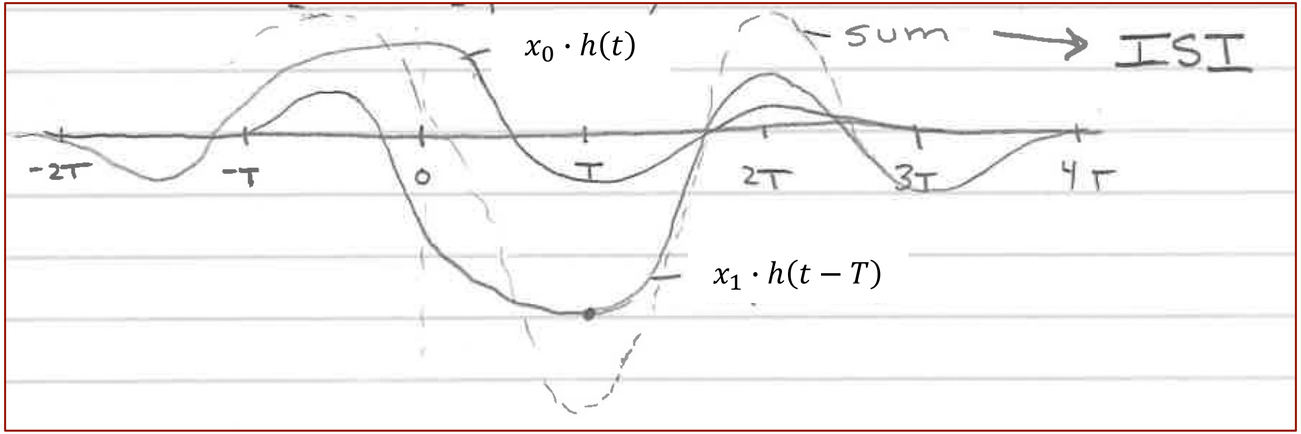


Input (sub-)symbol at time k
PAM real BB
QAM cplx BB

Bandlimited channel

$$\underbrace{h(t) = \varphi(t) * h_c(t)}_{pulse\ response}$$

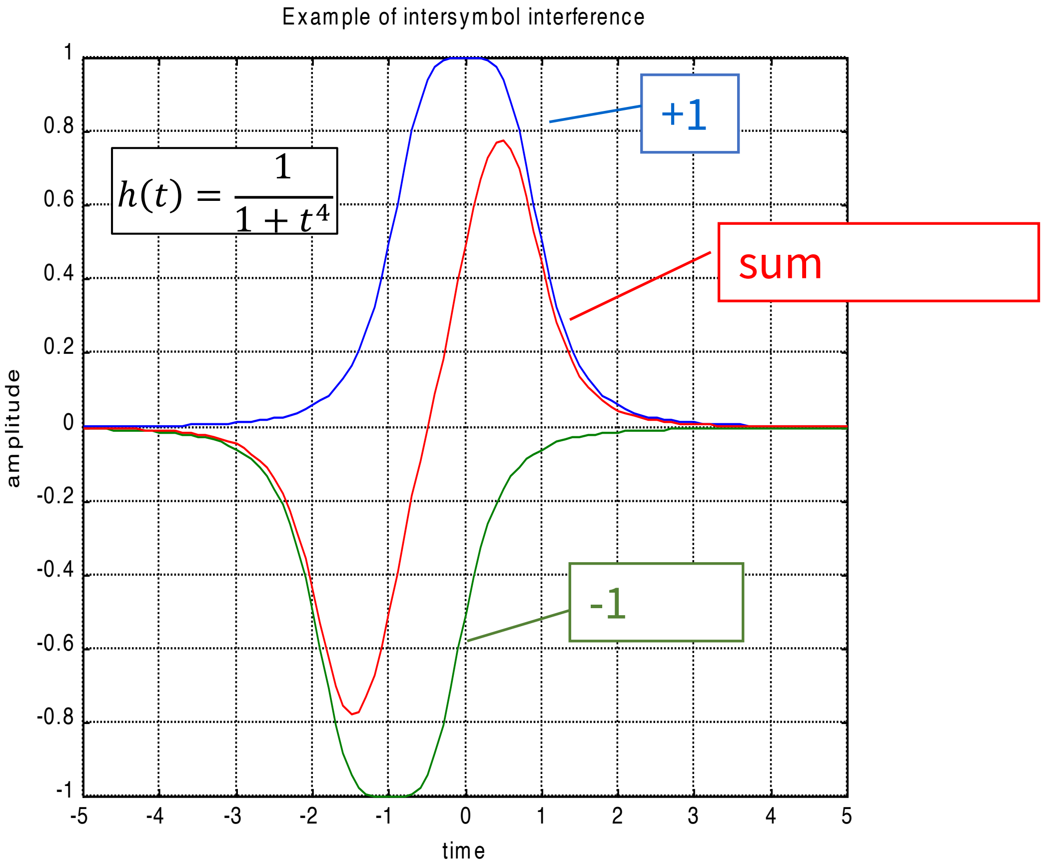
- Channel stretches basis functions
- Successive symbols
  - $x_0 \rightarrow x_0 \cdot h(t)$
  - $x_1 \rightarrow x_1 \cdot h(t - T)$
  - $x_2 \rightarrow x_2 \cdot h(t - 2T)$



Two successive symbols, opposite sign

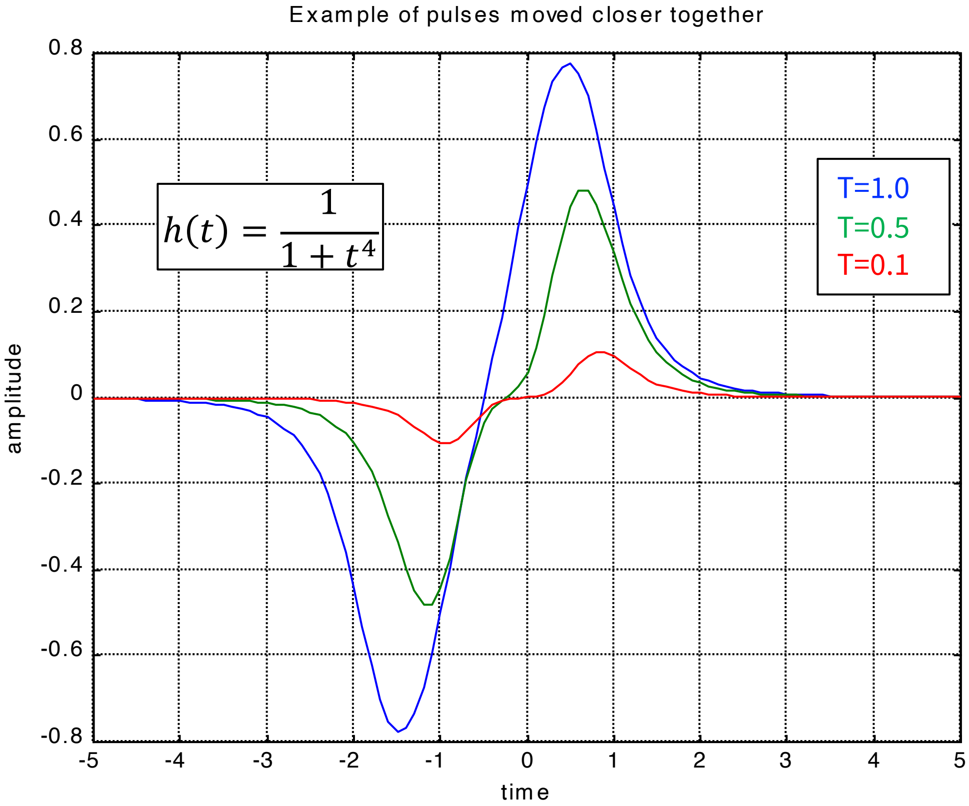
- Amplitude reduces
- Peak shifts
- d_{min} reduces

So how bad is it?



