Part I

Signal Processing and Detection
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Chapter 1

Fundamentals of Discrete Data Transmission

This chapter introduces a foundation for digital data transmission taken throughout this text. This foundation links basic mathematical concepts in probability theory, discrete and continuous fields, and basic optimum-decision concepts to transmission design. The consequent foundational understanding readies the designer to comprehend many simple wireline and wireless transmission systems, while also enabling the designer to progress to increasingly sophisticated methods in later chapters. Simple vector symbol representation of transmitters and receivers with definitions of optima and corresponding analysis methods appear here. These allow productive use of expressions introduced for data rates achieved, corresponding defined quality measures like probability of error, and a common framework for comparisons of different systems using nearest-neighbor and minimum-distance concepts defined within. More than 50 exercises at the end of this chapter also help the reader examine these foundational basics and illustrate the topics’ utility in a number of both practical designs and/or curious challenges.

Figure 1.1: Discrete data transmission.

Figure 1.1 illustrates discrete data transmission, which is the transmission of one message from a finite set of messages through a communication channel. A message sender at the transmitter communicates with a message recipient. The sender selects one message from the finite set, and the transmitter sends a corresponding symbol that uniquely represents this message through the communication channel. The receiver decides the message sent by observing the received symbol, which may not be the same as the transmitted symbol, and passes that decided message to the recipient.
This chapter concentrates on optimal detection for a single message transmission through the channel. Such single-message analysis is often called one-shot analysis. When a single message is sent, Section 1.1’s **encoder** maps that message to a symbol. Similarly with a single message, Section 1.1’s **detector** is the receiver device that makes the receiver’s decision. Section 1.1 also develops optimum detection that minimizes the probability of an erroneous receiver decision on which message was transmitted. A decoder maps the decision into the corresponding message. Chapter 2 addresses encoders that expand multiple successive coordinated symbol transmissions, while Chapter 3 addresses channel-induced inter-symbol interference.

A single bit, a digital sequence of bits, bytes (8-bit groups), or other groupings of a finite number of bits represent the message to be sent. More bits means a larger finite set of possible choices for the single message sent. The different bit combinations each uniquely represent individually the distinct discrete messages that pass to the channel. The bits themselves are usually not compatible with the direct transmission of messages through most communication channels. Thus the encoder converts the messages’ bits into appropriate **symbols** that the transmitter can send through the channel. The symbols depend on the permissible types of channel inputs, which are typically modeled as within a field or vector space. Section ?? models the channel with a conditional probability distribution on received symbols for each given transmitted symbol value.

The channel distorts the transmitted symbols both deterministically and randomly to produce the received symbols. Because the received symbol will usually not exactly equal the transmitted symbol, and so Section 1.1 develops the concept of the detector that makes a optimum decision based on the observed received symbol. Optimal decisions minimize the probability of message/symbol decision error. The transmitted symbols have probabilities equal to the probabilities of the messages that they represent. The optimal decision will depend only on the probabilistic model for the channel and the channel-input-message’s probability distribution. The general optimal decision specializes in many later sections’ important practical cases of interest. This probabilistic approach allows conceptual extension beyond data “transmission” to all types of recognition, detection, and matching problems that often go under more exotic modern names like “machine learning,” “search engine,” and/or “facial recognition.” A good part of life and education tries to learn, infer, or understand/receive some communication or information (as well as to transmit or store it so it is more easily understood by another), and this chapter provides basics that apply to all these basic communication areas.

Section 1.2 then expands the transmitter model to include **modulation**. A modulator converts the encoder-output symbols into continuous-time signals for transmission through a continuous-time channel. This chapter develops a theory of modulation and corresponding demodulation that links to Section ??’s discrete vector representation for any set of continuous-time signals. This “vector-channel” approach was pioneered for educational purposes by Wozencraft and Jacobs in their classic text [1] (Chapter 4). In fact, the first two sections of this chapter closely parallel their development (with some updating and rearrangement), before diverging in Sections 1.3 – ?? and in the remainder of this text. Section 1.2’s last subsection introduces the **multiple-input multiple-output (MIMO)** modulation that may generate multiple continuous-time signals for coordinated transmission through separate antennas’ or wires’ channels.

Section 1.3 investigates continuous-time channels, particularly the most common case of the additive Gaussian-noise channel, which maps easily into Section 1.1’s discrete-time vector model without loss of generality. Section 1.3 also develops simpler widely applicable methods to calculate and estimate average probability of error, \( P_e \), for a vector channel with Additive White Gaussian Noise (AWGN), particularly introducing and using nearest-neighbor and minimum-distance concepts. Section 1.3 also discusses several popular modulation formats and determine bounds for their probability of error with AWGN, including signals derived from rectangular lattices, a popular and practical signal-transmission method. Section 1.3 also addresses the extension of carrier modulated signals. Section 1.4 progresses to finite-field channels where the inputs and outputs belong to discrete finite sets and the concept of noise necessarily becomes discrete and part of the channel’s general conditional-probability model.

---

1Dependencies between successive message transmissions can be important also, but the study of such inter-message dependency is deferred to later chapters.
1.1 Discrete Data-Message Encoding and Decoding

This section mathematically and statistically models the basic transmitter, channel, and receiver through symbol vectors. Some results here will correspond to the transmitted symbol and corresponding received symbol being \( N \)-dimensional real-valued vectors, while Sections 1.3 and 1.4 will expand to complex vectors and other possible fields for the symbol values. Of importance is the study of the optimal detector. The optimal detector decides which of the discrete symbol vectors \( \mathbf{x}_i \), \( i = 0, ..., M - 1 \) was most likely transmitted based on the single observation of the received symbol vector \( \mathbf{y} \). Section 1.2 introduces MIMO (multiple-input-multiple-output) vector channels that fit precisely also into this section’s framework, but extend the vector dimensionality index to correspond to simultaneous transmission of symbols over \( L_x \) multiple-input, parallel, \( N \)-dimensional channels to \( L_y \) multiple received symbols that may each also be \( N \)-dimensional. The transmitted symbols may be viewed in aggregate then as a single transmitted symbol (chosen from a larger set of possible symbols over a larger dimensionality), and similarly the received symbols can also be viewed as single received symbol (also of larger dimensionality).

1.1.1 The Vector-Symbol Channel Model

The vector-symbol channel model appears in Figure 1.2. A message from the set of \( M \) possible messages \( m_i \), \( i = 0, ..., M - 1 \) is sent every \( T \) seconds, where \( T \) is the symbol period for the discrete data transmission system. Thus, messages are sent at the symbol rate of \( 1/T \) messages per second. The number of messages that can be sent is often measured in bits so that \( b = \log_2(M) \) bits are sent every symbol period. Thus, the data rate is \( R = b/T \) bits per second. The message is often considered to be a real integer equal to the index \( i \), in which case the message is abbreviated as \( m \) with possible values \( m \in \{0, ..., M - 1\} \). This chapter’s one-shot analysis will focus attention on a single symbol period over time \( t \in [0, T] \).

The encoder formats the messages for transmission over the vector-symbol channel by uniquely mapping each message \( m_i \) into its specific corresponding symbol vector \( \mathbf{x}_i \), typically an \( N \)-dimensional real data symbol chosen from a signal constellation \( \mathcal{C} \) that is the set of \( \mathcal{C} \geq M \) distinct points \( \mathcal{C} = \{\mathbf{x}_i \mid i = 0, ..., |\mathcal{C}| - 1\} \). In this chapter \( |\mathcal{C}| = M \), but it is possible in Chapter 2’s coded systems for the signal constellation to have more possible points than there are messages. The detector decides which message \( \hat{m}_i \) was sent from among the set of \( M \) possible messages \( \{m_i \mid i = 0, ..., M - 1\} \) that could have been transmitted over the vector channel. In the vector channel, \( \mathbf{x} \) is a random vector, with discrete probability distribution \( p_{\mathbf{x}}(i), i = 0, ..., |\mathcal{C}| - 1 \).

**Definition 1.1.1 (Symbol Transmission Definition Summary)** Table 1.1 summarizes the symbol-related definitions:

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**Figure 1.2:** Vector channel model.
### Table 1.1: Table of transmitted-symbol quantities’ definitions.

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 2^b$</td>
<td>The number of messages, corresponding to $b$ information bits</td>
</tr>
<tr>
<td>$T$</td>
<td>The symbol period; $1/T$ is the symbol rate</td>
</tr>
<tr>
<td>$R = \frac{b}{T}$</td>
<td>The data rate.</td>
</tr>
<tr>
<td>$x$</td>
<td>The transmitted symbol value (typically a real or complex vector)</td>
</tr>
<tr>
<td>$C$</td>
<td>The constellation consisting of all possible symbol values ${x_i</td>
</tr>
<tr>
<td>$p_m(i)$</td>
<td>The message’s probability distribution, $i = 0, ..., M - 1$</td>
</tr>
<tr>
<td>$p_x(i)$</td>
<td>The symbol values’ probability distribution</td>
</tr>
</tbody>
</table>

**INSERT SOME EXAMPLES HERE:**
1. real numbers + - 1 have noise added
2. Set of faces
3. delays in radar/lidar
4. search engine

An important concept for a signal constellation (with real or complex vector symbols) is its average energy:

**Definition 1.1.2 (Average Energy)** The average energy of a signal constellation is defined by

$$ E_x \overset{\Delta}{=} E[||x||^2] = \sum_{i=0}^{\|C\|-1} ||x_i||^2 \cdot p_x(i), $$ (1.1)

where $||x_i||^2$ is the squared-length of the vector $x_i$, $||x||^2 \overset{\Delta}{=} \sum_{n=1}^{N} x_n^2$. “$E$” denotes expected or mean value. The average energy is also closely related to the concept of average power, which is

$$ P_x \overset{\Delta}{=} \frac{E_x}{T}, $$ (1.2)

**corresponding to the amount of energy per symbol period.**

In the same symbol period, the transmitted symbol vector $x$ corresponds to a received symbol vector $y$, which is also an $N$-dimensional real vector.\(^3\) The received symbol’s conditional probability (given the input symbol), $p_y|x$, completely models the discrete data channel. The detector then translates the received symbol vector $y$ into a decision of the transmitted symbol $\hat{x}$. A decoder (which is part of the decision device) reverses the encoder process and converts the detector output $\hat{x}$ into the message corresponding to the decision $\hat{m}$.

The particular message symbol vector corresponding to $m_i$ is $x_i$ and has $n^{th}$ component $x_{in}$. The $n^{th}$ component of $y$ is denoted $y_n$, $n = 1, ..., N$. The random received symbol vector $y$ may have a continuous probability density or a discrete probability distribution $p_y(v)$, where $v$ is a dummy variable spanning all the possible $N$-dimensional vector values for $y$. The received symbol’s distribution is a function of the transmit-symbol and channel-transition-probability distributions:

$$ p_y(v) = \sum_{i=0}^{\|C\|-1} p_{y|x}(v, i) \cdot p_x(i), $$ (1.3)

---

2Electrical engineers may note power (and therefore energy) necessarily are also a function of line/antenna impedance. That impedance’s square root is presumed absorbed into the symbol’s value in cases where the symbol is viewed as a voltage level. This scaling would also be implied for received symbols.

3Section 1.2 will address the transformation of $y(t) \rightarrow y$ for continuous-time channels.
The average energy of the transmit symbol vector is
\[ \mathcal{E}_x = \sum_{i=0}^{C-1} \| x_i \|^2 \cdot p_x(i) \]  
(1.4)

The corresponding average energy for the received symbol vector is
\[ \mathcal{E}_y = \sum_v \| v \|^2 \cdot p_y(v) \]  
(1.5)

An integral replaces the sum in (1.3) and (1.5) for the case of a continuous density function \( p_y(v) \). As an example, consider the simple additive noise channel \( y = x + n \). In this case \( p_{y|x} = p_n(y-x) \), where \( p_n(\bullet) \) is the noise probability distribution, when \( n \) is independent of the input \( x \).

1.1.2 Optimum Data Detection

For the channel of Figure 1.2, the error probability is defined as the probability that the decoded message \( \hat{m} \) is not equal to the message that was transmitted:

**Definition 1.1.3 (Probability of Error)** The Error Probability is defined as
\[ P_e \triangleq P\{ \hat{m} \neq m \} \]  
(1.6)

The corresponding probability of being correct is therefore
\[ P_e = 1 - P_e = 1 - P\{ \hat{m} \neq m \} = P\{ \hat{m} = m \} \]  
(1.7)

The optimum data detector chooses \( \hat{m} \) to minimize \( P_e \), or equivalently, to maximize \( P_c \). The probability of being correct is a function of the particular transmitted message, \( m_i \).

1.1.2.1 The MAP Detector

The probability of a correct decision \( \hat{m} = m_i \), given the specific channel output vector \( y = v \), is
\[ P_c(\hat{m} = m_i, y = v) = p_{m/y}(m_i, v) \cdot p_y(v) = p_{x|y}(x_i, v) \cdot p_y(v) \]  
(1.8)

Thus the optimum decision device observes the particular received symbol \( y = v \) and, as a function of that symbol, chooses an \( \hat{m} = m_i \), \( i = 0, ..., M-1 \) that maximizes the probability of a correct decision in (1.8). This quantity \( p_{m/y} \) is referred to as the à posteriori probability for the vector channel. Summing (discrete \( v \) components or equivalently integrating when continuous \( v \)) over all \( v \) values, \( p_{m/y}(i, v) \cdot p_y(v) \) yields \( P_c \), which is maximized overall then too (since \( p_y(v) \geq 0 \)). \( P_c \) is then minimized. Thus, the optimum detector for the vector channel in Figure 1.2 is called the Maximum à Posteriori detector:

**Definition 1.1.4 (MAP Detector)** The Maximum à Posteriori (MAP) Detector is defined as the detector that chooses the message index \( i \) to maximize the à posteriori probability \( p_{m/y}(i, |v) \) given a received symbol \( y = v \).

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4The replacement of a continuous probability distribution function by a discrete probability distribution function (sometimes called a density mass function) is, in strictest mathematical terms, not advisable; however, we do so here, as this particular substitution prevents a preponderance of additional notation, and it has long been conventional in the data transmission literature. The reader is thus forewarned to keep the continuous or discrete nature of the probability density in mind in the analysis of any particular vector channel.
Subsection 1.1.4 provides the calculation of the corresponding optimum average error probability.

The MAP detector thus simply chooses the index \( i \) with the highest conditional probability \( p_m | y (i | v) \). When \( m \) and \( x \) are in 1-to-1 correspondence (as always in this chapter), then \(|C| = M \) and then \( p_x | y (i, v) = p_m | y (i, v) \) and \( p_x (i) = p_m (i) \). It is often convenient to represent the message by \( x \) when this is true. For every possible received vector \( y \), the designer of the detector can calculate the corresponding best index \( i \), which depends on the input distribution \( p_x (i) \). The \( a \ posteriori \) probabilities can be rewritten in terms of the \( a \ priori \) probabilities \( p_x (i) \) and the channel transition probabilities \( p_y | x \) by recalling the identity

\[
p_x | y (i | v) \cdot p_y (v) = p_y | x (v | i) \cdot p_x (i) \quad (1.9)
\]

Thus,

\[
P_x | y (i, v) = \frac{p_y | x (v, i) \cdot p_x (i)}{p_y (v)} = \sum_{j=0}^{M-1} \frac{p_y | x (v, j) \cdot p_x (j)}{p_y (v)}, \quad (1.10)
\]

for \( p_y (v) \neq 0 \). If \( p_y (v) = 0 \), then that particular output does not contribute to \( P_e \) and therefore is not of further concern. When maximizing (1.10) over \( i \), the denominator \( p_y (v) \) is a constant that is ignored.

Thus, Rule 1.1.1 below summarizes the following MAP detector rule in terms of the known probability densities of the channel \( (p_y | x) \) and of the input vector \( (p_x) \):

**Rule 1.1.1 (MAP Detection Rule)**

\[
\hat{m} \Rightarrow m_i \text{ if } \frac{p_y | m (v, i) \cdot p_m (i)}{p_y (v)} \geq \frac{p_y | m (v, j) \cdot p_m (j)}{p_y (v)} \forall j \neq i \quad (1.11)
\]

*If equality holds in (1.11), then the decision can be assigned to either message \( m_i \) or \( m_j \) without changing the minimized error probability.*

1.1.2.2 The Maximum Likelihood (ML) Detector

If all transmitted messages are of equal probability, that is if

\[
p_m (i) = \frac{1}{M} \quad \forall i = 0, ..., M - 1 \quad , \quad (1.12)
\]

then the MAP Detection Rule becomes the Maximum Likelihood Detection Rule:

**Rule 1.1.2 (ML Detection Rule)**

\[
\hat{m} \Rightarrow m_i \text{ if } \frac{p_y | m (v, i)}{p_y (v)} \geq \frac{p_y | m (v, j)}{p_y (v)} \forall j \neq i \quad . \quad (1.13)
\]

*If equality holds in (1.13), then the decision can be assigned to either message \( m_i \) or \( m_j \) without changing the probability of error.*

As with the MAP detector, the ML detector also chooses an index \( i \) for each possible received vector \( y = v \), but this index now only depends on the channel transition probabilities and is independent of the input distribution (by assumption). The ML detector essentially cancels the \( 1/M \) factor on both sides of (1.11) to get (1.13). This type of detector only minimizes \( P_e \) when the input data messages have equal probability of occurrence. As this requirement is often met in practice, ML detection is often used. Even when the input distribution is not uniform, ML detection is still often employed as a detection rule, because the input distribution may be unknown and thus assumed to be uniform. The **Minimax Theorem** sometimes justifies this uniform assumption:

\[ \text{The more general form of this identity is called “Bayes Theorem”; [2].} \]
Theorem 1.1.1 (Minimax Theorem) The ML detector minimizes the maximum possible average error probability when the input distribution is unknown if the conditional ML error probability $P_{e,ML/m=m_i}$ is independent of $i$.