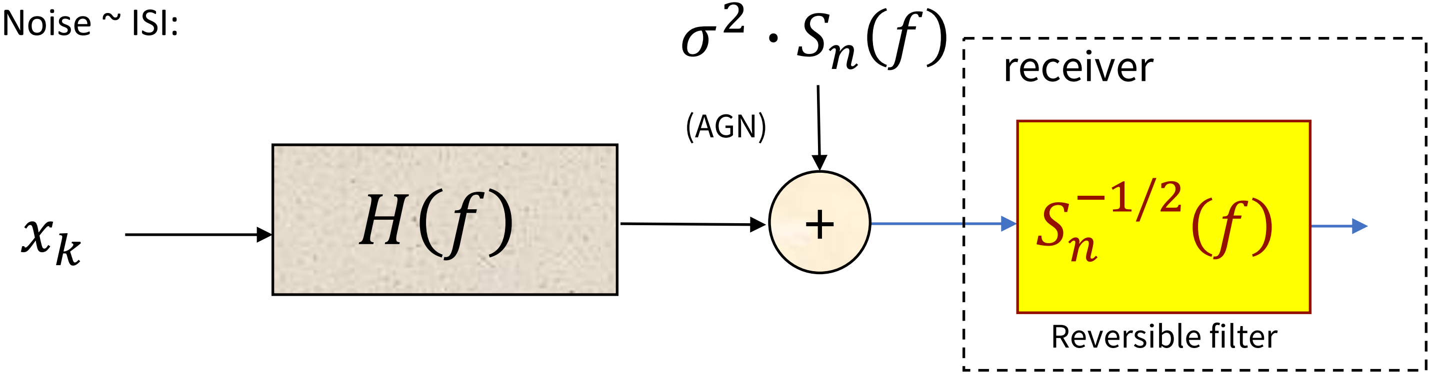
Depends on how fast

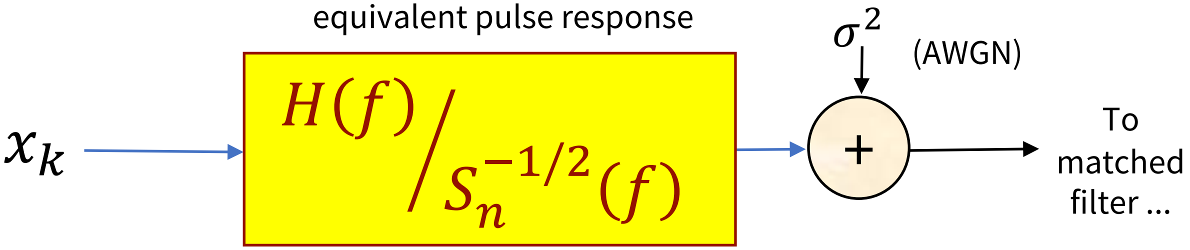
- As T decreases, R increases,
- but the amplitude shrinks further.
- This is not an equal-factor trade of bandwidth/dimensions for energy.
  - Designer is forced to take power.



Noise Equivalent Channel Review

- Colored Noise ~ ISI:





But WAIT!

- This filter only exists, and is 1-to-1 causal and causally invertible, **IF**

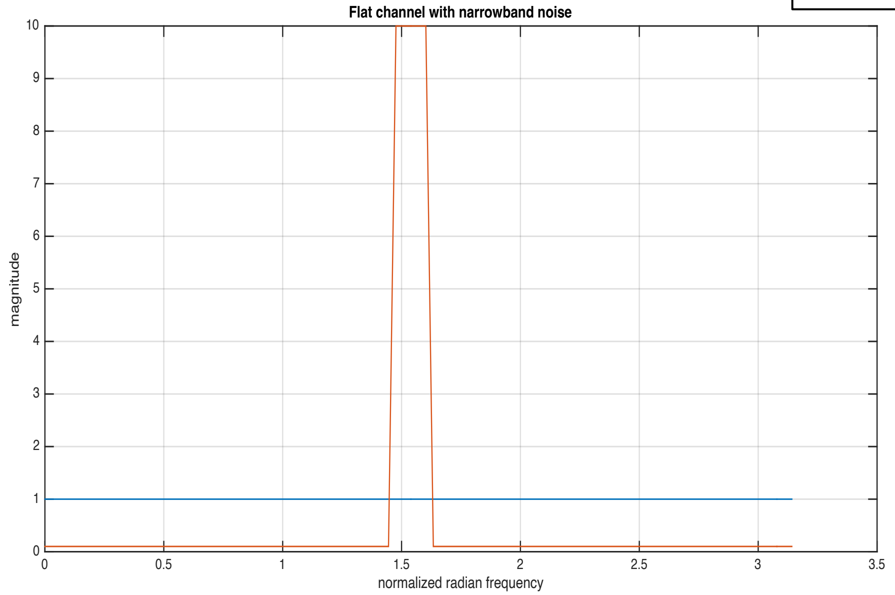
> **Theorem 1.3.6** [Paley-Wiener Criterion] *If*
>
> $$\int_{-\infty}^{\infty} \frac{|\ln \mathcal{S}_n(f)|}{1+f^2} df < \infty \quad , \tag{1.434}$$
>
> *then there exists a* $G(f)$ *satisfying* below *with a realizable inverse. (Thus the filter* $g(t)$ *is a 1-to-1 mapping).*

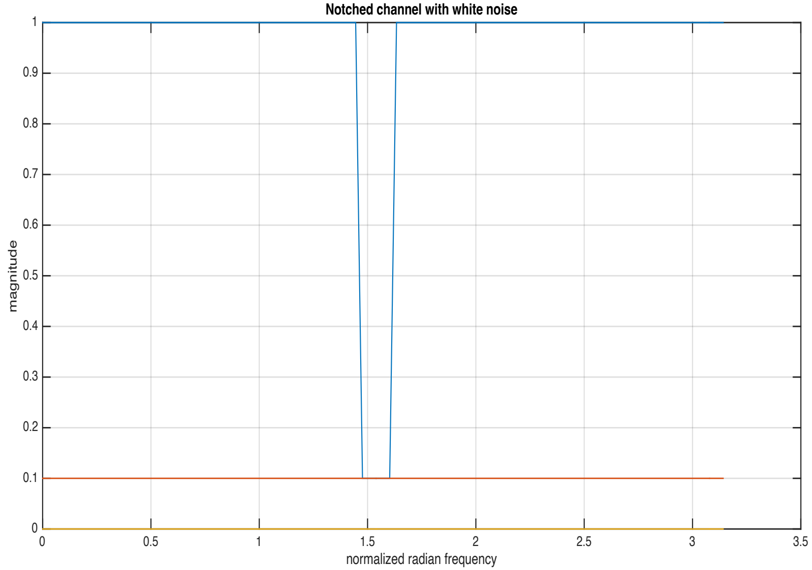
$$\left[\bar{\mathcal{S}}_n(f)\right]^{-1} = |G(f)|^2$$

- See Appendix D on canonical factorization of autocorrelation/power spectra:
  - Such a filter exists for any noise typically found in practice.
  - Notice this says “noise” – does not necessarily apply to systems that optimize transmit power spectra.