Proof:

First, if $P_{e,ML/i}$ is independent of $i$, then

$$P_{e,ML} = \sum_{i=0}^{M-1} p_X(i) \cdot P_{e,ML/i}$$

$$= P_{e,ML/i}$$

And so,

$$\max_{\{p_X\}} P_{e,ML} = \max_{\{p_X\}} \sum_{i=0}^{M-1} p_X(i) \cdot P_{e,ML/i}$$

$$= P_{e,ML}$$

Now, let $R$ be any receiver other than the ML receiver. Then,

$$\max_{\{p_X\}} P_{e,R} = \max_{\{p_X\}} \sum_{i=0}^{M-1} p_X(i) \cdot P_{e,R/i}$$

$$\geq \sum_{i=0}^{M-1} \frac{1}{M} P_{e,R/i} \text{ (because } \max_{\{p_X\}} P_{e,R} \geq P_{e,R} \text{ for given } \{p_X\} \text{ specifically uniform.)}$$

$$\geq \sum_{i=0}^{M-1} \frac{1}{M} P_{e,ML/i} \text{ (because the ML minimizes } P_e \text{ when } p_X(i) = \frac{1}{M} \text{ for } i = 0, \ldots, M - 1.)$$

$$= P_{e,ML}$$

So,

$$\max_{\{p_X\}} P_{e,R} \geq P_{e,ML}$$

$$= \max_{\{p_X\}} P_{e,ML}$$

The ML receiver minimizes the maximum $P_e$ over all possible receivers. QED.

The symmetry condition imposed by the Minimax Theorem is not always satisfied in practical situations; but the likelihood of an application where both the inputs are nonuniform in distribution and the ML conditional error probabilities are not symmetric is rare. Thus, ML receivers have come to be of nearly ubiquitous use in place of MAP receivers when detecting symbols. If the input probability distribution is not uniform, compression methods can be used to reduce the bit rate of the source so that it appears uniform at the new lower data rate; however such compression is beyond the scope of this text.

1.1.3 Decision Regions

In the case of either the MAP Rule in (1.11) or the ML Rule in (1.13), each and every possible value for the channel output $y$ maps into one of the $M$ possible transmitted messages. Thus, the vector space (or more generally the field of values for the transmitted symbol values) for $y$ is partitioned into $M$ regions corresponding to the $M$ possible decisions. Simple communication systems have well-defined boundaries (to be shown later), so the decision regions often coincide with intuition. Nevertheless, in some well-designed communications systems, the decoding function and the regions can be more difficult to visualize.
Definition 1.1.5 (Decision Region) The decision region using a MAP detector for each message $m_i$, $i = 0, ..., M - 1$, is defined as

$$D_i \triangleq \{ v \mid p_{y|m}(v, i) \cdot p_m(i) \geq p_{y|m}(v, j) \cdot p_m(j) \quad \forall \ j \neq i \} \ .$$  \hspace{1cm} (1.14)

With uniformly distributed input messages, the decision regions reduce to

$$D_i \triangleq \{ v \mid p_{y|m}(v, i) \geq p_{y|m}(v, j) \quad \forall \ j \neq i \} \ .$$  \hspace{1cm} (1.15)

In Figure 1.3, each of the four different two-dimensional transmitted vectors $x_i$ (corresponding to the messages $m_i$) has a surrounding decision region in which any received value for $y = v$ is mapped to the message $m_i$. In general, the decision regions need not be connected, and although such situations are rare in practice, they can occur (see Problem 1.12). Section 1.3 illustrates several examples of decision regions for the AWGN channel.

1.1.4 Optimum Average Error Probability Calculation

The probability of a correct decision, $P_c$, in Equation (1.8) is for a specific value of $m_i$. Use of the MAP detector corresponds to a specific (optimum) probability of correct decision, and corresponding minimum $P_e$ for those values of $v \in D_i$, and so can also be rewritten

$$P_c(\hat{m} = m_i, \ y = v \in D_i) \rightarrow P_{e/m=m_i}(v \in D_i/m_i) \ .$$  \hspace{1cm} (1.16)

The average error probability for a detector $\hat{m} = m_i$ with a optimum-decision-region (or really any decision region corresponding to the a specific) rule $D_i$ and corresponding $P_{e/m=m_i}(v/m_i)$ would then be

$$P_{e,max} \triangleq E[P_e] = \sum_{i=0}^{M-1} \left\{ \sum_{v \in D_i} P_{e|m=m_i}(v|m_i) \right\} \cdot p_{m_i} \ .$$  \hspace{1cm} (1.17)
Thus the minimum average $P_e$ for the MAP detector can be computed as

$$P_{e,\text{min}} \triangleq 1 - P_{c,\text{max}} = 1 - \sum_{i=0}^{M-1} \left( \sum_{v \in D_i} p_{y|m_i}(v|i) \cdot p_{m_i}(i) \right) \cdot p_{m_i}. \quad (1.18)$$

Several examples of (1.18)’s computation will occur for specific channels later in this chapter. Often in specific cases, the double sum/integration can be tightly bounded and simplified to a simple expressing involving the minimum separation between transmitted symbols and the average number of nearest neighboring transmitted symbols.

### 1.1.5 Irrelevant Components of the Channel Output

The discrete channel-output vector $y$ may contain information that does not help determine which of the $M$ messages has been transmitted. These irrelevant components may be discarded without loss of performance, i.e. the input detected and the associated probability of error remain unchanged. The received symbol $y$ can be separated into two sets of dimensions, those that carry useful information $y_1$ and those that do not carry useful information $y_2$. That is,

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (1.19)$$

Theorem 1.1.2 summarizes the condition on $y_2$ that guarantees irrelevance [1]:

#### Theorem 1.1.2 (Theorem on Irrelevance)

If

$$P(x/(y_1,y_2) = P(x/y_1), \quad (1.20)$$

or equivalently for the ML receiver,

$$P(y_2/(y_1,x) = P(y_2/y_1), \quad (1.21)$$

then $y_2$ is not needed in the optimum receiver, that is, $y_2$ is irrelevant.

Proof: For a MAP receiver, then clearly the value of $y_2$ does not affect the maximization of $P(x/(y_1,y_2)$ if $P(x/(y_1,y_2) = P(x/(y_1))$ and thus $y_2$ is irrelevant to the optimum receiver’s decision. Equation (1.20) can be written as

$$\frac{P(x,y_1,y_2)}{P(y_1,y_2)} = \frac{P(x,y_1)}{P(y_1)} \quad (1.22)$$

or equivalently via “cross multiplication”

$$\frac{P(x,y_1,y_2)}{P(x,y_1)} = \frac{P(y_1,y_2)}{P(y_1)}, \quad (1.23)$$

which is the same as (1.21). QED.

The reverse of the theorem of irrelevance is not necessarily true, as can be shown by counterexamples. Two examples (due to Wozencraft and Jacobs, [1]) reinforce the concept of irrelevance. In these examples, the two noise signals $n_1$ and $n_2$ are independent and a uniformly distributed input is assumed:

#### EXAMPLE 1.1.1 (Extra Irrelevant Noise)

Suppose $y_1$ is the noisy channel output shown in Figure 1.4.
In the first example, \( p_{y_2|y_1, x} = p_{y_2} = p_{y_1} \), thus satisfying the condition for \( y_2 \) to be ignored, as might be obvious upon casual inspection. The extra independent noise signal \( n_2 \) tells the receiver nothing given \( y_1 \) about the transmitted message \( x \). In the second example, the irrelevance of \( y_2 \) given \( y_1 \) is not quite as obvious as the signal is present in both the received channel output components. Nevertheless, \( p_{y_2|y_1, x} = p_{n_2}(v_2 - v_1) = p_{y_2|y_1} \).

In some other cases the output component \( y_2 \) should not be discarded. A classic example is the following case of “noise cancelation”:

**EXAMPLE 1.1.2 (Noise Cancelation)** Suppose \( y_1 \) is the noisy channel output shown in Figure 1.5 while \( y_2 \) may appear to contain only useless noise, it is in fact possible to reduce the effect of \( n_1 \) in \( y_1 \) by constructing an estimate of \( n_1 \) using \( y_2 \). Correspondingly, \( p_{y_2|y_1, x} = p_{n_2}(v_2 - (v_1 - x)) \neq p_{y_2|y_1} \).

**1.1.5.1 Reversibility**

An important result in digital communication is the **Reversibility Theorem**, which will be used several times over the course of this book. This theorem is, in effect, a special case of the Theorem on Irrelevance:
Theorem 1.1.3 (Reversibility Theorem) The application of an invertible transformation on the channel output vector \( y \) does not affect the performance of the MAP detector.

Proof: Using the Theorem on Irrelevance, if the channel output is \( y_2 \) and the result of the invertible transformation is \( y_1 = G(y_2) \), with inverse \( G^{-1}(y_1) \) then \( [y_1, y_2] = [y_1, G^{-1}(y_1)] \). Then, \( p_{x|y_1, y_2} = p_{x|y_1} \), which is the definition of irrelevance. Thus, either of \( y_1 \) or \( y_2 \) is sufficient to detect \( x \) optimally and attain the same minimum error probability or equivalently the same optimum performance. QED.

Equivalently, Figure 1.6 illustrates the reversibility theorem by constructing a MAP receiver for the output of the invertible transformation \( y_1 \) as the cascade of the inverse filter \( G^{-1} \) and the MAP receiver for the input of the invertible transformation \( y_2 \). The receiver for \( y_2 \) can sometimes be simpler to design than one for \( y_1 \). Later chapters will use this concept heavily to produce equivalent optimum receivers that might not otherwise appear equivalent.

### 1.1.6 Optimum Bit-Error Probability and Log Likelihood Decoding

Designers may be interested in minimizing the bit-error probability within a message instead of minimizing the symbol/message error. The message may then be viewed as having \( b \) bits specifically denoted by \( u_j \) with \( u_j = 0 \) or 1, and abbreviated by the vector \( u \). \( p_{u_j} \) is the probability that \( u_j = 1 \) and thus \( 1 - p_{u_j} \) is the probability that \( u_j = 0 \). For a set of MAP (or ML) detectors, one for each bit, can be designed with error criterion to minimize

\[
Pr\{\hat{u}_j \neq u_j\} = 1 - \sum_{u_j=0}^{1} \left[ \sum_{v \in \mathcal{D}_j} P_c(\hat{u}_j = u_j, v) \right] \cdot p_{u_j}
\]

(1.24)

\[
= 1 - \sum_{u_j=0}^{1} \left[ \sum_{v \in \mathcal{D}_j} p_{u_j|y}(u_j, y) \cdot p_y(v) \right] \cdot p_{u_j}.
\]

(1.25)

This probability of error is not necessarily equal for each bit, nor consequently equal to the minimized symbol-error probability, \( P_e \), although generally speaking minimizing the symbol error probability will usually lead to good probability of bit error. Calculation of the probability of error will require the probability distribution of the vector of bits \( p_{u|y} \) in place of \( p_{m|y} \). The notation \( u \ \setminus u_j \) means the
vector \( \mathbf{u} \) with the \( j^{th} \) bit removed. The bit-error optimum receiver will average other bits as explicitly indicated by writing \( p_{u_j|y(u_j, \mathbf{v})} \) for any received symbol \( \mathbf{v} \) as the sum of \( 2^k-1 \) terms:

\[
p_{u_j|y(u_j, \mathbf{v})} = \sum_{\mathbf{u} \backslash u_j} p_{\mathbf{u}|y(\mathbf{u}, \mathbf{v})}
\]

(1.26)

\[
P_{e,u_j} = 1 - \sum_{u_j=0}^{1} \left[ \underbrace{\sum_{\mathbf{v} \in \text{cal}D_j} \left\{ \sum_{\mathbf{u} \backslash u_j} p_{\mathbf{u}|y(\mathbf{u}, \mathbf{v})} \cdot p_y(\mathbf{v}) \right\}}_{\text{prob of bit error}} \right] \cdot p_{u_j}.
\]

(1.27)

Thus, the minimized probability of bit error for any bit can then be computed from the given conditional channel probability distribution and input bit probabilities (uniform in ML case), albeit the calculations may be tedious. These calculations may be simplified for specific channels, as evident in Section 1.3 for the additive white Gaussian noise channel and in Section 1.4 for the binary symmetric channel and binary erasure channel.

The bit-decision process can sometimes be simplified through the use of log likelihood ratios.

\[
LLR_{u_j}(\mathbf{v}) \triangleq \log \left( \frac{p_{u_j=1}(\mathbf{v})}{p_{u_j=0}(\mathbf{v})} \right) \quad \text{(1.28)}
\]

\[
= \log \left( \frac{P_{u_j=1}(\mathbf{v})}{1 - P_{u_j=1}(\mathbf{v})} \right) \quad \text{(1.29)}
\]

\[
= \log \left( \frac{\sum_{\mathbf{u} \backslash u_j} p_{\mathbf{u} | y(\mathbf{u} | u_j=1, \mathbf{v})} \cdot p_y(\mathbf{v})}{\sum_{\mathbf{u} \backslash u_j} p_{\mathbf{u} | y(\mathbf{u} | u_j=0, \mathbf{v})} \cdot p_y(\mathbf{v})} \right). \quad \text{(1.30)}
\]

A positive value of \( LLR_{u_j} \) causes the decision \( \hat{u}_j = 1 \), while a negative value leads to decision \( \hat{u}_j = 0 \). This type of decoding avoids the use of decision regions and maybe useful in systems where many bits are simultaneously decided, and there are relationships (called codes) between the bits that allow iterative construction of all the bits’ log-likelihood ratios that converges to final converged values that lead to optimum decisions for each bit.

### 1.1.7 \( P_e \) Calculation and The Bhattacharyya Bound

The Bhattacharyya Bound (or B-Bound) for error probability finds use in systems with coding (like those of Chapters 2, 8, and beyond). The messages will be presumed to be the indices themselves so that \( m \in \{0, ..., M - 1\} \) each with corresponding symbol value \( x_m \). The B-bound bounds the probability that a specific symbol \( x_{\tilde{m}} \) is chosen instead of a message \( m \), where \( m \) is correct. This error event is denoted \( \epsilon_{m\tilde{m}} \) with probability \( P\{\epsilon_{m\tilde{m}}\} \).

\begin{center}
\textbf{Theorem 1.1.4 (Bhattacharyya Bound)}
\end{center}

The error probability, using a maximum-likelihood decoder, that corresponds to choosing message \( \tilde{m} \) in place of message \( m \) is generally bounded according to the following expression:

\[
P\{\epsilon_{m\tilde{m}}\} \leq \sum_{\mathbf{v}} \sqrt{\frac{p_y(x, x_{\tilde{m}}) \cdot p_y(x, x_m)}{p_y(x, \mathbf{v})}} \quad \text{(1.31)}
\]

**Proof:** Let \( P\{\epsilon_{m\tilde{m}}\} \) denote the probability that message \( m \) is erroneously decided by a maximum likelihood decoder to be message \( \tilde{m} \). Then, the corresponding received symbol \( y = \mathbf{v} \) must be such that \( p_y(x, x_{\tilde{m}}) \geq p_y(x, x_m) \). The region of \( \mathbf{v} \) over which this error could occur is denoted \( D_{m\tilde{m}}(\mathbf{v}) \):

\[
D_{m\tilde{m}}(\mathbf{v}) \triangleq \left\{ \mathbf{v} : \frac{p_y(x, x_{\tilde{m}})}{p_y(x, x_m)} \geq 1 \right\}. \quad \text{(1.32)}
\]
Then,

\[ P\{\varepsilon_{\tilde{m}}\} = \sum_{v \in D_{\tilde{m}}(v)} p_{y/x}(v, x_{\tilde{m}}) = \sum_{v} f(v) \cdot p_{y/x}(v, x_{\tilde{m}}) , \quad (1.33) \]

where

\[ f(v) \triangleq \begin{cases} 1 & v \in D_{\tilde{m}}(v) \\ 0 & v \notin D_{\tilde{m}}(v) \end{cases} . \quad (1.34) \]

Further,

\[ f(v) \leq \left[ \frac{p_{y/x}(v, x_{\tilde{m}})}{p_{y/x}(v, x_{m})} \right]^{1/2} , \quad (1.35) \]

and thus (1.33) becomes

\[ P\{\varepsilon_{\tilde{m}}\} \leq \sum_{v} \sqrt{p_{y/x}(v, x_{\tilde{m}}) \cdot p_{y/x}(v, x_{m})} \quad (1.36) \]

QED.

A memoryless channel has the property

\[ p_{y/x} = \prod_{n=1}^{N} p_{y_{n}/x_{n}} . \quad (1.37) \]

Memoryless channels essentially have independent dimensions. For such channels the B-Bound takes a simpler form through the use of distribution of multiplication over addition:

\[ P\{\varepsilon_{\tilde{m}}\} \leq \sum_{v} \prod_{n=1}^{N} \sqrt{p_{y_{n}/x_{n}}(y_{n}, x_{\tilde{m}, n}) \cdot p_{y_{n}/x_{n}}(y_{n}, x_{m,n})} = \prod_{n=1}^{N} \sum_{y_{n}} \sqrt{p_{y_{n}/x_{n}}(y_{n}, x_{\tilde{m}, n}) \cdot p_{y_{n}/x_{n}}(y_{n}, x_{m,n})} . \quad (1.38) \]

The sum over each dimension’s output symbol values can be much less complex than the vector summation in the general case. Specialization to the case of bit-error probability and the vector \( u \) being treated as a symbol vector itself creates a special form. In this case, it is often convenient to investigate the messages \( m \) and \( \tilde{m} \) differing in \( d_{H} \) positions. If the probability of a bit error were the same on all dimensions and set equal to \( p \), then the bound has form:

\[ P\{\varepsilon_{\tilde{m}}\} \leq \prod_{n=1}^{N} \sum_{y_{n}} \sqrt{p_{y_{n}/x_{n}}(y_{n}, \tilde{u}_{n}) \cdot p_{y_{n}/x_{n}}(y_{n}, 0)} \quad (1.39) \]

\[ = \prod_{i=1}^{d_{H}} \sum_{y_{n}} \sqrt{p_{y_{n}/x_{n}}(y_{n}, 1) \cdot p_{y_{n}/x_{n}}(y_{n}, 0)} \quad (1.40) \]

\[ = \prod_{i=1}^{d_{H}} \sum_{y_{n}} \sqrt{p_{y_{n}/x_{n}}(y_{n}, 1) \cdot p_{y_{n}/x_{n}}(y_{n}, 0)} \quad (1.41) \]

\[ = [4p(1-p)]^{d_{H}/2} . \quad (1.42) \]
1.2 Data Modulation and Demodulation for Continuous-Time Channels

Figure 1.7 generalizes Figure 1.2 to the continuous-time situation. Continuous-time channels occur in many practical situations where the channel accepts only a continuous-time, or analog, waveform, \( x(t) \), called a signal. The corresponding received signal, \( y(t) \), is also continuous time. Examples include virtually all wireless channels where electromagnetic waveforms are physically transmitted, not actually the symbols. Examples also include virtually all forms of wireline (transmission-line or waveguide, optical or metallic) connections. These continuous-time channels require that the transmit-symbol set correspond uniquely to a set of continuous-time signals, \( \{ x_i(t) \}_{i=0}^{M-1} \), sometimes also called the signal set.

As in Figure 1.7, the modulator converts the symbol vector \( x \) into the continuous-time signal that the transmitter sends into the continuous-time channel. Correspondingly, the demodulator converts continuous-time received signal \( y(t) \) into the received-symbol vector \( y \), from which the detector tries to estimate \( x \), shown as \( \hat{x} \), and thus also into the message sent through the decoder. The estimated message, one from the message associated with the message source, then are provided by the receiver to the message “sink” (that is the user/thing that receives the message. A desirable property would be that the continuous-time channel can be completely represented by a discrete-time channel of Section 1.1. Indeed a very important practical channel, Section 1.3’s Additive White Gaussian Noise (AWGN) channel, can be so converted without loss into a discrete-time equivalent channel exactly like that in Figure 1.7. Binary Phase-Shift Keying (BPSK) is perhaps one of the simplest forms of modulation:

**EXAMPLE 1.2.1 (Binary Phase-Shift Keying (BPSK))** Figure 1.8 repeats Figure 1.1 with a specific linear time-invariant channel that has the transfer function indicated. This channel essentially passes signals between 100 Hz and 200 Hz with 150 Hz having the largest gain. Binary logic familiar to most electrical engineers transmits some positive voltage level (say perhaps 1 volt) for a 1 and another voltage level (say 0 volts) for a 0 inside integrated circuits. Clearly such a constant 1/0 transmission on this “DC-blocking” channel would not pass through Figure 1.8’s channel, leaving a received signal level of nearly 0 regardless of the channel-input signal’s constant voltage level. This zero received-signal level would complicate receiver detection of the correct message. Instead the two modulated signals \( x_0(t) = +\cos(2\pi150t) \) and \( x_1(t) = -\cos(2\pi150t) \) easily pass through this channel and are
more readily distinguishable by the receiver. This latter type of transmission is an example of BPSK. If the symbol period is 1 second and if successive transmission is used, the data rate would be 1 bit per second (1 bps).\textsuperscript{6} In more detail, the engineer could recognize the trivial vector encoder that converts the message bit of 0 or 1 into the real one-dimensional vectors \( x_0 = +1 \) and \( x_1 = -1 \). The modulator simply multiples this \( x_1 \) value by the function \( \cos(2\pi t) \).

\begin{align*}
  x_0(t) &= +\cos(2\pi \cdot 150 \cdot t) \\
  x_1(t) &= -\cos(2\pi \cdot 150 \cdot t)
\end{align*}

\textbf{Figure 1.8:} Example of channel for which 1 volt and 0 volt binary transmission is inappropriate, but BPSK modulation matches well.

Section 1.1’s vector representation however is common and leads to a single modulation-independent performance analysis of the data transmission (or storage) system. This section describes such a discrete vector representation of any continuous-time signal set and the conversion between the vector symbols and the continuous-time signals. This symbol-based analysis approach was pioneered by Wozencraft and Jacobs. Each modulation method selects a set of basis functions that link a constellation \( \{x_i\} \) with the continuous signals \( \{x_i(t)\} \). The modulation basis-function choice usually depends upon the channel. This section and Section 1.3 investigate and enumerate a number of different basis functions as well as the modulation-independent constellation designs that can be used with any modulation choice.

\subsection*{1.2.1 Signal Waveform Representation by Vectors}

The reader should be familiar with the infinite-series decomposition of continuous-time signals from the basic electrical-engineering study of Fourier series. For the transmission and detection of a message during a symbol period, this text considers the set of real-valued functions \( \{f(t)\} \) such that \( \int_0^T f^2(t)dt < \infty \) (technically known as the Hilbert space of continuous-time functions and abbreviated as \( L_2[0,T] \)). This infinite dimensional vector space has an inner product that measures a distance-scaled angle between two different functions \( f(t) \) and \( g(t) \),

\[ \langle f(t), g(t) \rangle = \int_0^T f(t) \cdot g(t)dt. \]

\textsuperscript{6}However, this chapter is mainly concerned with a single transmission. Each of this example's successive transmissions could be treated independently by ignoring transients at the beginning or end of any message transmission, because these transients would likely be negligible in time extent compared to a 1 second symbol period.
Subsection more formally addresses the inner product. 

An **orthonormal basis functions** allows formalization of the modulation concept:

**Definition 1.2.1 (Orthonormal Basis Functions)** A set of $N$ functions $\{\varphi_n(t)\}$ constitute an $N$-dimensional orthonormal basis if they satisfy the following property:

$$
\int_{-\infty}^{\infty} \varphi_m(t) \varphi_n(t) dt = \delta_{mn} = \begin{cases} 
1 & m = n \\
0 & m \neq n
\end{cases}. 
$$

(1.43)

The discrete-time function $\delta_{mn}$ will be called the **discrete delta function**. Any continuous-time function (or signal) $x(t) \in L^2(O,T)$ decomposes according to some set of $N$ orthonormal basis functions $\{\varphi_i(t)\}_{n=1}^{N}$ as

$$
x(t) = \sum_{n=1}^{N} x_n \cdot \varphi_n(t)
$$

where $\varphi_n(t)$ satisfy $\langle \varphi_n(t), \varphi_m(t) \rangle = 1$ for $n = m$ and 0 otherwise, often written $\langle \varphi_n(t), \varphi_m(t) \rangle = \delta_{nm}$.

The modulated signal $x(t)$ thus relates to the symbol vector $x$ through its dimensional components:

$$
x = \begin{bmatrix} 
x_1 \\
\vdots \\
x_N
\end{bmatrix}.
$$

The number of basis functions that represent the signal set $\{x_i(t)\}$ for a particular modulation choice may be infinite, i.e. $N$ may equal $\infty$, but are the same for each possible symbol value. Each signal $x_i(t)$ maps to a set of $N$ real numbers $\{x_{in}\}$; these real-valued scalar coefficients assemble into an $N$-dimensional symbol vector.

Figure 1.9 illustrates the signal $x(t)$ graphically for $N = 3$-dimensional symbol with axes defined by the modulation basis functions $\{\varphi_n(t)\}$.

---

$^7 \delta_{mn}$ is also called a “Kronecker” delta.
Such a geometric viewpoint advantageously enables the visualization of the distance between continuous-time functions using distances between the associated symbol vectors in $\mathcal{R}^N$, the space of $N$-dimensional real vectors when $x$ is a real vector. In fact, later developments show

$$\langle x_1(t), x_2(t) \rangle = \langle x_1, x_2 \rangle,$$

(1.44)

where Equation (1.44)’s right-hand side is the usual Euclidean inner product in $\mathcal{R}^N$ (discussed later in Definition 1.2.2). This continuous-time modulation representation formally extends to random processes using what is known as a “Karhunen-Loeve expansion,” where the values $x_n$ are considered random variables, and the functions $\varphi_n(t)$ are deterministic. Thus, the message index, $i$, usually does not appear, but the vector symbol value $x$ is randomly chosen according to the message distribution from the symbol set in use. Thus, $x_n$ refers to a random message component on the $n^{th}$ modulator basis function, and not the “$n^{th}$ message” as this text proceeds to avoid notational proliferation. The basis functions also extend for all time, i.e. on the infinite time interval $(-\infty, \infty)$, in which case the inner product becomes $\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt$. The modulator’s composition of random processes is fundamental to demodulation and detection in the presence of noise. Modulation constructively assembles random signals for the communication system from a set of basis functions $\{\varphi_n(t)\}$ and a set of symbol vectors $\{x_i\}$. The chosen basis functions and symbol vectors typically satisfy physical constraints of the system and determine performance in the presence of noise.

Figure 1.10 explicitly shows construction of a modulated waveform $x(t)$, where again, each distinct symbol constellation vector point corresponds to a different modulated waveform, but all the waveforms share the same set of basis functions.

![Figure 1.10: The modulator.](image)

The power available in any physical communication system limits the average amount of energy required to transmit each successive data symbol. With inner productions, definition 1.1.2’s average energy becomes

$$E_x = E[\langle x(t), x(t) \rangle] = E[\langle x, x \rangle] = E[\|x\|^2],$$

(1.45)

or equivalently the average length of the constellation’s symbol vectors. The minimization of $E_x$ intuitively places signal-constellation points near the origin; however, the distance between points shall relate
to the probability of correctly detecting the symbols in the presence of noise. The geometric problem of optimally arranging points in a vector space with minimum average energy while maintaining at least a minimum distance between each pair of points is the well-studied sphere-packing problem, said geometric viewpoint of communication formalized first in Shannon’s 1948 seminal famous work, A Mathematical Theory of Communication (Bell Systems Technical Journal). Chapter 2 addresses directly this coding challenge through the use of symbol constellations (no matter the explicit modulation-type details).

1.2.2 Modulator Examples

Returning to Example 1.2.1, the next example illustrates the utility of the basis-function concept:

**EXAMPLE 1.2.2 (BPSK revisited)** A more general form of BPSK’s basis functions, which are parameterized by variable \( T \), is

\[
\varphi_1(t) = \sqrt{\frac{2}{T}} \cos \left[ \frac{2\pi t}{T} + \frac{\pi}{4} \right]
\]

and

\[
\varphi_2(t) = \sqrt{\frac{2}{T}} \cos \left[ \frac{2\pi t}{T} - \frac{\pi}{4} \right]
\]

for \( 0 \leq t \leq T \) and 0 elsewhere. These two basis functions \((N = 2)\), \( \varphi_1(t) \) and \( \varphi_2(t) \), are shown in Figure 1.11.

The two basis functions are orthogonal to each other and both have unit energy, thus satisfying the orthonormality condition. The two possible modulated signals transmitted during the interval \([0, T]\) also appear in Figure 1.11, where

\[
x_0(t) = \varphi_1(t) - \varphi_2(t)
\]

and

\[
x_1(t) = \varphi_2(t) - \varphi_1(t).
\]

Thus, the data symbol vectors associated with the continuous-time signals are \( x_0 = [1 - 1]' \) and \( x_1 = [-1 1]' \) (a prime denotes transpose). The signal constellation appears in Figure 1.12.
The resulting waveforms are $x_0(t) = -\frac{2}{\sqrt{T}} \sin(\frac{2\pi t}{T})$ and $x_1(t) = \frac{2}{\sqrt{T}} \sin(\frac{2\pi t}{T})$. The name “binary phase-shift keying,” because the two waveforms are shifted in phase from each other. Other basis functions (and rotated versions of the constellation) could thus also be called BPSK. Since only two possible waveforms are transmitted during each $T$ second symbol period, the data rate is $R = \log_2(2) = 1$ bit per $T$ seconds. Thus to transmit at 1 million bits per second, or abreviated 1 Mbps, $T$ must equal $10^{-6}$ seconds or 1 $\mu$s. (Additional scaling may adjust the BPSK transmit power/energy level to some desired value, and then applies uniformly to all possible constellation points and transmit signals.)

Another set of basis functions is known as “FM code” (FM is ”Frequency Modulation”) in the storage industry and also as “Manchester Encoding” in data communications. This method is used to write (modulate) in many commercial disk storage products. It is also used in a quite different area known as “Ethernet” in what is called “10Base-T Ethernet” (the lowest ethernet speed commonly used in local area networks for the internet). The basis functions are approximated in Figure 1.13 – in practice, the sharp edges are somewhat smoother depending on the specific implementation. The two basis functions again satisfy the orthonormality condition. The data rate equals one bit per $T$ seconds; for a data transfer rate into the disk of 1 GByte/s or 8 Gbps, $T = 1/(8GHz) = 125$ps; by contrast at the different data rate of 10 Gbps in “10Gbase-SR Ethernet,” $T = 100$ ps. However, both modulation methods have the same signal constellation. Thus, for the FM/Manchester example, only two signal-constellation points are used, $x_0 = [1 \ 1]'$ and $x_1 = [-1 \ 1]'$, as shown in Figure 1.12, although the basis functions differ from the previous example. The resulting modulated waveforms appear in Figure 1.13 and correspond to the write currents that are applied to the head in the storage system.(Additional scaling may be used to adjust either the FM or Ethernet transmit power/energy level to some desired value, but this simply scales all possible constellation points and transmit signals by the same constant value.)

---

8In fact, Ethernet systems use 66/64 times higher symbol rate because of some overhead carried.
The common vector space representation (i.e., signal constellation) of the “Ethernet/FM” and “BPSK” examples allows the performance of a detector to be analyzed for either system in the same way, despite the gross differences in the overall systems.

In either of the systems in Example 1.2.2, a more compact representation of the signals with only one basis function is possible. (As an exercise, the reader should conjecture what this basis function could be and what the associated signal constellation would be.) Appendix A considers the construction of a minimal set of basis functions for a given set of modulated waveforms, which is often called “Gram-Schmidt” decomposition.

Two more examples briefly illustrate vector components $x_n$ that are not necessarily binary-valued.

**EXAMPLE 1.2.3 (Short-Haul non-coherent Fiber Ethernet 802.3bm - 2B1Q)**

This transmission system over fiber-optic cable uses $M = 4$ waveforms with one basis function $N = 1$. Thus, the system transmits $b = 2$ bits of information per $T$ seconds of channel use. The basis function is roughly approximated\(^{10}\) by $\varphi_1(t) = \sqrt{\frac{1}{T}} \text{sinc}(\frac{t}{T})$, where $1/T = 53.125$ GHz, and $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. This basis function is not time limited to the interval $[0,T]$. The associated signal constellation appears in Figure 1.14. Longer-distance fiber transmission (up to 2 km) may transmit at 1/2 this symbol rate (26.5625 GHz), so at roughly 50 Gbps in other related IEEE 802.3 “Ethernet” standards. 2 bits are transmitted using one 4-level (or “quaternary”) symbol every $T$ seconds, hence the name “2B1Q.”

---

9IEEE 802.3bm is a standard that contains specifications for short-length non-coherent transmission at (roughly) 100 Gbps on each of up to 8-16 parallel channels (8 wavelengths with each having two polarizations) on up to roughly 500m of fiber. IEEE 802.3 standards also use other constellations for alternatives on longer lengths of fiber.

10Actually $1/\sqrt{T}\text{sinc}(t/T)$, or some other “Nyquist” pulse shape is used, see Chapter 3 on Intersymbol Interference.
By contrast, telephone companies once heavily transmitted the much lower data rate 1.544 Mbps “T1 Service” symmetrically on twisted pairs between the switches, or between switches and a small business (such a signal often carries twenty-four 64 kbps digital voice signals plus overhead signaling information of 8 kbps). A single very different basis function (with a much lower symbol rate) and the same constellation appear in these methods. A method, known as HDSL (High-bit-rate Digital Subscriber Lines), uses 2B1Q with $1/T = 392$ kHz, and thus transmits a data rate of 784 kbps on each of two phone lines for a total of 1.568 Mbps (1.544 Mbps plus 24 kbps of additional HDSL management overhead). The range of this system is about 2 miles of twisted pair. More recent versions of this modulation type with $M = 8, 16, or 32$ and corresponding $T$ values to get 1.544 Mbps, and higher data rates, on a single twisted pair at different lengths that may be shorter than 2 miles. This is known as “Symmetric HDSL” or just “SDSL” or ITU standard G.991. The two very different transmission systems use the same constellation and can be analyzed identically.

A second example uses two dimensions, similar to BPSK:

**EXAMPLE 1.2.4 (32 Cross quadrature amplitude modulation)**  
Consider a signal set with 32 waveforms ($M = 32$) and with 2 basis functions ($N = 2$) for transmission of 32 signals per symbol. The two BPSK-like basis functions for this “quadrature amplitude modulation” (see Section 1.3 for formal definition) are $\varphi_1(t) = \sqrt{2/T} \cos \frac{\pi t}{T}$ and $\varphi_2(t) = \sqrt{2/T} \sin \frac{\pi t}{T}$ for $0 \leq t \leq T$ and 0 elsewhere. A raw bit rate of 12.0 Kbps\(^{12}\) occurs with a symbol rate of $1/T = 2400$ Hz. The signal constellation is shown in Figure 1.15; the 32 points are arranged in a rotated cross pattern, called 32 CR or 32 cross.

\(^{11}\)The ITU has published a set of modem standards numbered V.XX (where XX is some number). These are older standards, but illustrate this interesting odd-number-7 constellation that is not a perfect square. Essentially these “voiceband” modems use the analog end-to-end plain-old-telephone-service (POTS) connection for digital transmission. It is interesting to note that these standards, now almost a half-century old can be analyzed the same as very modern standards at much higher speeds on wired connections like cablemodems.

\(^{12}\)The actual user information rate might actually 9600 bps with the extra bits used for error-correction purposes as shown in Chapters 2 and 8.

25
5 bits are transformed into 1 of 32 possible 2-dimensional symbols.

The last two examples also emphasize another tacit advantage of the vector representation, namely that the details of the rates and carrier frequencies in the basis-function modulation format are implicit in the normalization of the basis functions. Thus, these functions do not appear in the description of the signal constellation, allowing Section 1.1’s results to apply.

1.2.3 Vector-Space Interpretation of the Modulated Waveforms

This section more formally defines the inner product of two time functions and/or of two $N$-dimensional vectors:

**Definition 1.2.2 (Inner Product)** The inner product of two (real) functions of time $u(t)$ and $v(t)$ is defined by

$$\langle u(t), v(t) \rangle \triangleq \int_{-\infty}^{\infty} u(t) \cdot v(t) dt.$$  \hspace{1cm} (1.46)

The inner product of two (real) vectors $u$ and $v$ is defined by

$$\langle u, v \rangle \triangleq u^* v = \sum_{n=1}^{N} u_n \cdot v_n,$$  \hspace{1cm} (1.47)

where $*$ denotes vector transpose (and conjugate vector transpose in Chapter 2 and beyond).