Nonwhite, Noise Equivalent to ISI

Equivalent channels

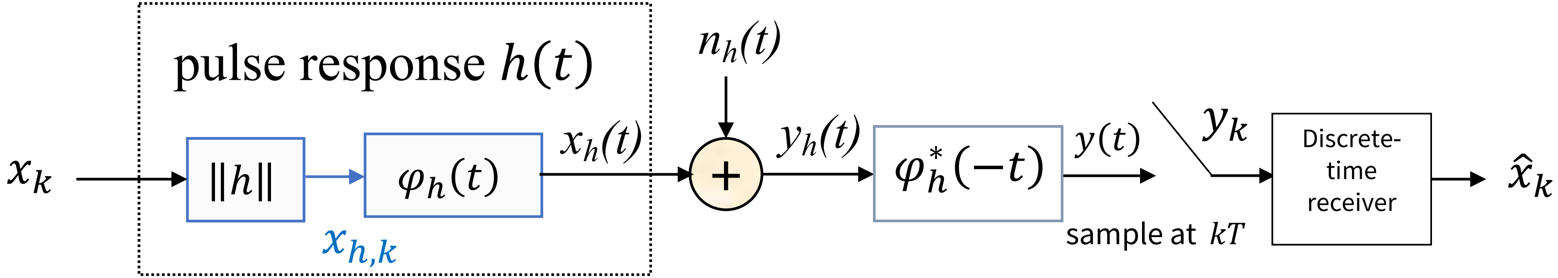




- Narrowband noise is equivalent to channel with frequency notch (significant ISI).
  - Sharp edge in radio band → long ISI effect in $h(t)$.

EE379 ISI Model

- The overall channel filtering includes the receiver's matched pulse-response filter:



$$q(t) = \varphi_h(t) * \varphi_h^*(-t)\ ;\ q_k = q(kT)\ ;\ Q(D) = \sum_k q_k \cdot D^k$$

- Recalled that the matched filter maximizes SNR.

- The sampled MF (at times kT , i.e. the symbol rate) is a sufficient statistic:
  - Nothing is lost in samples from continuous time – see proof in Section 3.1.

- Equivalent sampled model:

$$y_k = \underbrace{\|h\| \cdot x_k}_{\substack{\text{scaled input} \\ (desired)}} + \underbrace{n_k}_{\text{noise}} + \underbrace{\|h\| \cdot \sum_{m \neq k} x_m \cdot q_{k-m}}_{\text{ISI}}$$

Example pulse and peak distortion

$$h(t) = \sqrt{\frac{1}{T}} \cdot \left\{ sinc\left(\frac{t}{T}\right) + .25 \cdot sinc\left(\frac{t-T}{T}\right) - .125 \cdot sinc\left(\frac{t-2T}{T}\right) \right\}$$

- Pulse response has two nonzero ISI samples and no energy outside $f \in \left[-\frac{1}{2T}\ ,\ \frac{1}{2T}\right]$.
  - $\varphi_{h,k} = [1 \ \ .25 \quad -.125]/\sqrt{1+.25^2+.125^2}$
  - $q_k = \varphi_h(t) * \varphi_h^*(-t)|_{t=kT} = [-.1159 \ \ .2029 \quad 1 \quad .2029 \quad -.1159]$

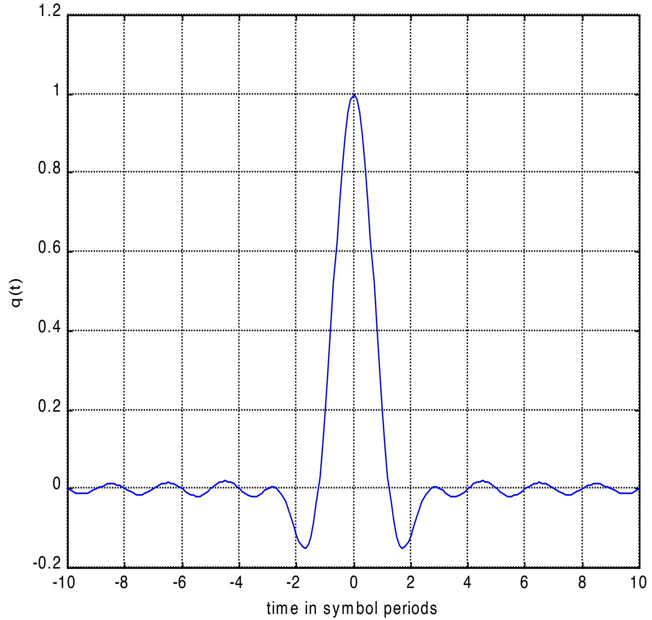
- Peak Distortion

$$\mathcal{D}_p \triangleq |x|_{max} \cdot \|h\| \cdot \sum_{m \neq 0} |q_m|$$

$$P_e \leq N_e \cdot Q\left[\frac{\|h\| \cdot \frac{d_{min}}{2} - \mathcal{D}_p}{\sigma}\right]$$

- With 4PAM and $|x|_{max} = 3$, this is

$$\mathcal{D}_p = 3 \cdot \|h\| \cdot (.1159 + .2029 + .2029 + .1159) = 3 \cdot \sqrt{1.078} \cdot .6376 \cong 2.0.$$



This is really large Pe degradation, and overly pessimistic.

Mean Square Distortion

- The worst-case pattern leading to peak distortion may have very low probability.
- Computing $\mathcal{D}_p$ over all patterns is complex.
- Instead use Mean-Square Distortion:

$$P_e \cong N_e \cdot Q\left[\frac{\|h\| \cdot d_{min}}{2 \cdot \sqrt{\sigma^2 + \mathcal{D}_{ms}}}\right]$$

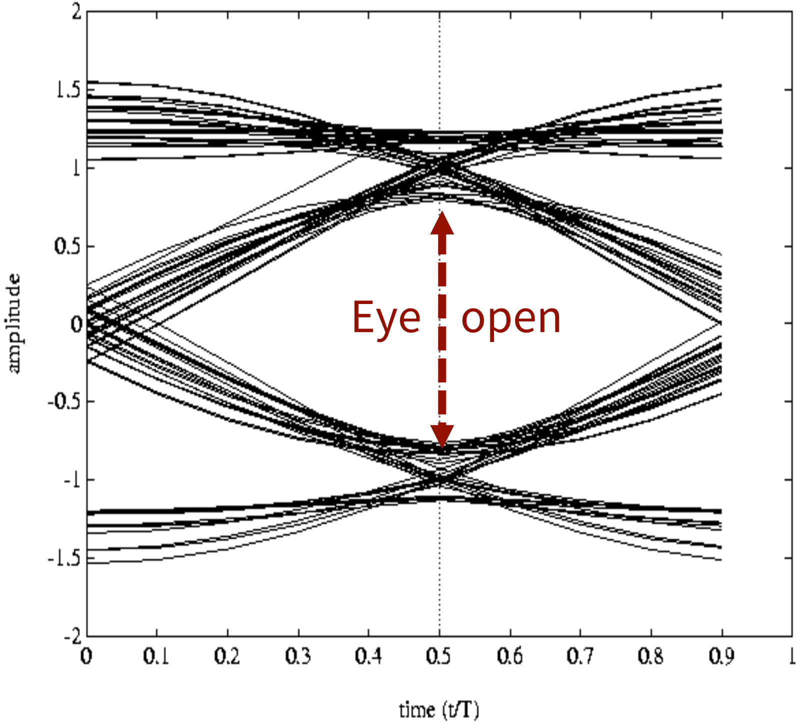
$$\mathcal{D}_{ms} \triangleq \mathbb{E}\left\{\left|\sum_{m \neq k} x_{h,m} \cdot q_{k-m}\right|^2\right\}$$

$$= \mathcal{E}_{\boldsymbol{x}} \cdot \|h\|^2 \cdot \textstyle\sum_{m \neq 0} |q_m|^2$$