The two inner products in the above definition are equal under the conditions in the following theorem:
Theorem 1.2.1 (Invariance of the Inner Product) If there exists a set of basis functions \( \varphi_n(t) \), \( n = 1, \ldots, N \) for some \( N \) such that \( u(t) = \sum_{n=1}^{N} u_n \cdot \varphi_n(t) \) and \( v(t) = \sum_{n=1}^{N} v_n \cdot \varphi_n(t) \) then

\[
\langle u(t), v(t) \rangle = \langle u, v \rangle .
\] (1.48)

where

\[
u \triangleq \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \text{ and } v \triangleq \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} .
\] (1.49)

The proof follows from

\[
\langle u(t), v(t) \rangle = \int_{-\infty}^{\infty} u(t) \cdot v(t) dt = \int_{-\infty}^{\infty} \sum_{n=1}^{N} \sum_{m=1}^{N} u_n \cdot v_m \cdot \varphi_n(t) \cdot \varphi_m(t) dt
\] (1.50)

\[
= \sum_{n=1}^{N} \sum_{m=1}^{N} u_n \cdot v_m \int_{-\infty}^{\infty} \varphi_n(t) \cdot \varphi_m(t) dt = \sum_{n=1}^{N} \sum_{m=1}^{N} u_n \cdot v_m \cdot \delta_{nm} = \sum_{n=1}^{N} u_n \cdot v_n
\] (1.51)

\[
= \langle u, v \rangle \text{ QED.}
\] (1.52)

Thus the inner product is “invariant” to the choice of basis functions and only depends on the components of the time functions along each of the basis functions. While the inner product is invariant to the choice of basis functions, the component values of the data symbols depend on basis functions. For example, for the 32CR example, one could recognize that the integral \( \frac{2}{T} \int_{0}^{T} \left[ 2 \cos \left( \frac{2\pi t}{T} \right) + \sin \left( \frac{\pi t}{T} \right) \right] \cdot \left[ \cos \left( \frac{\pi t}{T} \right) + 2 \sin \left( \frac{\pi t}{T} \right) \right] dt = 2 \cdot 1 + 1 \cdot 2 = 4 \).

Parseval’s Identity implies that the average energy of the signal constellation is invariant to the choice of basis functions, as long as they satisfy the orthonormality condition of Equation (1.43). As another 32CR example, one could recognize that the energy of the \([2,1]\) point is \( \frac{2}{T} \int_{0}^{T} \left[ 2 \cos \left( \frac{2\pi t}{T} \right) + \sin \left( \frac{2\pi t}{T} \right) \right]^2 dt = 2 \cdot 2 + 1 \cdot 1 = 5 \).

The individual basis functions themselves have a trivial vector representation; namely \( \varphi_n(t) \) is represented by \( \varphi_n = [0 \ 0 \ \ldots \ \ 1 \ \ldots \ 0]^* \), where the 1 occurs in the \( n^{th} \) position. Thus, the data symbol \( x_i \) has a representation in terms of the unit basis vectors \( \varphi_n \) that is

\[
x_i = \sum_{n=1}^{N} x_{in} \cdot \varphi_n .
\] (1.58)

The data-symbol component \( x_{in} \) can be determined as

\[
x_{in} = \langle x_i, \varphi_n \rangle ,
\] (1.59)
which, using the invariance of the inner product, becomes

\[ x_{in} = \langle x_i(t), \varphi_n(t) \rangle = \int_{-\infty}^{\infty} x_i(t) \cdot \varphi_n(t) dt \quad n = 1, ..., N. \tag{1.60} \]

Thus any set of modulated waveforms \{x_i(t)\} can be interpreted as a vector signal constellation, with the components of any particular vector \(x_i\) given by Equation (1.60). In effect, \(x_{in}\) is the projection of the \(i^{th}\) modulated waveform on the \(n^{th}\) basis function. Appendix A’s Gram-Schmidt procedure can be used to determine the minimum number of basis functions needed to represent any signal in the signal set.

### 1.2.4 Demodulation

As in Equation (1.60), the data symbol vector \(x\) can be recovered, component-by-component, by computing the inner product of \(x(t)\) with each of the \(N\) basis functions. This recovery is called **correlative demodulation** because the modulated signal, \(x(t)\), is “correlated” with each of the basis functions to determine \(x\), as Figure 1.16 illustrates.

![Figure 1.16: The correlative demodulator.](image)

The modulated signal, \(x(t)\), is first multiplied by each of the basis functions in parallel, and the outputs of the multipliers are then each passed into an integrator to produce a corresponding component of the data symbol vector \(x\). Practical realization of the multipliers and integrators may be difficult. Any physically implementable set of basis functions can only exist over the symbol period.\(^\text{13}\) Then the computation of \(x_n\) alternately becomes

\[ x_n = \int_{0}^{T} x(t) \cdot \varphi_n(t) dt. \tag{1.61} \]

The computation in (1.61) is more easily implemented by noting that it is equal to

\[ x(t) \ast \varphi_n(T - t)|_{t=T}, \tag{1.62} \]

\(^\text{13}\)This restriction to a finite time interval is later removed with the introduction of “Nyquist” Pulse shapes in Chapter 3, and the term “symbol period” will be correspondingly relaxed and expanded.
where ∗ indicates convolution. The signal’s component $x_n$ along the $n^{th}$ basis function is equivalent to the convolution (filter) of the waveform $x(t)$ with a filter $\varphi_n(T-t)$ at output sample time $T$. Such matched-filter demodulator is “matched” to the corresponding modulator basis function. Figure 1.17 illustrates matched-filter demodulation.

![Diagram of matched-filter demodulator]

Figure 1.17: The matched-filter demodulator.

Figure 1.17 illustrates a conversion between the data symbol and the corresponding modulated waveform such that the modulated waveform can be represented by a finite (or countably infinite as $N \to \infty$) set of components along an orthonormal set of basis functions. Sections 1.1 used and 1.3 will use this concept to analyze the performance of a particular modulation scheme on the AWGN channel.

### 1.2.5 MIMO Channel Basics

Multiple-Input-Multiple-Output (MIMO) channels also are vector channels. Figure 1.18 illustrates that the simplest MIMO cases may not use an adder in the modulator. Consequently, multiple signals pass through several parallel channels. In these simplest MIMO cases, the MIMO basis functions need only be normalized, and need not be necessarily orthogonal on the different parallel channels, because the infrastructure itself ensures the orthogonality (as indicated by the parallel dashed lines through the MIMO channel). There are thus $L$ “spatial” or “space-time” dimensions corresponding to the $L$ MIMO channels. At a basic mathematics level, “a dimension is a dimension” (space, time, frequency, or otherwise); however this text usually tries to use $N$ as an index that implies the dimensions arise from decomposing frequency-time at a single point in space while $L$ will usually apply to dimensions generated at different points in space and time while using the same frequency. It is possible that there are $N$ orthonormal basis functions used on each of the $L$ channels, leading to an overall dimensionality of $N_{total} = L \cdot N$ and a more complex channel/system. More complete discussion of such larger dimensionality appears in Chapters 3, 4, and beyond.
A vector basis function becomes

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_L(t) \end{bmatrix},$$

(1.63)

and a matrix of such basis functions stacked as column vectors is

$$\Phi \triangleq [\varphi_1(t) \ldots \varphi_N(t)] .$$

(1.64)

For the MIMO channel, the channel input then becomes

$$x(t) = \sum_{n=1}^{N} \varphi_n(t) \cdot x_n = \Phi \cdot x .$$

(1.65)

The $\varphi_n(t)$ could be a common set of basis functions used on all the parallel channels\(^{14}\). In the simplest MIMO case,

$$\varphi_1(t) = \begin{bmatrix} \varphi_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} ,$$

(1.66)

while similarly all $N$ such simple basis function vectors have only one non-zero component in the $n^{th}$ place. This makes each MIMO dimension a separate channel.

An example could be a system that has $L = N$ highly directional transmit antennas that each point at another set of $L = N$ highly directional receive antennas. In effect, each transmit antenna has an input component $x_n$ of a transmit vector $x_n$ with on the $n^{th}$ normalized basis function vector $\varphi_n(t)$ that passes only to the corresponding $n^{th}$ output antenna. These types of systems are used in

\(^{14}\)If different parallel channels had different functions, the total set would be the union of all sets as long as orthonormality is retained in creating the set of larger functions for modulation across all channels.
licensed wireless-band systems like "Long-Term-Evolution" (LTE) from the 3GPP standards group for 4G wireless communication and also in most of the modern Wi-Fi IEEE 802.11 standards (n, ac, ax, ad, ay). The receivers as a set have corresponding components \( y_n \), which can be aggregated into a channel-output vector \( y \). Similarly, \( L \) parallel wires could be used between common end points to increase speed. For instance, the IEEE 803.3z 1 Gbps Ethernet standard uses 4 parallel twisted pairs that each carry 250 Mbps of individual throughput (the actual data rate is 312.5 Mbps because an extra 25% is used for overhead that is not counted as user data in the quotation of 250 Mbps speed to user). These sets of \( L = 4 \) wires are often called “cat-5” cables, connecting with the familiar RJ45 connectors for ethernet (if one looks closely, 8 wires or 4 twisted pairs are in those connectors). Yet another example occurs in the above mentioned IEEE 802.3 ethernet fiber standards for 40 and 100 Gbps where 4 wavelengths on the same fiber (with no interference between them) each carry 1/4 of the overall data rate. Sometimes there is leakage between the channels, known as crosstalk, which is similar to the “intersymbol interference” addressed in Chapter 3, but crosstalk is better termed as “intra-symbol” interference. This topic is addressed in Chapters 4, 5, as well as later chapters. Important here is that the MIMO channel also fits into the vector-channel analysis that is common then to all forms of transmission in this book.

The inner product of vector functions simply generalizes to (a superscript of * denotes transpose here)

\[
\langle f(t), g(t) \rangle = \int_0^T f^*(t) \cdot g(t) \, dt ,
\] (1.67)

basically a sum of integrals instead of a single integral previously\(^{15}\). Inner products of the components on the (now) vector basis functions again equal the sum-of-integral inner products. This entire section could be reread with the basis vector functions replacing the scalar basis functions, and the modulated signal being a vector of transmitted time-domain waveforms \( x(t) \) that results in a vector of channel output waveforms \( y(t) \). For basis functions, MIMO orthogonality need not always apply across independent links (the independence assuring the effective equivalent of orthogonality, but usually the functions are normalized.

\(^{15}\)It is sometimes convenient in vector functions also to write the inner product as \( \langle f(t), g(t) \rangle = \text{trace} \left\{ \int_0^T f(t) \cdot g^*(t) \, dt \right\} \), which the astute reader may notice is the same. Indeed, any generalized norm can be used and the entire theory revisited for those familiar with vector spaces and norms/inner-products.
1.3 The Additive White Gaussian Noise (AWGN) Channel

Figure 1.19’s AWGN is perhaps the most important, and certainly the most analyzed, continuous-time communication channel.

![AWGN Channel Diagram](image)

The AWGN channel sums the modulated signal \( x(t) \) with an uncorrelated Gaussian noise \( n(t) \) to produce the received signal \( y(t) \) (at the channel output or equivalently input to the receiver). The stationary\(^{16}\) Gaussian noise is assumed to be uncorrelated with itself (or “white”) for any non-zero time offset \( \tau \), that is

\[
E[n(t)n(t-\tau)] = \frac{N_0}{2} \cdot \delta(\tau),
\]

and has zero mean, \( E[n(t)] = 0 \). For the MIMO case, white noise generalizes to identically distributed, independent AWGNs added to each output dimension\(^{17}\) Colored noise is considered in Section 1.3.7.

The assumption of white Gaussian noise is valid in the very common situation where the noise is predominantly determined by analog front-end receiver thermal noise. Such noise has a power spectral density given by the Boltzman equation:

\[
N(f) = \frac{hf}{e^{\frac{hf}{kT}} - 1} \approx kT \text{ for “small” } f < 10^{12},
\]

where Boltzman’s constant is \( k = 1.38 \times 10^{-23} \) Joules/degree Kelvin, Planck’s constant is \( h = 6.63 \times 10^{-34} \) Watt-s\(^2\), and \( T \) is the temperature on the Kelvin (absolute) scale. This power spectral density is approximately -174 dBm/Hz (10\(^{-17.4}\) mW/Hz) at room temperature (larger in practice). The Gaussian distribution assumption is a consequence of the addition of many small contributing noise sources, thus invoking the Central Limit Theorem\(^{18}\).

This section’s long AWGN development begins with Subsection 1.3.1 that shows that Section 1.2’s modulation and demodulation process and consequent discrete vector-symbol transmission channel completely represents the AWGN; that is, there is no loss with respect to optimum performance even though continuous time is replaced by a discrete set of vector-symbol values. Subsection 1.3.1 also introduces the important concept of a signal-to-noise ratio, and its maximization, which is a recurring theme both in this text book and in good transmission design and analysis in practice. Subsection 1.3.2 then progresses to develop many performance-analysis simplifications that are possible with the AWGN, particularly error probability bounds that are tight and depend only on distance between constellation symbol vectors.

---

\(^{16}\)The Gaussian noise is strict sense stationary (See Annex E for a discussion of stationarity types)

\(^{17}\)All proofs in this section then generalize easily to the case where scalar \( x, y, \) and \( \varphi \) are generalized to vectors with the more general definition of inner product at the end of Subsection 1.2.5.

\(^{18}\)The Central Limit Theorem is presumed known to the reader and basically says that the sum of many independent random variables tends towards a Gaussian distribution.
and the number of nearest neighbors. These simplifications also recur throughout this book and in practical design. This leads to Subsection 1.3.3’s discussion of fair comparison, a topic somewhat unique to this text and that reinforces a view of transmission that recognizes dimensionality in all its forms (often an area where area experts have different opinions because this fair comparison area is overlooked or misunderstood). Subsection 1.3.4 enumerates and evaluates many commonly encountered constellations and designs. Subsections 1.3.5 and 1.3.6 extend to complex channels. Complex symbol vectors replace previous sections’ real symbol vectors to simplify and extend analysis to channels where an exterior carrier is used to translate signals to and from an an appropriate frequency band. The use of complex arithmetic effectively makes the carrier superfluous to simplify analysis. The text can then proceed with complex signals, symbols, and various systems that process them with complex symbols replacing and/or generalizing real symbols. Subsection 1.3.7 then addresses bandlimits or filtered AWGN’s and the closely related concept of “colored” (not white) additive Gaussian noise.

1.3.1 Continuous-Time AWGN Conversion to a Vector AWGN Channel

In the absence of additive noise in Figure 1.19, \( y(t) = x(t) \), and the demodulation process in Subsection 1.2.3 would exactly recover the transmitted signal. This section shows that for the AWGN Channel, this demodulation process provides sufficient information to determine optimally the transmitted signal. The demodulator’s components \( y_l = \langle y(t), \varphi_l(t) \rangle \), \( l = 1, ..., N \) comprise a vector channel output, \( y = [y_1, ..., y_N]' \) that is equivalent for detection purposes to \( y(t) \). The analysis can thus convert the continuous channel \( y(t) = x(t) + n(t) \) to a discrete vector channel model,

\[
y = x + n \tag{1.70}
\]

where \( n \triangleq [n_1 \ n_2 \ ... \ n_N] \) and \( n_l \triangleq \langle n(t), \varphi_l(t) \rangle \). The received symbol vector at the demodulator output is the sum of the vector equivalent of the modulated signal and a vector of the demodulated noise. However, the exact noise sample function may not be reconstructed from \( n \),

\[
n(t) \neq \sum_{l=1}^{N} n_l \cdot \varphi_l(t) \triangleq \hat{n}(t) \quad , \tag{1.71}
\]

or equivalently,

\[
y(t) \neq \sum_{l=1}^{N} y_l \cdot \varphi_l(t) \triangleq \hat{y}(t) \quad . \tag{1.72}
\]

There may exist a component of \( n(t) \) that is orthogonal to the space spanned by the basis functions \( \{\varphi_1(t) ... \varphi_N(t)\} \). This unrepresented noise component is

\[
\tilde{n}(t) \triangleq n(t) - \hat{n}(t) = y(t) - \hat{y}(t) \quad , \tag{1.73}
\]

from which a lemma quickly follows:

**Lemma 1.3.1 (Uncorrelated noise samples)** The AWGN Noise samples in the demodulated noise vector are independent and of equal variance \( \frac{\sigma_0^2}{2} \).

**Proof:** Write

\[
E[n_k n_l] = E \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(t) \cdot n(s) \cdot \varphi_k(t) \cdot \varphi_l(s) dt \, ds \right] \tag{1.74}
\]

\[
= \frac{\sigma_0^2}{2} \int_{-\infty}^{\infty} \varphi_k(t) \cdot \varphi_l(t) dt \tag{1.75}
\]

\[
= \frac{\sigma_0^2}{2} \cdot \delta_{kl} \quad \text{QED.} \tag{1.76}
\]
Section 1.1’s MAP-detector development could have replaced \(y\) by \(y(t)\) everywhere and the development would have proceeded identically with the tacit inclusion of the time variable \(t\) in the probability densities (and also assuming stationarity of \(y(t)\) as a random process). The Theorem of Irrelevance would hold with \([y_1 \ y_2]\) replaced by \([\hat{y}(t) \ \hat{n}(s)]\), as long as the relation (1.21) holds for any pair of time instants \(t\) and \(s\). In a non-mathematical sense, the unrepresented noise is useless to the receiver, so there is nothing of value lost in the vector demodulator, even though some of the channel output noise is not represented. The following algebra demonstrates that \(\hat{n}(s)\) is irrelevant:

First,

\[
E[\hat{n}(s) \cdot \hat{n}(t)] = E\left[\hat{n}(s) \cdot \sum_{l=1}^{N} n_l \cdot \varphi_l(t)\right] = \sum_{l=1}^{N} \varphi_l(t) \cdot E[\hat{n}(s) \cdot n_l].
\]

and,

\[
E[\hat{n}(s) \cdot n_l] = E[(n(s) - \hat{n}(s)) \cdot n_l] = E\left[\int_{-\infty}^{\infty} n(s) \cdot \varphi_l(\tau) \cdot n(\tau)d\tau\right] - E\left[\sum_{k=1}^{N} n_k \cdot n_l \cdot \varphi_k(s)\right]
\]

\[
= \frac{N_0}{2} \int_{-\infty}^{\infty} \delta(s - \tau) \cdot \varphi_l(\tau)d\tau - \frac{N_0}{2} \cdot \varphi_l(s)
\]

\[
= \frac{N_0}{2} \cdot [\varphi_l(s) - \varphi_l(s)] = 0.
\]

Second,

\[
P_{x|\hat{y}(t),\hat{n}(s)} = \frac{P_{x,\hat{y}(t),\hat{n}(s)}}{P_{\hat{y}(t),\hat{n}(s)}} \quad (1.82)
\]

\[
= \frac{P_{x,\hat{y}(t)} \cdot P_{\hat{n}(s)}}{P_{\hat{y}(t)} \cdot P_{\hat{n}(s)}} \quad (1.83)
\]

\[
= \frac{P_{x,\hat{y}(t)}}{P_{\hat{y}(t)}} \quad (1.84)
\]

\[
= P_{x|\hat{y}(t)}. \quad (1.85)
\]

Equation (1.85) satisfies the theorem of irrelevance, and thus the receiver need only base its decision on \(\hat{y}(t)\), or equivalently, only on the received vector \(y\). The vector AWGN Channel is equivalent to the continuous-time AWGN channel.