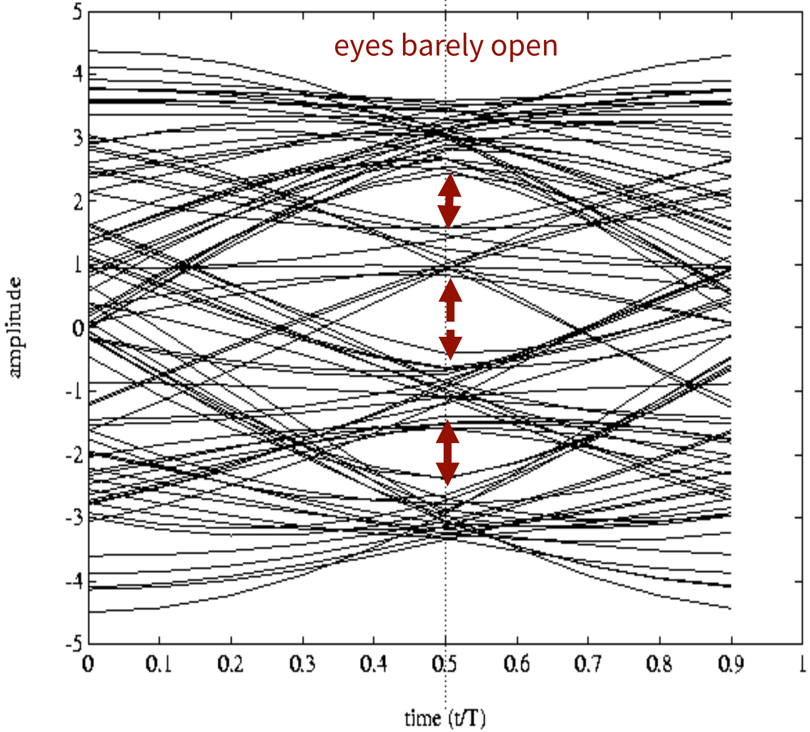
Eye Diagrams

- High-speed oscilloscope triggers on the symbol rate (or graph in software/matlab), $h(t) = {}^1\!/_{1+(3t/T)^2}$.

Binary: Lorentzian pulse



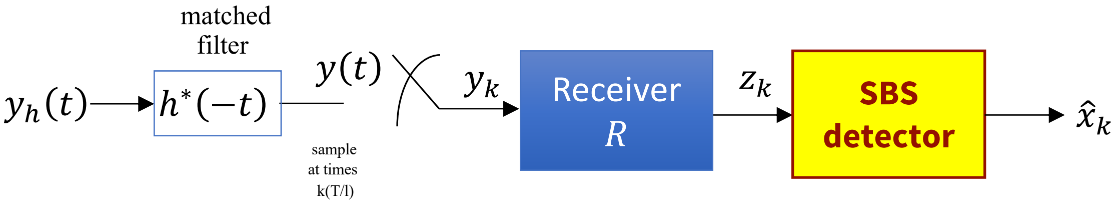
4PAM: Lorentzian pulse



Receiver SNR

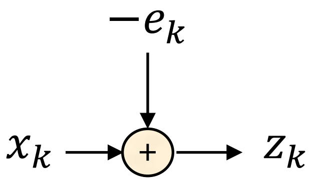
Section 3.2

Receiver to minimize ISI impact



SBS Det Pretends AWGN precedes it

Equivalent AWGN at time k



$$SNR = \frac{\mathcal{E}_x}{\mathbb{E}\left[|z_k - x_k|^2\right]}$$

- **SBS = Symbol By Symbol** (detector) – SBS is not necessarily ML, nor optimum, but maybe much cheaper.
  - All the QAM/PAM, coding formulas apply to this new channel $\bar{b} \cong \frac{1}{2} \cdot \log_2(1+SNR/\Gamma)$, so receiver converts to AWGN (or tries to).
- Receiver may be any, some, or all of:
  - linear filter that tries to Minimize Mean-Square Error (max SNR),
  - nonlinear processor that successively detects over sequence removing past ISI, also MMSE,
  - neural network (recurrent: two stages, first is single ReLU, second provides feed back of past state(s) influence,
  - Viterbi Detector (or other sequence detector) for ISI's trellis, which is NOT an SBS (more in L17).

Beware of coding theorist's fallacy, 1/T fixed

MMSE Receiver Biases

- MMSE has a "feature" or bias:

Definition 3.2.1 [Receiver SNR] *The* **receiver SNR**, SNR_R *for any receiver* R *with (pre-decision) output* z_k, *and decision regions based on* x_k *(see Figure 3.14) is*

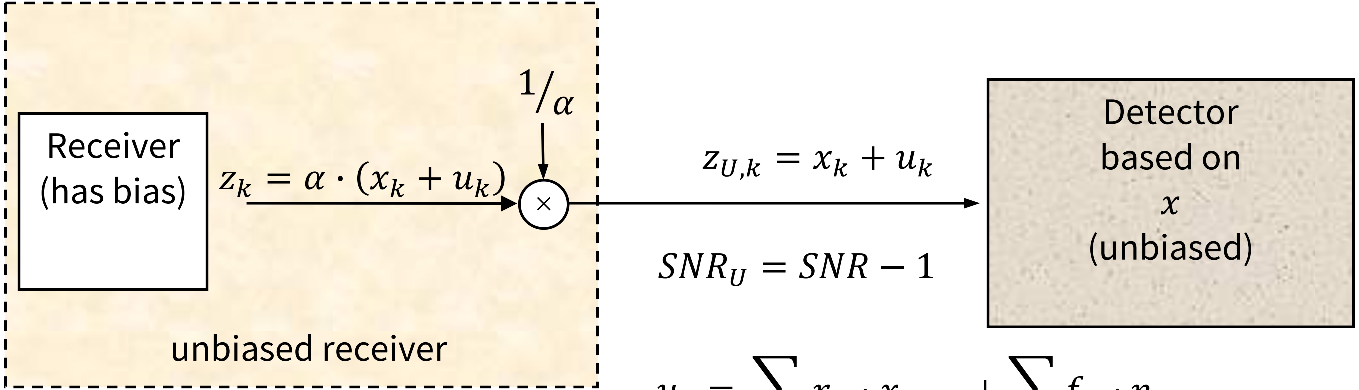
$$SNR_R \stackrel{\Delta}{=} \frac{\mathcal{E}_{\boldsymbol{x}}}{E|e_k|^2} \quad , \tag{3.35}$$

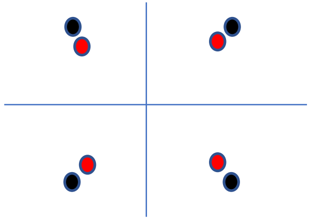
where $e_k \stackrel{\Delta}{=} x_k - z_k$ *is the* **receiver error**. *The denominator of (3.35) is the* **mean-square error**[7], $MSE{=}\mathbb{E}\left[|e_k|^2\right]$ *that is the same as the mean-square distortion. When* $\mathbb{E}\left[z_k|x_k\right] = x_k$, *the receiver is* **unbiased** *(otherwise* **biased**) *with respect to the decision regions for* x_k.