**Rule 1.3.1 (The Vector AWGN Channel)** The vector AWGN channel is given by

\[
y = x + n
\]

and is equivalent to the channel illustrated in Figure 1.19. The noise vector \(n\) is an \(N\)-dimensional Gaussian random vector with zero mean, equal-variance, uncorrelated components in each dimension. The noise distribution is

\[
p_n(u) = (\pi N_0)^{-\frac{N}{2}} \cdot e^{-\frac{1}{N_0}\|u\|^2} = \left(2\pi \sigma^2\right)^{-\frac{N}{2}} \cdot e^{-\frac{1}{2\sigma^2}\|u\|^2}.
\]

Application of \(y(t)\) to either the correlative demodulator of Figure 1.16 or to the matched-filter demodulator of Figure 1.17, generates the desired vector channel output \(y\) at the demodulator output. The following section specifies the decision process that produces an estimate of the input message, given the output \(y\), for the AWGN Channel.
1.3.1.1 Optimum Detection with the AWGN Channel

For the vector AWGN Channel in (1.86),

$$p_{y|x}(v|i) = p_n(v - x_i), \quad (1.88)$$

where $p_n$ is the vector noise distribution in (1.87). Thus for the AWGN Channel, the MAP Decision Rule becomes

$$\hat{m} \Rightarrow m_i \text{ if } e^{-\frac{1}{N_0} \|v - x_i\|^2} \cdot p_x(i) \geq e^{-\frac{1}{N_0} \|v - x_j\|^2} \cdot p_x(j) \quad \forall \, j \neq i, \quad (1.89)$$

where the common factor of $(\pi N_0)^{-\frac{N}{2}}$ has been canceled from each side of (1.89). As noted earlier, if equality holds in (1.89), then the decision can be assigned to any of the corresponding messages without change in minimized probability of error. The log of (1.89) is the preferred form of the MAP Decision Rule for the AWGN channel:

**Rule 1.3.2 (AWGN MAP Detection Rule)**

$$\hat{m} \Rightarrow m_i \text{ if } \|v - x_i\|^2 - N_0 \cdot \ln\{p_x(i)\} \leq \|v - x_j\|^2 - N_0 \cdot \ln\{p_x(j)\} \quad \forall \, j \neq i \quad (1.90)$$

If the channel input messages are equally likely, the natural-log terms on both sides of (1.90) cancel, yielding the AWGN ML Detection Rule:

**Rule 1.3.3 (AWGN ML Detection Rule)**

$$\hat{m} \Rightarrow m_i \text{ if } \|v - x_i\|^2 \leq \|v - x_j\|^2 \quad \forall \, j \neq i \quad (1.91)$$

The ML detector for the AWGN channel in (1.91) has the intuitively appealing physical interpretation that the decision $\hat{m} = m_i$ corresponds to choosing the data symbol $x_i$ that is closest, in terms of the Euclidean distance, to the vector received symbol $y = v$. Without noise, the received vector is $y = x_i$, the transmitted symbol, but the additive Gaussian noise results in a received symbol most likely in the neighborhood of $x_i$. The noise’s Gaussian distribution implies the probability of a received point decreases as the distance from the transmitted point increases.

As an example consider the decision regions for binary data transmission over the AWGN Channel illustrated in Figure 1.20. The ML Receiver decides $x_1$ if $y = v \geq 0$ and $x_0$ if $y = v < 0$. (One might have guessed this answer without need for theory.) With $d$ defined as the distance $\|x_1 - x_0\|$, the decision regions are offset in the MAP detector by $\frac{\sigma^2}{\pi} \cdot \ln\{\frac{p_x(j)}{p_x(i)}\}$ with the decision boundary shifting.
towards the data symbol of lesser probability, as illustrated in Figure 1.21. Unlike the ML detector, the MAP detector accounts for the à priori message probabilities.

![Binary MAP detector](image)

**Figure 1.21:** Binary MAP detector.

Figure 1.22 illustrates the decision region for a two-dimensional example of the QPSK signal set, which uses the same basis functions as the V.32 example (Example 1.2.4), but with $M = 4$ and constellation as shown in Figure 1.22. The points in the signal constellation are all assumed to be equally likely.

![QPSK decision regions](image)

**Figure 1.22:** QPSK decision regions.

### 1.3.1.2 General Receiver Implementation

While the decision regions in the above examples appear simple to describe, an implementation may be more complex. This section investigates general receiver structures and the detector implementation. The MAP detector minimizes the quantity (the quantity $y$ now replaces $v$ avertng strict mathematical notation, because probability density functions are used less often in the subsequent analysis):

$$\|y - x_i\|^2 - N_0 \cdot \ln\{p_x(i)\} \quad (1.92)$$

over the $M$ possible messages, indexed by $i$. The quantity in (1.92) expands to

$$\|y\|^2 - 2\langle y, x_i \rangle + \|x_i\|^2 - N_0 \ln\{p_x(i)\} \quad . \quad (1.93)$$

Minimization of (1.93) can ignore the $\|y\|^2$ term. The MAP decision rule then becomes

$$\hat{m} \Rightarrow m_i \; if \; \langle y, x_i \rangle + c_i \geq \langle y, x_j \rangle + c_j \; \forall j \neq i \quad , \quad (1.94)$$
where \( c_i \) is the constant (independent of \( y \))

\[
c_i = N_0 \frac{\ln \{ p_x(i) \} - \| x_i \|^2}{2}.
\] (1.95)

A system design can precompute the constants \( \{ c_i \} \) from the transmitted symbols \( \{ x_i \} \) and their probabilities \( p_x(i) \). The detector thus only needs to implement the \( M \) inner products, \( \langle y, x_i \rangle = 0, \ldots, M-1 \). When all the data symbols have the same energy (\( \mathbb{E}_x = \| x_i \|^2 \forall i \)) and are equally probable (i.e. MAP = ML), then the constant \( c_i \) is independent of \( i \) and can be eliminated from (1.94). The ML detector thus chooses the \( x_i \) that maximizes the inner product (or correlation) of the received value for \( y = v \) with \( x_i \) over \( i \).

There exist two common implementations of the MAP receiver in Equation (1.94). The first, shown in Figure 1.23, called a “basis detector,” computes \( y \) using a matched filter demodulator. This MAP receiver computes the \( M \) inner products of (1.94) digitally (an \( M \times N \) matrix multiply with \( y \)), adds the constant \( c_i \) of (1.95), and picks the index \( i \) with maximum result. Finally, a decoder translates the index \( i \) into the desired message \( m_i \). Often in practice, the signal constellation is such (see Section 1.3.6 for examples) that the max-and-decode functionality reduces to simple truncation of each received symbol-vector component.

![Figure 1.23: Basis detector (\( y(t) \) would be just \( L_x = N \) parallel signals \( y(t) \) in the MIMO case).](image)

The second demodulator form eliminates Figure 1.23’s matrix multiply by exploiting directly the inner product equivalences between the discrete vectors \( x_i, y \) and the continuous-time functions \( x_i(t) \) and \( y(t) \). That is

\[
\langle y, x_i \rangle = \int_0^T y(t) x_i(t) dt = \langle y(t), x_i(t) \rangle.
\] (1.96)

Equivalently,

\[
\langle y, x_i \rangle = y(t) * x_i(T - t) |_{t=T}
\] (1.97)

where \( * \) indicates convolution. This type of detector is called a “signal detector” and appears in Figure 1.24.
EXAMPLE 1.3.1 (pattern recognition as a signal detector) Pattern recognition is a
digital signal processing procedure that is used to detect whether a certain signal is present.
A specific pattern-recognition application occurs when an aircraft/drone converts pictures of
the ground into electrical signals, and these signals then permit analysis to determine the
presence of certain objects in the pictures. The Gaussian noise would represent the various
imperfections in the camera’s accuracy and any conversion to electrical signals. This is a
communication channel in disguise where the two inputs are the usual terrain of the ground
and the terrain of the ground including the object to be detected. A signal detector consisting
of two filters that are essentially the time reverse of each of these possible input signals, with
a comparison of the outputs (after adding any necessary constants), allows detection of the
presence of the object or pattern. There are many other examples of pattern recognition in
voice/command recognition or authentication, facial recognition, written character scanning,
and so on.

The above example/discussion illustrates that many digital-transmission-theory principles are common
to other fields of digital signal processing and computer science.

1.3.1.3 Signal-to-Noise Ratio (SNR) Maximization with a Matched Filter

SNR measures system-performance as the ratio of signal power (message) to unwanted noise power. A
discrete (continuous) channel’s output SNR is defined as the ratio of the received signal’s energy (power)
to the mean-square noise value (power). The AWGN’s SNR will be the same for both continuous-
and discrete-time. Figure 1.25’s matched filter satisfies the SNR maximization property, which the
following theorem summarizes:
Theorem 1.3.1 (SNR Maximization) For the system shown in Figure 1.25, the filter \( h(t) \) that maximizes the signal-to-noise ratio at sample time \( T_s \) is given by the matched filter \( h(t) = x(T_s - t) \).

Proof: Compute the SNR at sample time \( t = T_s \) as follows.

\[
\text{Signal Energy} = \left[ x(t) \cdot h(T_s - t) \right]^2 \quad (1.98)
\]

\[
\text{Noise Energy} = E \left[ \int_{-\infty}^{\infty} n(t) \cdot h(T_s - t) dt \int_{-\infty}^{\infty} n(s) \cdot h(T_s - s) ds \right] \quad (1.100)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{N_0}{2} \cdot \delta(t - s) \cdot h(T_s - t) \cdot h(T_s - s) dtds \quad (1.101)
\]

\[
= \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(T_s - t) dt \quad (1.102)
\]

\[
= \frac{N_0}{2} \|h\|^2 \quad (1.104)
\]

The signal-to-noise ratio, defined as the ratio of the signal power in (1.99) to the noise power in (1.104), equals

\[
\text{SNR} = \frac{2}{N_0} \cdot \frac{\langle x(t), h(T_s - t) \rangle^2}{\|h\|^2} \quad (1.105)
\]

The “Cauchy-Schwarz Inequality” states that

\[
\langle x(t), h(T_s - t) \rangle^2 \leq \|x\|^2 \|h\|^2 \quad (1.106)
\]

with equality if and only if \( x(t) = k \cdot h(T_s - t) \), where \( k \) is some arbitrary constant. Thus, by inspection, (1.105) is maximized over all choices for \( h(t) \) when \( h(t) = x(T_s - t) \). The filter \( h(t) \) is “matched” to \( x(t) \), and the corresponding maximum SNR (for any \( k \)) is

\[
\text{SNR}_{\text{max}} = \frac{2}{N_0} \cdot \|x\|^2 \quad (1.107)
\]

An example use of the matched-filter’s SNR-maximization property occurs in time-delay estimation, which is used for instance in radar and lidar:
EXAMPLE 1.3.2 (Time-delay estimation) Radar and lidar systems emit electromagnetic pulses and measure those pulse’s reflection from objects within radar range. The reflection’s delay determines the object’s distance, with longer delay corresponding to longer distance. By processing the radar’s received signal with a filter matched to the radar pulse shape, the signal level measured in the presence of a presumably fixed background AWGN will appear largest relative to the noise at twice the round-trip delay (so $2T_s$) when the filter is matched to the pulse. Any other filter would reduce the accuracy of the estimate. Thus, the ability to determine the exact time instant at which the maximum pulse returned is improved by the use of the matched filter, allowing more accurate estimation of the object’s position.

1.3.2 Error Probability for the AWGN Channel

This section discusses average error-probability computation when the optimum detector incorrectly detects the transmitted message on an AWGN channel. From the previous section, the AWGN channel is equivalent to a vector channel with output given by

$$y = x + n.$$  \hspace{1cm} (1.108)

The computation of $P_e$ often assumes that the inputs $x_i$ are equally likely, or $p_{x_i}(i) = \frac{1}{M}$. Under this assumption, the optimum detector is the ML detector, which has decision rule

$$\hat{m} \Rightarrow m_i \text{ if } \|v - x_i\|^2 \leq \|v - x_j\|^2 \forall j \neq i.$$  \hspace{1cm} (1.109)

The $P_e$ associated with this rule depends on the signal constellation $\{x_i\}$ and the noise variance $\frac{N_0}{2}$. Two general invariance theorems in Subsection 1.3.2.1 facilitate the computation of $P_e$. The exact $P_e$,

$$P_e = \frac{1}{M} \cdot \sum_{i=0}^{M-1} P_{c/i}$$  \hspace{1cm} (1.110)

may be difficult to compute, so convenient and accurate bounding procedures in Subsections 1.3.2.2 through 1.3.2.4 can alternately approximate $P_e$.

1.3.2.1 AWGN Invariance to Rotation and Translation

The symbol constellation’s orientation with respect to the coordinate axes and to the origin does not affect the $P_e$ of the ML detector on the AWGN. This result follows because (1) the error depends only on relative distances between points in the symbol constellation, and (2) AWGN is spherically symmetric in all directions. First, the ML receiver’s error probability is invariant to any rotation of the signal constellation, as summarized in the following theorem:

**Theorem 1.3.2 (Rotational Invariance)** If all the data symbols in a symbol constellation are rotated by an orthogonal transformation, that is $\bar{x}_i \leftarrow Qx_i$ for all $i = 0, ..., M - 1$ (where $Q$ is an $N \times N$ matrix such that $QQ^T = Q^TQ = I$), then the ML receiver’s error probability remains unchanged on an AWGN channel.

**Proof:** First, an AWGN remains statistically equivalent after rotation by $Q'$: A rotated Gaussian random vector is $\bar{n} = Q'n$. ($\bar{n}$ is Gaussian since a linear combination of Gaussian random variables remains a Gaussian random variable). A Gaussian random vector is completely specified by its mean and covariance matrix: The mean is $E[\bar{n}] = 0$ since $E[n_i] = 0, \forall i = 0, \ldots, N - 1$. The covariance matrix is $E[\bar{n}\bar{n}^T] = Q'E[nn^T]Q = \frac{N_0}{2}I$. Thus,
\* is statistically equivalent to \( n \). The channel output for the rotated signal constellation is now \( \hat{y} = \hat{x} + n \) as illustrated in Figure 1.26. The corresponding decision rule is based on the distance from the received symbol vector \( \hat{y} \) to the rotated constellation symbol vector(s) \( \hat{x} \).

\[
\| \hat{y} - \hat{x} \|^2 = \hat{y} - \hat{x}
\]

(1.112)

\[
= \begin{pmatrix} v - x_i \end{pmatrix}^T Q Q^T \begin{pmatrix} y - x_i \end{pmatrix}
\]

(1.113)

\[
= \| y - x_i \|^2,
\]

(1.114)

where \( y = x + Q' n \). Since \( \tilde{n} = Q' n \) has the same distribution as \( n \), and the distances measured in (1.114) are the same as in the original unrotated symbol constellation, the ML detector for the rotated constellation is the same as the ML detector for the original (unrotated) constellation in terms of all distances and noise variances. Thus, the error probability must be identical. QED.

An example of the QPSK constellation appears in Figure 1.22, where \( N = 2 \). With \( Q \) as a 45° rotation matrix,

\[
Q = \begin{bmatrix}
\cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\
-\sin \frac{\pi}{4} & \cos \frac{\pi}{4}
\end{bmatrix},
\]

(1.115)

then Figure 1.27 shows the rotated constellation and decision regions. From Figure 1.27, clearly the rotation has not changed the detection problem and has only changed the labeling of the axes, effectively giving another equivalent set of orthonormal basis functions. Since rotation does not change the squared length of any data symbols, the average energy remains unchanged. The invariance does depend on the noise components being uncorrelated with one another, and being of equal variance, as in (1.76); for other noise correlations (i.e., \( n(t) \) not white, see Section 1.3.7), rotational invariance does not hold. Figure 1.28 summarizes rotational invariance. All three constellations in figures 1.27 and 1.28 have identical \( P_e \) when used with identical AWGN.

Error probability is also invariant to translation by a constant vector amount for the AWGN, because again \( P_e \) depends only on relative distances and the noise remains unchanged.

**Theorem 1.3.3 (Translational Invariance)** If all the data symbols in a signal constellation are translated by a constant vector amount, that is \( \tilde{x}_i \leftarrow x_i - a \) for all \( i = 0, ..., M-1 \), then the ML error probability remains unchanged on an AWGN channel.
Figure 1.27: QPSK rotated by 45°.

Figure 1.28: Rotational invariance summary.
An important use of the Theorem of Translational Invariance is the **minimum energy translate** of a constellation:

**Definition 1.3.1 (Minimum Energy Translate)** The minimum energy translate of a constellation is defined as that constellation obtained by subtracting the constant vector \( E\{x\} \) from each data symbol in the constellation.

To show that the minimum energy translate has the minimum energy among all possible translations of the signal constellation, the average energy of the translated signal constellation is written as

\[
E_{x-a} = \sum_{i=0}^{M-1} \|x_i - a\|^2 \cdot p_x(i)
= \sum_{i=0}^{M-1} \left[ \|x_i\|^2 - 2\langle x_i, a \rangle + \|a\|^2 \right] \cdot p_x(i)
= E_x + \|a\|^2 - 2\langle E\{x\}, a \rangle
\]  

From (1.117), the energy \( E_{x-a} \) is minimized over all possible translates \( a \) if and only if \( a = E\{x\} \), so

\[
\min E_{x-a} = \sum_{i=0}^{M-1} \left[ \|x_i - E\{x\}\|^2 \cdot p_x(i) \right] = E_x - \|E(x)\|^2 .
\]  

Thus, as transmitter energy (or power) is often a quantity to be preserved, the engineer can always translate the signal constellation by \( E\{x\} \), to minimize the required energy without affecting performance. (However, there may be practical reasons, such as complexity and synchronization, where some designs avoid this translation.)

### 1.3.2.2 Union Bounding

Specific examples of calculating \( P_e \) appear in the next two subsections. This subsection illustrates this calculation for binary \((M = 2)\) symbols in \( N \) dimensions in upper-bounding error-probability.