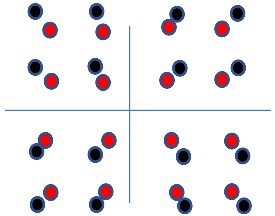
- The minimized MSE reduces the desired x_k amplitude (often slightly) because that also reduces noise,
  - in a way that minimizes MSE.
- The min P_e, which arises from the ML detector $\max_{x_k} p_{z_k/x_k}$, is unbiased – so MMSE is not quite ML.\,
  - unless, as we'll see in 379B, the code in use has zero gap – where indeed they are the same.

Receiver Removal of Bias

- Fortunately, removing the bias is easy:







$$u_k = \sum_{m \neq 0} r_m \cdot x_{k-m} + \sum_m f_m \cdot n_{k-m}$$

So, $\mathbb{E}[^{z_k}/_{x_k}] = x_k$

- Best value of α when the receiver minimizes MSE is $\frac{SNR}{SNR-1}$ (see proof in Section 3.2 or Appendix D).
- This boosts desired signal as well as distortion/noise, but in a way so that receiver's decision regions based on p_{z_k/x_k} are correct.
- The correct new SNR is always (with any MMSE receiver bias removal) $SNR_U = SNR - 1$ and it is the maximum unbiased SNR (see Theorem 3.2.1).

MMSE Conclusion / Use

- So it is ok to use MMSE, just remove the bias before entering the SBS detector.

- Maximum value occurs when there is no ISI, so that all signal energy at receiver MF output sampler is

$$SNR_{MMSE,U} \leq \frac{\bar{\mathcal{E}}_x \cdot \|h\|^2}{\sigma^2} \triangleq SNR_{MFB} \ .$$

Matched-Filter Bound SNR

- Bound attained with no ISI, so $Q(D)$=1 in:

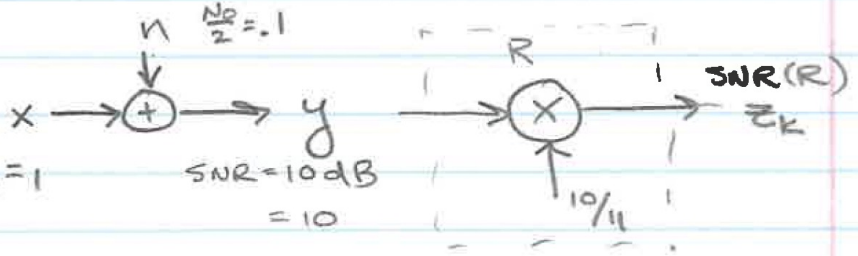
$$Y(D) = \|h\| \cdot Q(D) \cdot X(D) + N(D)\ .$$

- Argument of Q-function? (not same Qs)
  - For PAM and SQ QAM: the MFB then is $MFB = \sqrt{\frac{3}{4^b-1} \cdot SNR_{MFB}}\ ,$
  - Cross, see Section 3.2

Trival, but helpful, Example

- The MMSE SNR would exceed the MFB SNR if bias were not removed

6. Example:



ⓐ. $E\left[|x_k - z_k|^2\right] = \left(1 - 10/11\right)^2 \cdot 1 + \left(\frac{10}{11}\right)^2 \cdot (.1) =$

$= \frac{1}{121} \cdot 1 + \frac{10}{121} = \frac{1}{11}$

$\underline{SNR(R) = 11 > 10}$ – Best MSE/SNR R

ⓑ. remove bias

$z_k \rightarrow \boxed{11/10} \rightarrow z_{u,k}$ $\quad SNR_u(R) = 11 - 1 = \underline{\underline{10}}$

Nyquist Pulse Shaping

Section 3.3

Nyquist's Criterion

- When does a channel have no ISI? $q_k = 1 \quad ; \quad Q(D) = 1$
- The function $q(t)$ passes through zero at all the right times (except time 0).
- Well, ok, but what does that mean in continuous time/frequency?

$$
\begin{aligned}
q(kT) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega)\cdot e^{\jmath\omega kT}\cdot d\omega \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{\frac{(2n-1)\pi}{T}}^{\frac{(2n+1)\pi}{T}} Q(\omega)\cdot e^{\jmath\omega kT}\cdot d\omega \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} Q(\omega+\frac{2\pi n}{T})\cdot e^{\jmath(\omega+\frac{2\pi n}{T})kT}\cdot d\omega \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} Q_{eq}(\omega)\cdot e^{\jmath\omega kT}\cdot d\omega \quad ,
\end{aligned}
$$

$$Q_{eq}(\omega) \triangleq \sum_{n=-\infty}^{\infty} Q(\omega+\frac{2\pi n}{T}) = 1$$

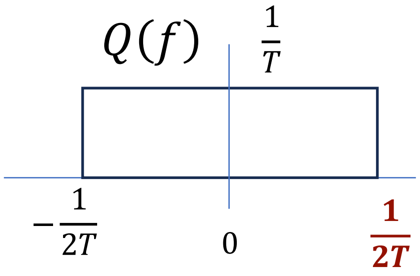
Equivalent (aliased) frequency response is "flat."

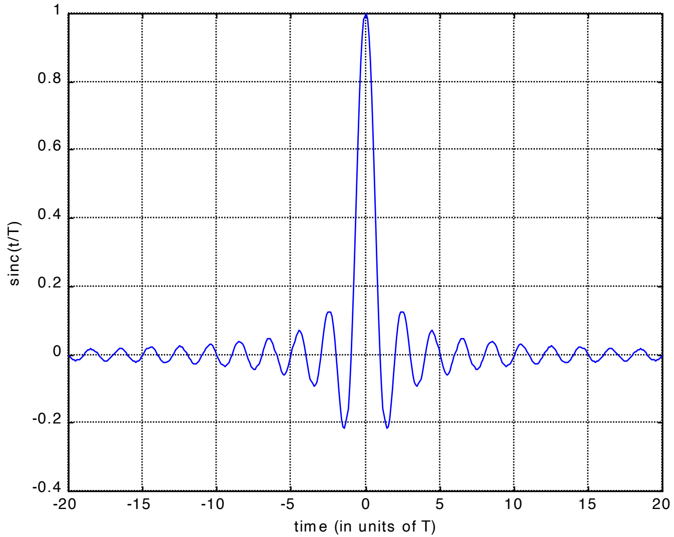
Theoretical simple example

- Sinc function:

$$q(t) = \operatorname{sinc}\left(\frac{t}{T}\right)$$

- There is no aliasing.