Suppose a system has two signals in \( N \) dimensions, as illustrated for \( N = 1 \) dimension in Figure 1.20 with an AWGN channel. Then ML-detector error probability is the probability that the noise vector \( n \)'s component on the line connecting the two data symbols is greater than half the distance along this line. In this case, the noisy received vector \( y \) lies in the incorrect decision region, resulting in an error. Since the noise is white Gaussian, its projection in any dimension, in particular, the segment of the line connecting the two data symbols, is of variance \( \sigma^2 = \frac{N_0}{2} \), as was discussed in the proof of Theorem 1.3.2.

Thus,

\[
P_e = P\{\langle n, \varphi \rangle \geq \frac{d}{2} \} ,
\]

where \( \varphi \) is a unit norm vector along the line between \( x_0 \) and \( x_1 \) and \( d \triangleq \|x_0 - x_1\| \). This error probability is

\[
P_e = \int_{\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{u^2}{2\sigma^2}} du
= \int_{\frac{d}{2\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du
= Q\left[ \frac{d}{2\sigma} \right] .
\]
The Q-function is defined in Appendix B of this chapter, but basically computes the indefinite integral in Equation (1.120). As \( \sigma^2 = \frac{N_0}{2} \), (1.121) can also be written

\[
P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right).
\]

(1.122)

1.3.2.2.1 Minimum Distance  Every signal constellation has an important characteristic known as the minimum distance:

**Definition 1.3.2 (Minimum Distance, \( d_{\min} \))** The minimum distance, \( d_{\min}(x) \) measures the smallest separation between any two data symbols in a constellation \( x \triangleq \{x_i\}_{i=0}^{M-1} \). The argument \( (x) \) is often dropped when the specific signal constellation is obvious from the context, thus leaving

\[
d_{\min} \triangleq \min_{i \neq j} \|x_i - x_j\| \ \forall \ i,j.
\]

(1.123)

Equation (1.121) is useful in the following theorem’s proof of ML-detector error-probability bound for any constellation with \( M \) data symbols (on the AWGN Channel):

**Theorem 1.3.4 (Union Bound)** An error probability bound for the ML detector on the AWGN Channel, with an \( M \)-point constellation with minimum distance \( d_{\min} \), is

\[
P_e \leq (M - 1) \cdot Q\left(\frac{d_{\min}}{2\sigma}\right).
\]

(1.124)

**Proof:** The Union Bound defines an “error event” \( \varepsilon_{ij} \) as the event where the ML detector chooses \( \hat{x} = x_j \) while \( x_i \) is the correct transmitted data symbol. The conditional error probability given that \( x_i \) was transmitted is then

\[
P_{e/i} = P\{\varepsilon_{i0} \cup \varepsilon_{i1} \cup \ldots \cup \varepsilon_{i,i-1} \cup \varepsilon_{i,i+1} \cup \ldots \cup \varepsilon_{i,M-1}\} = P\bigcup_{j \neq i}^{M-1} \varepsilon_{ij}. \]

(1.125)

Because the error events in (1.125) are mutually exclusive (meaning if one occurs, the others cannot), the probability of the union is the sum of the probabilities, and also bounded by the sum of the noise-component error events (which are not necessarily mutually exclusive because the noise might be so large as to have components in multiple directions to be larger than half the distance)

\[
P_{e/i} = \sum_{j=0}^{M-1} P\{\varepsilon_{ij}\} \leq \sum_{j=0}^{M-1} P_2(x_i, x_j),
\]

(1.126)

where

\[
P_2(x_i, x_j) \triangleq P\{y \text{ is closer to } x_j \text{ than to } x_i\},
\]

(1.127)

because

\[
P\{\varepsilon_{ij}\} \leq P_2(x_i, x_j).
\]

(1.128)

As illustrated in Figure 1.29, \( P\{\varepsilon_{ij}\} \) is the probability the received vector \( y \) lies in the shaded decision region for \( x_j \) given the symbol \( x_i \) was transmitted.
The incorrect decision region for the probability $P_2(x_i, x_j)$ includes part (shaded red in Figure 1.29) of the region for $P(e_{ik})$, which illustrates the inequality in Equation (1.128). Thus, the union bound overestimates $P_{e/i}$ by summing the results of integrating pairwise on possibly overlapping half-planes. Using the result in (1.121),

$$P_2(x_i, x_j) = Q\left[\frac{\|x_i - x_j\|}{2\sigma}\right].$$  \hspace{1cm} (1.129)

Substitution of (1.129) into (1.126) results in

$$P_{e/i} \leq \sum_{j=0}^{M-1} Q\left[\frac{\|x_i - x_j\|}{2\sigma}\right],$$  \hspace{1cm} (1.130)

and thus averaging over all transmitted symbols

$$P_e \leq \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} Q\left[\frac{\|x_i - x_j\|}{2\sigma}\right] p_x(i).$$  \hspace{1cm} (1.131)

$Q(x)$ is monotonically decreasing in $x$, and thus since $d_{\min} \leq \|x_i - x_j\|$, 

$$Q\left[\frac{\|x_i - x_j\|}{2\sigma}\right] \leq Q\left[\frac{d_{\min}}{2\sigma}\right].$$  \hspace{1cm} (1.132)

Substitution of (1.132) into (1.131), and recognizing that $d_{\min}$ is not a function of the indices $i$ or $j$, one finds the desired result

$$P_e \leq \sum_{i=0}^{M-1} (M - 1)Q\left[\frac{d_{\min}}{2\sigma}\right] p_x(i) = (M - 1)Q\left[\frac{d_{\min}}{2\sigma}\right].$$  \hspace{1cm} (1.133)

QED.

Since the constellation contains $M$ points, the factor $M - 1$ equals the maximum number of neighboring constellation points that can be at distance $d_{\min}$ from any particular constellation point.
1.3.2.2 Examples  The union bound can be tight (or exact) in some cases, but it is not always a good approximation to the actual $P_e$, especially when $M$ is large. Two examples for $M = 8$ show situations where the union bound is a poor approximation to the actual probability of error. These two examples also naturally lead to the “nearest neighbor” bound of the next subsection.

Figure 1.30: 8 Phase Shift Keying (8PSK).

EXAMPLE 1.3.3 (8PSK)  The constellation in Figure 1.30 is often called “eight phase” or “8PSK”. For the maximum likelihood detector, the 8 decision regions correspond to sectors bounded by straight lines emanating from the origin that bisect the circle’s arcs between constellation points. The union bound for 8PSK is

$$P_e \leq 7Q\left[ \frac{\sqrt{E_x \sin(\pi/8)}}{\sigma} \right],$$

(1.134)

and $d_{\text{min}} = 2\sqrt{E_x \sin(\pi/8)}$.

Figure 1.31 magnifies the detection region for one of the 8 data symbols.

Figure 1.31: 8PSK $P_e$ bounding.

By symmetry the analysis would proceed identically, no matter which point is chosen, so $P_{e/i} = P_e$. An error can occur if the AWGN component along either of the two directions shown is greater than $d_{\text{min}}/2$. These two events are not mutually exclusive, although the variance of the noise along either vector (with unit vectors along each defined as $\varphi_1$ and $\varphi_2$) is $\sigma^2$. Thus,

$$P_e = P\{\| < n, \varphi_1 > \| > \frac{d_{\text{min}}}{2}\} \cup \{\| < n, \varphi_2 > \| > \frac{d_{\text{min}}}{2}\}$$

(1.135)

$$\leq P\{n_1 > \frac{d_{\text{min}}}{2}\} + P\{n_2 > \frac{d_{\text{min}}}{2}\}$$

(1.136)
which is a tighter “union bound” on the probability of error. Also

\[ P\{|\|n_1\| > \frac{d_{\min}}{2}\} \leq P_e, \]  \hspace{1cm} (1.138)

yielding a lower bound on \( P_e \), thus the upper bound in (1.137) is tight. The bound in (1.137) overestimates the \( P_e \) by integrating the two half planes, which overlap as Figure 1.30 depicts. The lower bound of (1.138) only integrates over one half plane that does not completely cover the shaded region. The multiplier in front of the Q function in (1.137) equals the number of “nearest neighbors” for any one data symbol in the 8PSK constellation.

The following second example illustrates problems in applying the union bound to a 2-dimensional signal constellation with 8 or more signal points on a rectangular grid (or lattice):

**EXAMPLE 1.3.4 (8AMPM)** Figure 1.32 illustrates an 8-point signal constellation called “8AMPM” (amplitude-modulated phase modulation), or “8 Square”. The union bound for \( P_e \) yields

\[ P_e \leq 7 \cdot Q\left[ \frac{\sqrt{2}}{\sigma} \right]. \]  \hspace{1cm} (1.139)

By rotational invariance, Figure 1.33’s rotated 8AMPM constellation has the same \( P_e \) as Figure 1.32’s unrotated constellation. The decision boundaries shown are pessimistic at the corners of the constellation, so the \( P_e \) derived from them will be an upper bound. For notational brevity, let \( Q \triangleq Q[d_{\min}/2\sigma] \). The probability of a correct decision for 8AMPM is

\[ P_c = \sum_{i=0}^{7} P_{c/i} \cdot p_x(i) = \sum_{i\neq 1,4} P_{c/i} \cdot \frac{1}{8} + \sum_{i=1,4} P_{c/i} \cdot \frac{1}{8} \]  \hspace{1cm} (1.140)

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Thus $P_e$ is upper bounded by

$$P_e = 1 - P_c < 3.25Q \left[ \frac{d_{\min}}{2\sigma} \right], \quad (1.144)$$

which is tighter than the union bound in (1.139). As $M$ increases for constellations like 8AMPM, the accuracy of the union bound degrades, since the union bound calculates $P_e$ by pairwise error events and thus redundantly includes the probabilities of overlapping half-planes. It is desirable to produce a tighter bound. The multiplier on the Q-function in (1.144) is the average number of nearest neighbors (or decision boundaries) $= \frac{1}{4}(4+3+3+3) = 3.25$ for the constellation. This rule of thumb, the Nearest-Neighbor Union bound (NNUB), often used by practicing data transmission engineers, is formalized in the next subsection.

### 1.3.2.3 The Nearest Neighbor Union Bound

The **Nearest Neighbor Union Bound** (NNUB) provides a tighter bound on a constellation’s associated error probability by lowering the multiplier of the Q-function. The factor $(M - 1)$ in the original union bound is often too large for accurate performance prediction as in the preceding section’s two examples. The NNUB requires more computation. However, it is easily approximated.

The NNUB’s development uses the average number of nearest neighbors:

**Definition 1.3.3 (Average Number of Nearest Neighbors)** The average number of nearest neighbors, $N_c$, for a signal constellation is defined as

$$N_c = \sum_{i=0}^{M-1} N_i \cdot p_X(i), \quad (1.145)$$
where $N_i$ is the number of neighboring constellation points of the point $x_i$, that is the number of other signal constellation points sharing a common decision region boundary with $x_i$. Often, $N_e$ is approximated by

$$N_e \approx \sum_{i=0}^{M-1} \hat{n}_i \cdot p_x(i), \quad (1.146)$$

where $\hat{n}_i$ is the set of points at minimum distance from $x_i$, whence the often used name “nearest” neighbors. This approximation is often very tight and facilitates computation of $N_e$ when signal constellations are complicated (i.e., coding is used - see Chapters 2, 8, and beyond).

Thus, $N_e$ also measures the average number of sides of the decision regions surrounding any point in the constellation. These decision boundaries can be at different distances from any given point and thus might best not be called “nearest.” $N_e$ is used in the following theorem:

**Theorem 1.3.5 (Nearest Neighbor Union Bound)** The probability of error for the ML detector on the AWGN channel, with an $M$-point signal constellation with minimum distance $d_{\text{min}}$, is bounded by

$$P_e \leq N_e \cdot Q \left[ \frac{d_{\text{min}}}{2\sigma} \right]. \quad (1.147)$$

In the case that $N_e$ is approximated by counting only “nearest” neighbors, then the NNUB becomes an approximation to probability of symbol error, and not necessary an upper bound.

**Proof:** Note that for each signal point, the distance to each decision-region boundary must be at least $d_{\text{min}}/2$. The probability of error for point $x_i$, $P_{e/i}$, is upper bounded by the union bound as

$$P_{e/i} \leq N_i \cdot Q \left[ \frac{d_{\text{min}}}{2\sigma} \right]. \quad (1.148)$$

Thus,

$$P_e = \sum_{i=0}^{M-1} P_{e/i} \cdot p_x(i) \leq Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \sum_{i=0}^{M-1} N_i \cdot p_x(i) = N_e \cdot Q \left[ \frac{d_{\text{min}}}{2\sigma} \right]. \quad (1.149)$$

QED.

The previous Examples 1.3.3 and 1.3.4 show that the $Q$-function multiplier in each case is exactly $N_e$ for that constellation. As constellation becomes more complicated with coding and larger $N$ in Chapters 2 and 8, the number of nearest neighbors is commonly approximated as only those neighbors who also are at minimum distance, and $N_e$ is then approximated by (1.146). With this approximation, the $P_e$ expression in the NNUB consequently becomes only an approximation rather than a strict upper bound.

1.3.2.4 Alternative Performance Measures

The optimum receiver design minimizes the symbol error probability $P_e$. Other closely related measures of performance can also be used. An important measure used in practical system design is the **Bit Error Rate**. Most digital communication systems encode the message set $\{m_i\}$ into bits. Thus engineers are interested in the average number of bit errors expected. The average bit error probability will depend on the specific binary labeling applied to the signal points in the constellation. The quantity $n_b(i,j)$ denotes the number of bit errors corresponding to a symbol error when the detector incorrectly chooses $m_j$ instead of $m_i$, while $P\{\varepsilon_{ij}\}$ denotes the corresponding symbol-error probability.

The bit error rate $P_b$ obeys the following bound:
Definition 1.3.4 (Bit Error Rate) The bit error rate is

\[ P_b \triangleq \sum_{i=0}^{M-1} \sum_{j=0}^{N_i} p_x(i) \cdot P\{\varepsilon_{ij}\} \cdot n_b(i, j) \]  \hspace{1cm} (1.150)

where \( n_b(i, j) \) is the number of bit errors for the particular choice of encoder when symbol \( i \) is erroneously detected as symbol \( j \). This quantity, despite the label using \( P \), is not strictly a probability.

The bit error rate will always be approximated for the AWGN in this text by:

\[ P_b \approx \sum_{i=0}^{M-1} p_x(i) \cdot \sum_{j=1}^{N_i} n_b(i, j) \]  \hspace{1cm} (1.151)

\[ P_b \leq Q\left[ \frac{d_{\min}}{2\sigma} \right] \sum_{i=0}^{M-1} p_x(i) \sum_{j=1}^{N_i} n_b(i, j) \]  \hspace{1cm} (1.152)

\[ P_b \approx N_b \cdot Q\left[ \frac{d_{\min}}{2\sigma} \right] \]  \hspace{1cm} (1.155)

where

\[ n_b(i) \triangleq \sum_{j=1}^{N_i} n_b(i, j) \]  \hspace{1cm} (1.153)

and the Average Total Bit Errors per Error Event, \( N_b \), is defined as:

\[ N_b = \sum_{i=0}^{M-1} p_x(i) \cdot n_b(i) \]  \hspace{1cm} (1.154)

An expression similar to the NNUB for \( P_b \) is

\[ P_b \approx N_b \cdot Q\left[ \frac{d_{\min}}{2\sigma} \right] \]  \hspace{1cm} (1.155)

where the approximation comes from Equation (1.151), which is an approximation because of the reduction in the number of included terms in the sum over other points. This approximation’s accuracy is good as long as those terms corresponding to distant neighbors (with distance \( \geq d_{\min} \)) have small value in comparison to nearest neighbors, which is a reasonable assumption for good constellation designs.

The bit error rate is sometimes a more uniform measure of performance because it is independent of \( M \) and \( N \). On the other hand, \( P_e \) is a block error probability (with block length \( N \)) and can correspond to more than one bit in error (if \( M > 2 \)) over \( N \) dimensions. Both \( P_e \) and \( P_b \) depend on the same distance-to-noise ratio (the argument of the Q function). While the notation for \( P_b \) is commonly expressed with a \( P \), the bit error rate is not a probability and could exceed unity in value in aberrant cases. A better measure that is a probability is to normalize the bit-error rate by the number of bits per symbol: Normalization of \( P_b \) produces a probability measure because it is the average number of bit errors divided by the number of bits over which those errors occur - this probability is the desired average probability of bit error:
**Lemma 1.3.2 (Average probability of bit error $\bar{P}_b$.)** The average probability of bit error is defined by

$$\bar{P}_b = \frac{P_b}{b}.$$  

(1.156)

The corresponding average total number of bit errors per bit is

$$\bar{N}_b = \frac{N_b}{b}.$$  

(1.157)

The average bit error rate can exceed one, but the average probability of bit error never exceeds one.

Furthermore, $P_e$ comparisons between systems with different dimensionality is not fair (for instance to compare a 2B1Q system operating at $P_e = 10^{-7}$ against a multi-dimensional design consisting of 10 successive 2B1Q dimensions decoded jointly as a single symbol also with $P_e = 10^{-7}$, the latter system really has $10^{-8}$ errors per dimension and so is better.) A more fair measure of symbol error probability normalizes the measure by the dimensionality (or number of dimensions per symbol) of the system to compare systems with different block lengths.

**Definition 1.3.5 (Normalized Error Probability $\bar{P}_e$.)** The normalized error probability is defined by

$$\bar{P}_e = \frac{P_e}{N}.$$  

(1.158)

The normalized average number of nearest neighbors is:

**Definition 1.3.6 (Normalized Number of Nearest Neighbors)** The normalized number of nearest neighbors, $\bar{N}_e$, for a signal constellation is defined as

$$\bar{N}_e = \sum_{i=0}^{M-1} \frac{N_i}{N} \cdot p_x(i) = \frac{N_e}{N}.$$  

(1.159)

Thus, the NNUB is

$$\bar{P}_e \leq \bar{N}_e \cdot Q \left[ \frac{d_{\text{min}}}{2\sigma} \right].$$  

(1.160)

**EXAMPLE 1.3.5 (8AMPM)** The average number of bit errors per error event for 8AMPM using the octal labeling indicated by the subscripts in Figure 1.32 is computed by

$$N_b = \sum_{i=0}^{7} \frac{1}{8} n_b(i)$$

$$= \frac{1}{8} [(1 + 1 + 2) + (3 + 1 + 2 + 2) + (2 + 1 + 1) + (1 + 2 + 3) + (3 + 2 + 2 + 1) + (1 + 1 + 2) + (3 + 1 + 2) + (1 + 2 + 1)]$$

$$= \frac{44}{8} = 5.5.$$  

(1.164)
Then
\[ P_b \approx 5.5 \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] . \]  

Also,
\[ \bar{N}_e = \frac{3.25}{2} = 1.625 \]  

so that
\[ \bar{P}_e \leq 1.625 \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] , \]  

and
\[ \bar{P}_b \approx \frac{5.5}{3} \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] . \]  

Thus the bit error rate is somewhat higher than the normalized symbol error probability. Careful assignment of bits to symbols can reduce the average bit error rate slightly.

### 1.3.2.5 Block Error Measures

Higher-level engineering of communication systems may desire knowledge of message errors within packets of several messages cascaded into a larger message. An entire packet may be somewhat useless if any part of it is in error. Thus, the concept of a symbol from this perspective of analysis may be the entire packet of messages. The **probability of “block” or “packet” error** is truly identical to the probability of symbol error already analyzed as long as the entire packet is considered as a single symbol. Generally these measures are included in what engineers call **Quality of Service (QoS)**. QoS may include other measures such as the actual data rate for a given performance level, system outages (total loss of connectivity for some period of time), and the delay introduced by modulation and encoding. This subsection focuses only on error measures.

If a packet contains \( B \) bits, each of which independently has average probability of bit error \( \bar{P}_b \), then the probability of packet (or block) error is often approximated by \( B \cdot \bar{P}_b \). Clearly then if one says the probability of packet error is \( 10^{-7} \) and there are 125 bytes per packet, or 1000 bits per packet, then the average probability of bit error would then be \( 10^{-10} \). Low packet error rate is thus a more stringent criterion on the detector performance than is low average probability of bit error. Nonetheless, analysis can proceed exactly as in this section. As \( B \) increases, the approximation above of \( P_e = B \cdot \bar{P}_b \) can become inaccurate as we see below.

Telecommunications systems often use an erred second to measure performance. An erred second is any second in which any bit error occurs. Obviously, fewer erred seconds is better. A given fixed number of error seconds translates into increasingly lower probability of bit error as the data rate of the channel increases. An **error-free second** is a second in which no error occurs. If a second contains \( B \) independent bits, then the exact probability of an error-free second is
\[ P_{efs} = (1 - \bar{P}_b)^B \]  

while the exact probability of an errored second is
\[ P_es = 1 - P_{efs} = \sum_{i=1}^{B} \binom{B}{i} (1 - \bar{P}_b)^{B-i} \bar{P}_b^i . \]  

Dependency between bits and bit errors will change the above formulas’ exact nature, but analysis often ignores any such dependency. More common in telecommunications is the derived concept of **percentage error-free seconds**, which is the percentage of seconds that are error free. Thus, if a detector has \( \bar{P}_b = 10^{-7} \) and the data rate is 10 Mbps, then one might naively guess that almost every second contains errors according to \( P_e = B \cdot \bar{P}_b \), and the percentage of error free seconds is thus very low. To be exact, \( P_{efs} = (1 - 10^{-7})^{10^7} = .368 \), so that the link has 36.8% error free seconds, so actually about 63% of the seconds have errors. Typically large telecommunications networks strive for **five nines** reliability, which translates into 99.999% error-free seconds. At 10 Mbps, this means that the detector
has \( \tilde{P}_b = 1 - e^{10^{-7}\ln(99999)} = 2.3 \cdot 10^{-12} \). At lower data rates, five nines is less stringent on the error probability.

Advanced data networks, often designed for bit error rates above \( 10^{-12} \) operate with external “error detection and retransmission” protocols. Retransmission causes delay that may not be acceptable for the data network, at least at the physical layer. More sophisticated coding methods in later chapters will address means to reduce error probability within delay limits, which really means somehow increasing the number of dimensions used but minimally so for system objectives. In any case, the average symbol and bit error probabilities are often fundamental to all other measures of network performance and can be used by the serious communication engineer to evaluate carefully a system’s performance.

### 1.3.3 General Classes of Constellations and Modulation

This subsection describes three constellation classes (and sometimes associated modulation) that abound in digital data transmission. Each of these three classes represent different geometric approaches to constellation construction. Three successive subsections examine the usual basis-function choice(s) constellation class and develop corresponding general expressions for the average error probability \( P_e \) for the AWGN Channel. Subsection 1.3.3.2 discusses cubic constellations (Section 1.3.6 also investigates some important extensions to the cubic constellations). Subsection 1.3.3.6 examines orthogonal constellations, while Subsection 1.3.3.6.6 studies circular constellations.

To compare constellations, some constellation measures are developed first. The cost of modulation depends upon transmitted power (energy per unit time). A unit of time translates to a number of dimensions, given a certain system bandwidth, so the energy per dimension is essentially a measure of power. Given a wider bandwidth, the same unit in time and power will correspond to proportionately more dimensions, but a lower power spectral density. While somewhat loosely defined, a system with symbol period \( T \) and bandwidth\(^{20} \) \( W \), has a number of dimensions available for signal constellation construction that is

\[
N = 2 \cdot W \cdot T \quad \text{dimensions.} \tag{1.171}
\]

The reasons for this approximation will become increasingly apparent, but all this subsection’s methods follow this simple rule with reasonable and obvious bandwidth definition. Field systems all follow Equation (1.171), or have fewer dimensions than this practical maximum, even though it may be possible to construct signal sets with slightly more dimensions theoretically. The number of dimensions in any case is a combined measure of the system resources of bandwidth and time - thus, performance measures and energy are thus often normalized by \( N \) for fair comparison. The data rate concept thus generalizes to the **number of bits per dimension**:

\[
\text{Definition 1.3.7 (Average Number of Bits Per Dimension) The average number of bits per dimension, } \bar{b}, \text{ for a signal constellation } x, \text{ is}
\]

\[
\bar{b} \triangleq \frac{b}{N} \,. \tag{1.172}
\]

The related previously defined quantity, data rate, is

\[
R = \frac{b}{T} \,. \tag{1.173}
\]

Using (1.171), one can compute that

\[
2\bar{b} = \frac{R}{W} \,, \tag{1.174}
\]

\(^{20}\)It is theoretically not possible to have finite bandwidth and finite time extent, but in practice this can be approximated closely.
the modulation and constellation method’s **spectral efficiency**. Spectral efficiency is often used by transmission engineers to measure transmission design (how much data rate per unit of bandwidth). Spectral efficiency is often described in terms of the unit bits/second/Hz, which is really a measure of double the number of bits/dimension through Equation 1.174. Engineers often abbreviated the term bits/second/Hz to say bits/Hz, which is an (unfortunately) often used and confusing term because the units are incorrect. Nonetheless, experienced engineers automatically translate the verbal abbreviation bits/Hz to the correct units and interpretation, bits-per-second/Hz, or simply double the number of bits/dimension. An assumption in (1.174) is that \( N = 2WT \), which is only true when \( L = 1 \) - that is there is no MIMO in use. When there are \( L \) parallel spatial channels in use, the spectral efficiency will be the sum of the spectral efficiencies of all \( L \) subchannels, essentially meaning that free spatial dimensions improve spectral efficiency.

The concept of power also generalizes to **energy per dimension**:

\[
\mathcal{E}_x = \frac{\mathcal{E}_x}{N}. \tag{1.175}
\]

A previously defined and related quantity is the **average power**,

\[
P_x = \frac{\mathcal{E}_x}{T}. \tag{1.176}
\]

Clearly \( N \) cannot exceed the actual number of dimensions in the constellation, but the constellation may require fewer dimensions for a complete representation. For example the two-dimensional constellation in Figure 1.12 can be described using only one basis vector simply by rotating the constellation by 45 degrees. The average power, which was also defined earlier, is a scaled quantity, but consistently defined for all constellations. In particular, the normalization of basis functions often absorbs gain into the signal constellation definition that may tacitly conceal complicated calculations based on transmission-channel impedance, load matching, and various non-trivially calculated analog effects. These effects can also be absorbed into bandlimited channel models as is the case in Chapters 2, 3, 4, ?? and ??.

The energy per dimension allows the comparison of constellations with different dimensionality. The smaller the \( \mathcal{E}_x \) for a given \( P_x \) and \( \bar{b} \), the better the design. The concatenation of two successively transmitted \( N \)-dimensional signals taken from the same \( N \)-dimensional signal constellation as a single \( 2N \)-dimensional signal causes the resulting \( 2N \)-dimensional constellation, formed as a Cartesian product of the constituent \( N \)-dimensional constellations, to have the same average energy per dimension as the \( N \)-dimensional constellation. Thus, simple concatenation of a constellation with itself does not improve the design. However, Chapter 2 shows that careful packing of signals in increasingly larger dimensional signals sets can lead to a reduction in the energy per dimension required to transmit a given set of messages.

The average power is the usual measure of energy per unit time and is useful when sizing a modulator’s power requirements or in determining scale constants for analog filter/driver circuits in the actual implementation. The power can be set equal to the square of the voltage over the load resistance.

The **noise energy per dimension** for an \( N \)-dimensional AWGN channel is

\[
\bar{\sigma}^2 = \frac{\sum_{i=1}^{N} \sigma_i^2}{N} = \sigma^2 = \frac{N_0}{2}. \tag{1.177}
\]

While AWGN is inherently infinite dimensional, by the theorem of irrelevance, error-probability calculation need only consider the noise components in the same \( N \) dimensions as the signal constellation.

The previously defined SNR can now also be written for the AWGN as
The UB and NNUB show the AWGN performance of a constellation depends on the minimum distance between any two vectors in the constellation. Increasing the distance between points in a particular constellation increases the average energy per dimension of the constellation. The “Constellation Figure of Merit.”\textsuperscript{21} combines the energy per dimension and the minimum distance measures:

\begin{align}
\zeta_x & \equiv \left( \frac{d_{\text{min}}}{2} \right)^2 \bar{E}_x, \\
& \text{a unit-less quantity, defined only when } \bar{b} \geq 1.
\end{align}

The CFM $\zeta_x$ will measure the quality of any constellation used with an AWGN channel. A higher CFM $\zeta_x$ generally results in better performance. The CFM should only be used to compare systems with equal numbers of bits per dimension $\bar{b} = b/N$, but can be used to compare systems of different dimensionality.

A different measure, known as the “energy per bit,” measures performance in systems with low average bit rate of $\bar{b} \leq 1$ (see Chapter 10).

\begin{align}
\mathcal{E}_b & = \frac{\bar{E}_x}{\bar{b}} = \frac{\bar{E}_x}{b}. \\
& \text{This measure is only defined when } \bar{b} \leq 1 \text{ and has no meaning in other contexts.}
\end{align}

Margin’s are often quoted in transmission design as they give a level of confidence to designers that unforeseen noise increases or signal attenuation will not cause the system performance to become unacceptable. Chapters 2 and 4 will investigate margin further, while an example appears here.

**EXAMPLE 1.3.6 (Margin in DSL)** Digital Subscriber Line systems deliver 100’s of kilobits to 10’s of megabits of data over telephone lines and use sophisticated adaptive modulation systems described later in Chapters 4 and 5. The two modems are located at the ends of the telephone line between the telephone-company edge and the customer’s premise.

\textsuperscript{21}G. D. Forney, Jr., 8/89 IEEE Journal on Selected Areas in Communication.
However, DSLs ultimately also have an error probability specified by a relation of the form 
\[ N_e \cdot Q(d_{\text{min}}/2\sigma). \]
Because noise sources can be unpredictable on telephone lines, which tend to sense everything from other phone lines’ signals to radio signals to refrigerator doors and fluorescent and other lights, and because customer-location additional wiring to the modem can be poor grade or long, a margin of at least 6 dB is mandated at the data rate of service offered if the customer is to be allowed service. This 6 dB essentially allows performance to be degraded by a combined factor of 4 in increased noise before costly manual maintenance or repair service would be necessary.

1.3.3.1 Fair Comparisons

Two transmission systems’ fair comparison requires consideration of the following 5 parameters:

1. data rate \( R = \frac{b}{T}, \)
2. power \( \mathcal{E}_x/T, \)
3. total bandwidth \( W, \)
4. total time or symbol period \( T, \) and
5. probability of error \( \bar{P}_b \) (or \( \bar{P}_e \)).

(For MIMO systems, the number of parallel channels should be held the same also.)

Any fair comparison thus holds 4 of the 5 parameters constant while varying the 5th. However, a simplification can be achieved with dimensionality as the normalizer instead of \( W \) and \( T. \) In this case, a fair comparison uses

1. bits per dimension \( \bar{b}, \)
2. energy per dimension \( \bar{\mathcal{E}}_x, \) and
3. probability of error per dimension (or \( \bar{P}_e). \)

Any two of these 3 can be held constant and the 3rd compared. Transmission history is replete with examples of engineers who should have known better not keeping 4 of the 5, or the simpler 2 of the 3 constant, before comparing the last. This simpler fair comparison is an advantages of the dimensional-normalization. The constellation figure of merit presumes \( \bar{b} \) fixed and then looks at the ratio of \( d_{\text{min}}^2 \) to \( \bar{\mathcal{E}}_x \), essentially holding \( \bar{\mathcal{E}}_x \) fixed and looking at \( \bar{P}_e \) (equivalent to \( d_{\text{min}} \) if nearest neighbors are ignored on the AWGN). The normalization essentially prevents an excess of symbol period or bandwidth from letting one modulation method look better than another, tacitly including the third and fourth parameters (bandwidth and symbol period) from the list of parameters in a comparison.

The CFM – when it is well defined – can be used approximately for fair comparison if the \( \bar{P}_e \) is held constant, because essentially this ratio is of a function of \( \bar{b} \) and \( \bar{\mathcal{E}}_x \) (so in effect holding \( \bar{b} \) constant relative to normalized energy). However, it is best in general to use the 3 quantities and directly hold 2 fixed while comparing the third.

1.3.3.2 Cubic Constellations

Cubic constellations are commonly used on simple data communication channels. Some examples of cubic signal constellations are shown in Figure 1.34 for \( N = 1, 2, \) and 3. The construction of a cubic constellation directly maps a sequence of \( N = b \) bits into the components of the basis vectors in a corresponding \( N \)-dimensional signal constellation. For example, the bit stream \( \ldots 010010 \ldots \) may be grouped as the sequence of two-dimensional vectors \( \ldots (01)(00)(10) \ldots \) The resulting constellation is uniformly scaled in all dimensions, and may be translated or rotated in the \( N \)-dimensional space it occupies.

The simplest cubic constellation appears in Figure 1.34, where \( N = b = \bar{b} = 1. \) This constellation is known as “binary signaling”, since only two possible signals are transmitted using one basis function \( \varphi_1(t). \) Several examples of binary signaling are described next.
1.3.3.3 Binary Antipodal Signaling

In binary antipodal signaling, the two possible values for $x = x_1$ are equal in magnitude but opposite in sign, e.g. $x_1 = \pm \frac{d}{2}$. As for all binary signaling methods, the average error probability is

$$P_e = P_b = Q \left[ \frac{d_{\min}}{2\sigma} \right].$$

(1.181)

The CFM for binary antipodal signaling equals $\zeta_x = (d/2)^2/[(d/2)^2] = 1$.

Particular types of binary antipodal signaling differ only in the basis-function choice $\varphi_1(t)$ and have the same analyses. In practice, these basis functions may include “Nyquist” pulse shaping waveforms to avoid intersymbol interference. Chapter 3 further discusses waveform shaping and intersymbol interference. Besides the time-domain shaping, the basis function $\varphi_1(t)$ shapes the power spectral density of the resultant modulated waveform. Thus, different basis functions may use different bandwidths, and so fair comparison rules should be applied.

**Definition 1.3.13 (Binary Phase Shift Keying)** Binary Phase Shift Keying (BPSK) uses a sinusoid to modulate the sequence of data symbols $\{\pm \sqrt{E}\}$.

$$\varphi_1(t) = \begin{cases} \sqrt{\frac{2}{T}} \sin \frac{2\pi t}{T} & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

(1.182)

This representation uses the minimum number of basis functions $N = 1$ to represent BPSK, rather than $N = 2$ as in Example 1.2.2.
**Definition 1.3.14 (Bipolar (NRZ) transmission)** Bipolar signaling, also known as “baseband binary” or “Non-Return-to-Zero (NRZ)” signaling, uses a square pulse to modulate the data symbols \{±\sqrt{E_x}\}.

\[
\varphi_1(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases} \tag{1.183}
\]

**Definition 1.3.15 (Manchester Coding (Bi-Phase Level))** Manchester Coding, also known as “biphase level” (BPL) or, in magnetic and optical recording, as “frequency modulation,” uses a sequence of two opposite phase square pulses to modulate each data symbol. In NRZ signaling long runs of the same bit result in a constant output signal with no transitions until the bit changes. Since timing recovery circuits usually require some transitions, Manchester or BPL guarantees a transition occurs in the middle of each bit (or symbol) period \(T\). The basis function is:

\[
\varphi_1(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t < T/2 \\
-\frac{1}{\sqrt{T}} & T/2 \leq t < T \\
0 & \text{elsewhere}
\end{cases} \tag{1.184}
\]

The power spectral density of the modulated signal is related to the Fourier transform \(\Phi_1(f)\) of the pulse \(\varphi_1(t)\). The Fourier transform of the NRZ square pulse is a sinc function with zero crossings spaced at \(\frac{1}{T}\) Hz. The basis function for BPL in Equation (1.184) requires approximately twice the bandwidth of the basis function for NRZ in Equation (1.183), because the Fourier transform of the biphase pulse is a sinc function with zero crossings spaced at \(\frac{2}{T}\) Hz. Similarly BPSK requires double the bandwidth of NRZ. Both BPSK and BPL are referred to as “rate 1/2” since \(\bar{b} = \frac{1}{2}\), and thus spectral efficiency of BPSK is 1 bit/s/Hz. This means that for the same bandwidth they permit only half the transmitted transmission rate compared with NRZ, which has a spectral efficiency of 2 bits/s/Hz, or equivalently \(\bar{b} = 1\).