Nyquist Freqency

- Cannot be realized in practice, only approximated (with typically long delay)

Excess Bandwidth

- Percent excess bandwidth

Definition 3.3.2 **[Percent Excess Bandwidth]** *The* **percent excess bandwidth** α *follows from a strictly band-limited* $Q(\omega)$ *and is the highest frequency in* $Q(\omega)$ *for which there is nonzero energy transfer. That is*

$$Q(\omega) = \begin{cases} \textit{nonzero} & |\omega| \leq (1+\alpha)\cdot\frac{\pi}{T} \\ 0 & |\omega| > (1+\alpha)\cdot\frac{\pi}{T} \end{cases} \quad . \tag{3.72}$$

- Can be approximated more easily if $0 < \alpha \leq 1$:

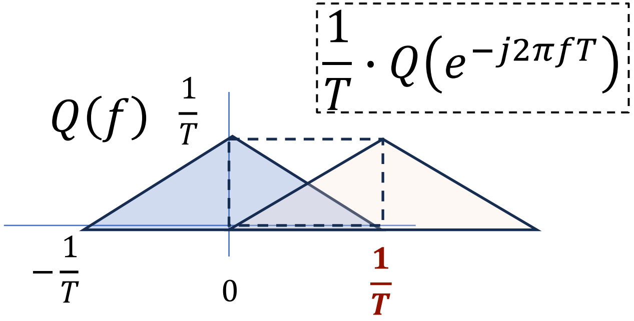
$$\begin{aligned} 1 &= Q(e^{-\jmath\omega T}) \\ &= \frac{1}{T}\left\{Q\left(\omega+\frac{2\pi}{T}\right)+Q\left(\omega\right)+Q\left(\omega-\frac{2\pi}{T}\right)\right\} \quad -\frac{\pi}{T}\leq\omega\leq\frac{\pi}{T} \end{aligned}$$

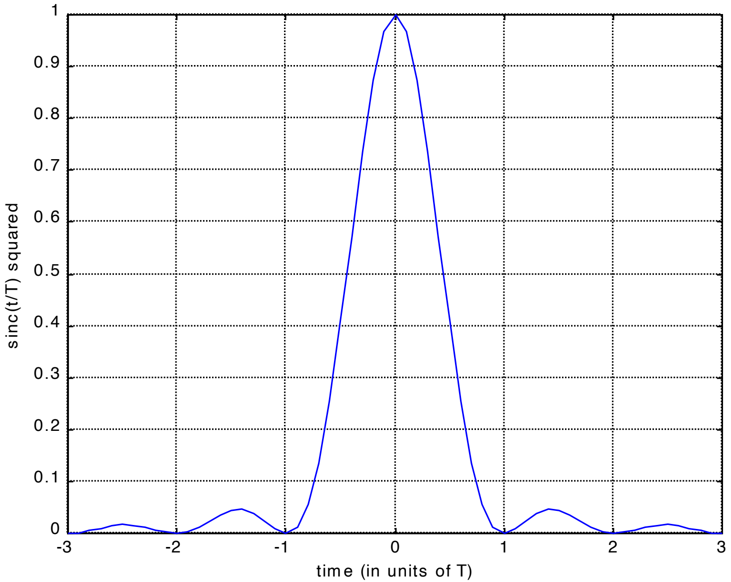
Vestigial Symmetry

- Positive frequencies

$$
\begin{aligned}
1 &= Q(e^{-\jmath\omega T}) \\
&= \frac{1}{T}\left\{Q(\omega) + Q\left(\omega - \frac{2\pi}{T}\right)\right\}
\end{aligned}
$$

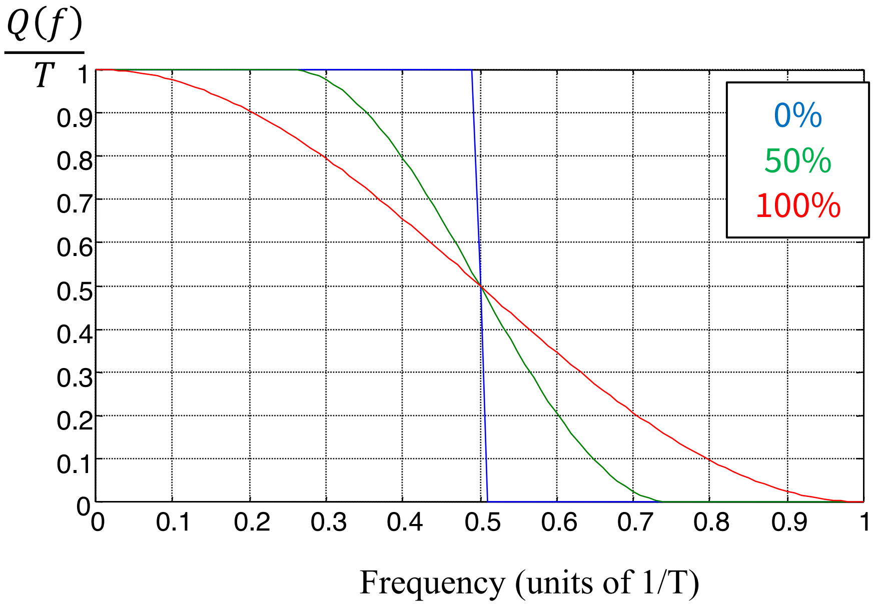
- 100% excess bandwidth example is $q(t) = sinc^2(t)$

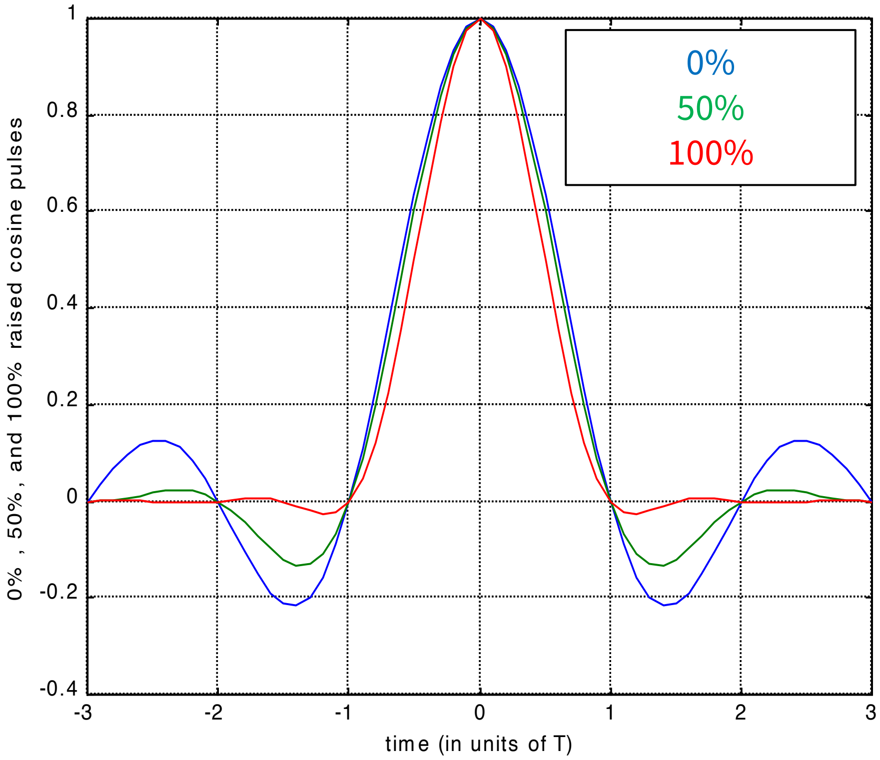




Raised Cosine Rolloff (RCR) Pulses

- **RCR pulse responses** have small excess bandwidth, but decay rapidly.
- They decay with t^3.





Raised Cosine Pulses mathmatically

Definition 3.3.3 **[Raised-Cosine Pulse Shapes]** *The raised cosine family of pulse shapes (indexed by* $0 \le \alpha \le 1$*) is given by*

$$q(t) = sinc\left(\frac{t}{T}\right) \cdot \left[\frac{\cos\left(\frac{\alpha\pi t}{T}\right)}{1 - \left(\frac{2\alpha t}{T}\right)^2}\right] \quad , \tag{3.79}$$

and have Fourier Transforms

$$Q(\omega) = \begin{cases} T & |\omega| \le \frac{\pi}{T}\cdot(1-\alpha) \\ \frac{T}{2}\cdot\left[1 - \sin\left(\frac{T}{2\alpha}\cdot\left\{|\omega| - \frac{\pi}{T}\right\}\right)\right] & \frac{\pi}{T}\cdot(1-\alpha) \le |\omega| \le \frac{\pi}{T}\cdot(1+\alpha) \\ 0 & \frac{\pi}{T}\cdot(1+\alpha) \le |\omega| \end{cases} \quad . \tag{3.80}$$

- Really want square root in xmit and again in rcvr matched filter (because with no ISI, this maximizes SNR).

$$\sqrt{Q(\omega)} = \begin{cases} \sqrt{T} & |\omega| \le \frac{\pi}{T}\cdot(1-\alpha) \\ \sqrt{\frac{T}{2}}\cdot\left[1 - \sin\left(\frac{T}{2\alpha}\cdot\left(|\omega| - \frac{\pi}{T}\right)\right)\right]^{1/2} & \frac{\pi}{T}\cdot(1-\alpha) \le |\omega| \le \frac{\pi}{T}\cdot(1+\alpha) \\ 0 & \frac{\pi}{T}\cdot(1+\alpha) \le |\omega| \end{cases}$$

$$\varphi_h(t) = \frac{4\alpha}{\pi\sqrt{T}} \cdot \frac{\cos\left([1+\alpha]\frac{\pi t}{T}\right) + \frac{T\cdot\sin\left([1-\alpha]\frac{\pi t}{T}\right)}{4\alpha t}}{1 - \left(\frac{4\alpha t}{T}\right)^2}$$

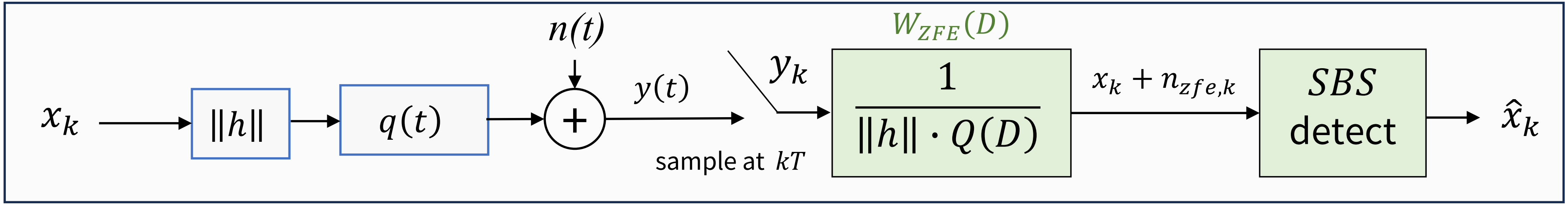
Zero-Forcing Equalization (ZFE)

Section 3.4

ZFE – Just filter with channel inverse

- The ZFE **Forces** ISI to **Zero.**

Don't confuse Q's ! (sorry)



- ZFE ignores the noise; this may lead to noise-energy increase (``**noise enhancement**'').
- SBS ignores and views its input as AWGN with SNR_{ZFE} (uses AWGN's ML detector).

$$P_{ZFE,e} \approx N_e \cdot Q\left(\frac{d_{\min}}{2\sigma_{ZFE}}\right)$$

- Noise may no longer be white, $n(t) \triangleq n_h(t) * \varphi_h^*(-t)$.

$$\bar{R}_{nn}(D) = \frac{\mathcal{N}_0}{2} \cdot Q(D)$$

ZFE Output Noise $Z(D) = X(D) + N_{ZFE}(D)$

- The ZFE further filters the noise, so $r_{ZFE,k} \triangleq \mathbb{E}\left[n_{ZFE,l} \cdot n^*_{ZFE,l}\right] = r_{nn,k} * w_{ZFE,k} * w^*_{ZFe,-k}$.

$$\bar{R}_{ZFE}(D) = \frac{\mathcal{N}_0}{2} \cdot \frac{Q(D)}{\|h\|^2 \cdot Q^2(D)} = \frac{\mathcal{N}_0}{2} \cdot \frac{W_{ZFE}(D)}{\|h\|}$$

$$W_{ZFE}(D) = \frac{1}{\|h\| \cdot Q(D)}$$

- Calculate $\sigma^2_{ZFE} = \frac{\mathcal{N}_0}{2} \cdot \frac{w_{ZFE,0}}{\|h\|}$; where $w_{ZFE,0}$ is the time-zero value of $w_{ZFE,k}$.

- Integral of the inverse-ISI function is important:

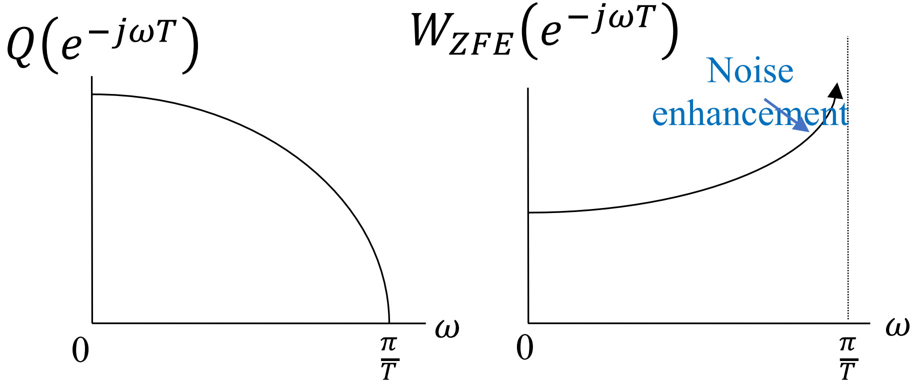
$$\gamma^{-1}_{ZFE} = \frac{T}{2\pi} \cdot \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{d\omega}{Q(e^{-j\omega T})} = w_{ZFE,0} \cdot \|h\|$$

- SNR of SBS then is $SNR_{ZFE} = \frac{\bar{\mathcal{E}}_x}{\sigma^2_{ZFE}} = SNR_{MFB} \cdot \gamma_{ZFE}$.

Loss Factor γ_{ZFE}

Noise Enhancement

- Typical channels often are lowpass;
  - because they don't transfer infinite energy.
- There may also be in-band noise , multipath, or DC coupling (real baseband only) that cause bandpass, bandstop, or highpass effects.
- Noise "enhances" where channel attenuates.



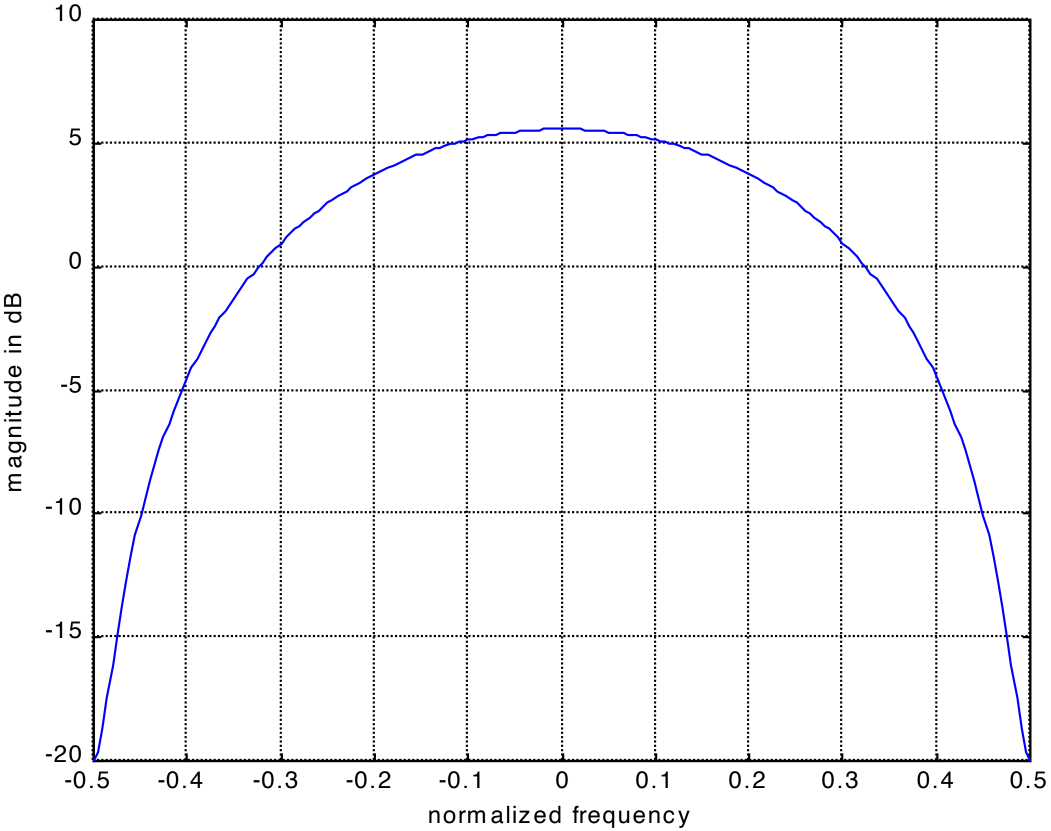
My favorite Channel : $H(D) = 1 + .9 \cdot D^{-1}$

$$H(\omega) = \begin{cases} \sqrt{T} \cdot \left(1 + .9 e^{\jmath \omega T}\right) & |\omega| \leq \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases} \qquad \Phi_h(\omega) = \begin{cases} \sqrt{\frac{T}{1.81}} \cdot \left(1 + .9 \cdot e^{\jmath \omega T}\right) & |\omega| \leq \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases}$$

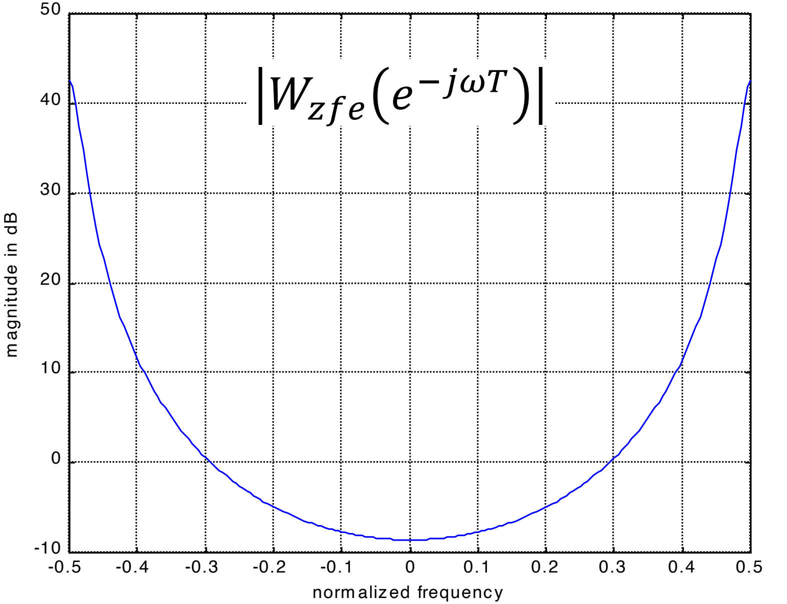
$$\|h\|^2 = 1.81 \quad \sigma^2 = 0.181 \qquad SNR_{MFB} = 10 \ (= 10 \text{ dB also})$$

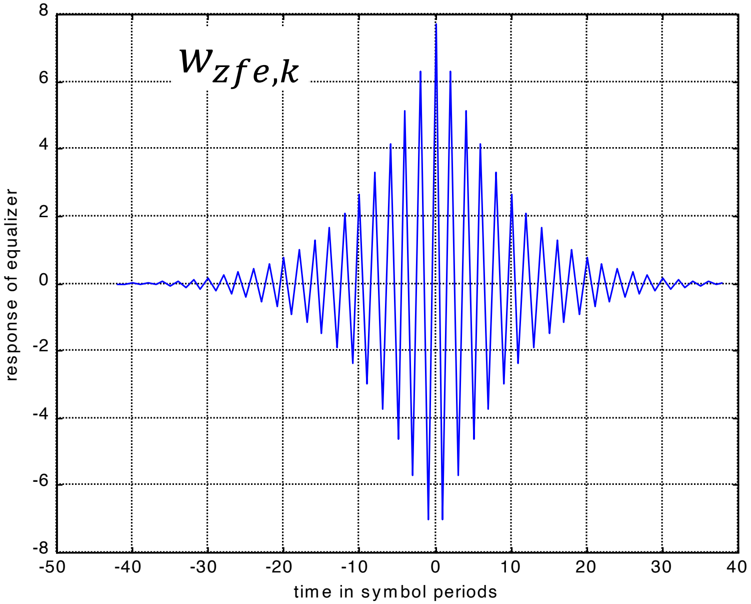
Example channel continued

- Real baseband, so is symmetric about DC.
- Lowpass
- Analysis ignores slight discontinuity at Nyquist frequency (makes math easier for students to follow, has negligible effect otherwise).
- It has significant ISI (2nd large tap).
- The SNR is already not so good.
  - SNRmfb is 10 dB.
  - So, design needs a code outside the receiver!
    - One designed for AWGN.



Corresponding ZFE Spectrum





- Edge noise increases 50dB (with respect to center).
- Very long response (delay, or complexity if realized as FIR).
- The integral of $W_{zfe}(e^{-j\omega T})$ can be done in closed form for this example → $\sigma^2_{ZFE} = 5.26 \cdot \sigma^2$ (see Example 3.4.1)

$$\gamma_{ZFE} = 10\log_{10}(1.81 \cdot 5.26) \approx 9.8\text{dB}$$

Big Loss!

End Lecture 13