For the AWGN channel using binary antipodal signaling, Subsection 1.1.7’s Bhattacharya Bound for this memoryless channel is with \(N\) successive uses and input messages that differ in \(d_H\) positions:

\[
P\{\varepsilon_{m\bar{m}}\} \leq \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(y_n-x_{m,n})^2} \cdot e^{-\frac{\bar{x}}{2\sigma^2}y_n} dy_n \tag{1.185}
\]

\[
= \prod_{n=1}^{d_H} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}y_n^2} dy_n \cdot e^{-\frac{\bar{x}}{2\sigma^2}y_n} \tag{1.186}
\]

\[
= e^{-d_H\frac{\bar{x}}{2\sigma^2}} \tag{1.187}
\]

### 1.3.3.4 On-Off Keying (OOK)

**On-Off Keying**, used in direct-detection optical data transmission, as well as in “gate-to-gate” transmission in most digital circuits, uses the same basis function as bipolar transmission.

\[
\varphi_1(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases} \tag{1.188}
\]

Unlike bipolar transmission, however, one of the levels for \(x_1\) is zero, while the other is nonzero (\(\sqrt{2E_x}\)). Because of the asymmetry, this method includes a DC offset, i.e. a nonzero mean value. The CFM is
ζx = .5, and thus OOK is 3 dB inferior to any type of binary antipodal transmission. The comparison between signal constellations is 10 log10[ζx,OOK/ζx,NRZ] = 10 log10(0.5) = −3 dB.

As for any binary signaling method, OOK has

\[ P_e = P_b = Q \left[ \frac{d_{\min}}{2\sigma} \right] \]  

(1.189)

1.3.3.5 Vertices of a Hypercube (Block Binary)

Binary signaling in one dimension generalizes to the corners of a hypercube in N-dimensions, hence the name “cubic constellations.” The hypercubic constellations all transmit an average of \( \overline{b} = 1 \) bit per dimension. For two dimensions, the most common modulation type is QPSK:

1.3.3.5.1 Quadrature Phase Shift Keying (QPSK) The two-dimensional QPSK basis functions are

\[
\varphi_1(t) = \begin{cases} 
\sqrt{\frac{2}{T}} \cdot \cos \frac{2\pi t}{T} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases}
\]  

(1.190)

\[
\varphi_2(t) = \begin{cases} 
\sqrt{\frac{2}{T}} \cdot \sin \frac{2\pi t}{T} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases}
\]  

(1.191)

The transmitted signal is a linear combination of both an inphase (cos) component and a quadrature (sin) component. The four possible data symbol vectors are

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} 
\sqrt{\frac{E_x}{2}} \cdot [-1 - 1]' \\
\sqrt{\frac{E_x}{2}} \cdot [-1 + 1]' \\
\sqrt{\frac{E_x}{2}} \cdot [+1 - 1]' \\
\sqrt{\frac{E_x}{2}} \cdot [+1 + 1]' 
\end{cases}
\]  

(1.192)

The additional basis function does not require any extra bandwidth with respect to BPSK, and the average energy \( E_x \) remains unchanged. While the squared minimum distance \( d_{\min}^2 \) has decreased by a factor of two, the number of dimensions has doubled, thus the CFM for QPSK is \( \zeta_x = 1 \) again, as with BPSK. However, QPSK’s \( \overline{b} = 1 \), so twice as much information per dimension as BPSK. If instead \( \overline{E}_x \) is the same for BPSK and QPSK, then \( P_c \) will be the same (same \( d_{\min}^2 \)), but \( b_{BPSK} = \frac{1}{2} b_{QPSK} = 1 \), so QPSK is better. The fair comparison notes that QPSK is making better use of its resources (BPSK essentially wastes a dimension by putting no energy on it).

Exact performance evaluation first computes the average probability of a correct decision \( P_c \), and then \( P_e = 1 - P_c \). Analysis here is for maximum likelihood detection on the AWGN channel with equally probable signals. By constellation symmetry, \( P_{c|i} \) is identical \( \forall i = 0, \ldots, 3 \).

\[
P_c = \sum_{i=0}^{3} P_{c|i} \cdot p_{x}(i) = P_{c|i}
\]  

(1.193)

\[
P_c = \left( 1 - Q \left[ \frac{d_{\min}}{2\sigma} \right] \right) \left( 1 - Q \left[ \frac{d_{\min}}{2\sigma} \right] \right)
\]  

(1.194)

\[
P_c = 1 - 2Q \left[ \frac{d_{\min}}{2\sigma} \right] + \left( Q \left[ \frac{d_{\min}}{2\sigma} \right] \right)^2.
\]  

(1.195)

The noises independence in the two dimensions allows progress from (1.193) to (1.194). The probability of a correct decision requires both noise components to fall within the decision region (see Figure 1.27), which has probability equal to the product of probabilities in (1.194). Thus

\[
P_e = 1 - P_c
\]  

(1.196)

\[
P_e = 2Q \left[ \frac{d_{\min}}{2\sigma} \right] - \left( Q \left[ \frac{d_{\min}}{2\sigma} \right] \right)^2 < 2Q \left[ \frac{d_{\min}}{2\sigma} \right],
\]  

(1.197)
where \( d_{\text{min}} = \sqrt{2\sigma x} = 2e^{1/2} \). For reasonable error rates \( (P_e < 10^{-2}) \), the \( \left( Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \right)^2 \) term in (1.197) is negligible, and the bound on the right, which is also the NNUB, is tight. With a “reasonable” mapping of bits to data symbols (e.g. the Gray code \( 0 \rightarrow -1 \) and \( 1 \rightarrow +1 \)), the probability of a bit error \( \bar{P}_b = \bar{P}_e \) for QPSK. \( P_e \) for QPSK is twice \( P_e \) for BPSK, but \( \bar{P}_e \) is the same for both systems.

### 1.3.3.5.2 Block Binary

For hypercubic signal constellations in three or more dimensions, \( N \geq 3 \), the signal points are the vertices of a hypercube centered on the origin. In this case, the error probability generalizes to

\[
P_e = 1 - \left( 1 - Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \right)^N < NQ \left[ \frac{d_{\text{min}}}{2\sigma} \right].
\]

(1.198)

where \( d_{\text{min}} = 2e^{1/2} \). The basis functions are usually given by \( \varphi_n(t) = \varphi(t - nT) \), where \( \varphi(t) \) is the square pulse given in (1.183). The transmission of one symbol with the hypercubic constellation requires a time interval of length \( NT \). Alternatively, scaling of the basis functions in time can retain a symbol period of length \( T \), but the narrower pulse will require \( N \) times the bandwidth as the \( T \) width pulses. For this case again \( \zeta_x = 1 \). As \( N \rightarrow \infty \), \( P_e \rightarrow 1 \). While the probability of any single dimension being correct remains constant and less than one, as \( N \) increases, the probability of all dimensions being correct decreases.

Ignoring the higher order terms \( Q^i, i \geq 2 \), the average probability of error is approximately \( \bar{P}_e \approx Q(d_{\text{min}}/(2\sigma)) \), which equals \( \bar{P}_e \) for binary antipodal signaling. This example illustrates that increasing dimensionality does not always reduce the probability of error unless the signal constellation has been carefully designed. As block binary constellations are just a concatenation of several binary transmissions, the receiver can equivalently decode each of the independent dimensions separately. However, with a careful selection of the transmitted signal constellation, it is possible to drive the probability of both a message error \( P_e \) and a bit error \( P_b \) to zero with increasing dimensionality \( N \), as long as the average number of transmitted bits per unit time does not exceed a fundamental rate known as the “capacity” of the communication channel (Chapter 8).

### 1.3.3.6 Orthogonal Constellations

In orthogonal signal sets, the dimensionality increases linearly with the number of points \( M \propto N \) in the signal constellation, which results in a decrease in the number of bits per dimension \( \bar{b} = \log_2(M) = \frac{\log_2(\alpha N)}{N} \).

#### 1.3.3.6.1 Block Orthogonal

Block orthogonal signal constellations have a dimension, or basis function, for each signal point. The block orthogonal signal set thus consists of \( M = N \) orthogonal signals \( x_i(t) \), that is

\[
\langle x_i(t), x_j(t) \rangle = \mathcal{E}_x \cdot \delta_{ij}.
\]

(1.199)

Block orthogonal signal constellations appear in Figure 1.35 for \( N = 2 \) and \( 3 \). The signal constellation vectors are, in general,

\[
x_i = \begin{bmatrix} 0 \ldots 0 & \sqrt{\mathcal{E}_x} & 0 \ldots 0 \end{bmatrix} = \sqrt{\mathcal{E}_x} \cdot \varphi_{i+1}.
\]

(1.200)

The CFM should not be used on block orthogonal signal sets because \( \bar{b} < 1 \), but the fair comparison of 2 of 3 of \( \bar{b}, \mathcal{E}_x \), and \( \bar{P}_e \) can be used and Block Orthogonal will not compare well.
As examples of block orthogonal signaling, consider the following two dimensional signal sets.

**Definition 1.3.16 (Return to Zero (RZ) Signaling)** RZ uses the following two basis functions for the two-dimensional signal constellation shown in Figure 1.35:

\[
\begin{align*}
\varphi_1(t) &= \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases} \\
\varphi_2(t) &= \begin{cases} 
\frac{1}{\sqrt{T}} & 0 \leq t < T/2 \\
-\frac{1}{\sqrt{T}} & T/2 \leq t < T \\
0 & \text{elsewhere}
\end{cases}
\end{align*}
\]

(1.201) (1.202)

“Return to zero” indicates that the transmitted voltage (i.e. the real value of the signal waveform) always returns to the same value at the beginning of any symbol interval. Equivalently, for the same energy/dimension and normalized error probability, RZ provides half the data rate of NRZ.

As for any binary signal constellation, and thus for RZ,

\[
P_b = P_e = Q \left[ \frac{d_{\min}}{2\sigma} \right] = Q \left[ \sqrt{\frac{E_x}{2\sigma^2}} \right].
\]

(1.203)

RZ is 3 dB inferior to binary antipodal signaling, and uses twice the bandwidth of NRZ.

**Definition 1.3.17 (Frequency Shift Keying (FSK))** Frequency shift keying uses the following two basis functions for the two dimensional signal constellation shown in Figure 1.35.

\[
\begin{align*}
\varphi_1(t) &= \begin{cases} 
\frac{\pi}{T} \sin \frac{\pi t}{T} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases} \\
\varphi_2(t) &= \begin{cases} 
\frac{\pi}{T} \sin \frac{2\pi t}{T} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases}
\end{align*}
\]

(1.204) (1.205)

The term “frequency-shift” indicates that the sequence of “1’s” and “0’s” in the transmitted data shifts between two different frequencies, \(1/(2T)\) and \(1/T\).
As for any binary signal constellation,

\[
P_b = P_e = Q \left[ \frac{d_{\min}}{2\sigma} \right] = Q \left[ \sqrt{\frac{E_x}{2\sigma^2}} \right].
\]  

(1.206)

FSK is also 3 dB inferior to binary antipodal signaling. FSK and RZ have the same performance and performance analysis, although the basis functions appear different.

FSK can be extended to higher dimensional signal sets \(N > 2\) by adding the following basis functions \((i \geq 3)\):

\[
\varphi_i(t) = \begin{cases} 
\sqrt{2} T \sin \frac{\pi t}{T} & 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases}
\]  

(1.207)

The required bandwidth necessary to realize the additional basis functions grows linearly with \(N\) for this FSK extension. Such FSK systems do not fairly compare well with RZ or cubic constellations. They may be used for other reasons.

1.3.3.6.2 \(P_e\) Computation for Block Orthogonal

The computation of \(P_e\) for block orthogonal constellations returns to the discussion of the signal detector in Figure 1.24. Because all the signals are equally likely and of equal energy, the constants \(c_i\) can be omitted (because they are all the same constant \(c = c\)). In this case, the MAP receiver becomes

\[
\hat{m} \Rightarrow m_i \text{ if } \langle y, x_i \rangle \geq \langle y, x_j \rangle \forall j \neq i,
\]  

(1.208)

By the symmetry of the block orthogonal signal constellation, \(P_{e/i} = P_e\) or \(P_{c/i} = P_c\) for all \(i\). For convenience, the analysis calculates \(P_c = P_{c|0=0}\), in which case the \(i^{th}\) elements of \(y\) are

\[
y_0 = \sqrt{E_x} + n_0
\]  

(1.209)

\[
y_i = n_i \forall i \neq 0.
\]  

(1.210)

If a decision is made that message 0 was sent, then \(\langle y, x_0 \rangle \geq \langle y, x_i \rangle\) or equivalently \(y_0 \geq y_i \forall i \neq 0\). The probability of this decision being correct is

\[
P_{c/0} = P\{y_0 \geq y_i \forall i \neq 0| \text{ given 0 was sent}\}.
\]  

(1.211)

If \(y_0\) takes on a particular value \(v\), then since \(y_i = n_i \forall i \neq 0\) and since all the noise components are independent,

\[
P_{c/0, y_0=v} = P\{n_i \leq v, \forall i \neq 0\} = \prod_{i=1}^{N-1} P\{n_i \leq v\} = [1 - Q(v/\sigma)]^{N-1}.
\]  

(1.212)

(1.213)

(1.214)

The last equation uses the fact that the \(n_i\) are independent, identically distributed Gaussian random variables \(N(0, \sigma^2)\). Finally, recalling that \(y_0\) is also a Gaussian random variable \(N(\sqrt{E_x}, \sigma^2)\).

\[
P_c = P_{c/0} = \int_{-\infty}^{\infty} (2\pi\sigma^2)^{-1/2} \cdot e^{-\frac{1}{2\sigma^2} (v-\sqrt{E_x})^2} \cdot [1 - Q(v/\sigma)]^{N-1} dv,
\]  

yielding

\[
P_c = 1 - \int_{-\infty}^{\infty} (2\pi\sigma^2)^{-1/2} \cdot e^{-\frac{1}{2\sigma^2} (v-\sqrt{E_x})^2} \cdot [1 - Q(v/\sigma)]^{N-1} dv.
\]  

(1.215)

(1.216)

This function must be evaluated numerically using a computer, which is the case in Figure 1.36.
A simpler calculation yields the NNUB, which also coincides with the union bound because the number of nearest neighbors $M-1$ equals the total number of neighbors to any point for block orthogonal constellations. The NNUB is given by

$$P_e \leq (M-1)Q\left(\frac{d_{\text{min}}^2}{2\sigma^2}\right) = (M-1)Q\left(\sqrt{\frac{E_x}{2\sigma^2}}\right).$$  (1.217)

Figure 1.36 shows that as $N$ gets large, performance improves without increase of SNR, but at the expense of a lower $\bar{b}$. This illustrates the possibility for driving $P_e \to 0$ at the cost of diminishing data rate. Chapter 2 will show that for finite SNR, the data rate need not diminish as long as it is below a theoretically computed maximum called the capacity. Block orthogonal signaling is not necessarily a good method to obtain high reliability.

### 1.3.3.6.3 Simplex Constellation

For block orthogonal constellations, the mean value of the signal constellation is nonzero, that is $E[x] = (\sqrt{E_x}/M)[1 \ 1 \ ... \ 1]$. Translation of the constellation by $-E[x]$ minimizes the average constellation energy without changing the average error probability. The translated constellation, known as the simplex constellation, is

$$x_i^s = \left[-\frac{\sqrt{E_x}}{M}, ..., -\frac{\sqrt{E_x}}{M}, \sqrt{E_x} \cdot (1 - \frac{1}{M}), -\frac{\sqrt{E_x}}{M}, ..., -\frac{\sqrt{E_x}}{M}\right]^\prime,$$  (1.218)

where the term $\sqrt{E_x} \cdot (1 - \frac{1}{M})$ occurs in the $i^{th}$ position. The superscript $s$ distinguishes the simplex constellation $\{x_i^s\}$ from the block orthogonal constellation $\{x_i\}$ from which the simplex constellation is constructed. The simplex constellation’s average energy is

$$E_x^s = \frac{M-1}{M} E_x,$$  (1.219)

which provides significant energy savings for small $M$ (over block orthogonal). The constellation’s symbol vectors, however, are no longer orthogonal.

$$\langle x_i^s, x_j^s \rangle = (x_i - E[x])^\prime (x_j - E[x])$$  (1.220)
\[
\begin{align*}
E_x \cdot \delta_{ij} &\quad - (E[x_i, (x_i + x_j)]) + \frac{E_x}{M} \\
\quad &\quad = E_x \cdot \delta_{ij} - 2 \frac{E_x}{M} + \frac{E_x}{M} \\
\quad &\quad = E_x \cdot \delta_{ij} - \frac{E_x}{M} 
\end{align*}
\]

By the Theorem of Translational Invariance, \(P_e\) of simplex constellations equals \(P_e\) of block orthogonal constellation given in (1.216) and bounded in (1.217), albeit the simplex constellations at slightly lower energy consumption.

### 1.3.3.6.4 Pulse Duration & Position Modulation

Another signal set in which the signals are not orthogonal, as usually described, is pulse duration modulation (PDM). The number of signals points in the PDM constellation increases linearly with the number of dimensions, as for orthogonal constellations. PDM is commonly used, with some modifications, in read-only optical data storage (i.e. compact disks and CD-ROM). In optical data storage, data is recorded by the length of a hole or “pit” burned into the storage medium. The signal set can be constructed as illustrated in Figure 1.37. The minimum width of the pit (4T in the figure) is much larger than the separation (\(T\)) between the different PDM signal waveforms. The signal set is evidently not orthogonal.

A second performance-equivalent (to PDM) set of waveforms, is known as a Pulse Position Modulation (PPM), and appears in Figure 1.38. The PPM Constellation is a block-orthogonal constellation, which has the previously derived \(P_e\). The average energy of the PDM constellation clearly exceeds that of the PPM constellation, which in turn exceeds that of a corresponding simplex constellation. Nevertheless, constellation energy minimization is usually not important when PDM is used; for example in optical storage, the optical channel physics mandate the minimum “pit” duration and the resultant “energy” increase is not of primary concern.

### 1.3.3.6.5 Biorthogonal Signal Constellations

A variation on block-orthogonal modulation is biorthogonal modulation, which doubles the size of the signal set from \(M = N\) to \(M = 2N\) by including the negative of each of the data symbol vectors in the signal set. From this perspective, QPSK has both a biorthogonal constellation and a cubic constellation.
The error-probability analysis for biorthogonal constellations parallels that for block orthogonal constellations. As with orthogonal signaling, because all the signals are equally likely and of equal energy, the constants $c_i$ in the signal detector in Figure 1.24 can be omitted, and the MAP receiver becomes

$$\hat{m} \Rightarrow m_i \text{ if } \langle y, x_i \rangle \geq \langle y, x_j \rangle \ \forall \ j \neq i .$$

By symmetry $P_{c/i} = P_c$ or $P_{c/i} = P_c$ for all $i$. Let $i = 0$. Then

$$y_0 = \sqrt{E_x} + n_0$$
$$y_i = n_i \ \forall \ i \neq 0 .$$

If $x_0$ was sent, then a correct decision is made if $\langle y, x_0 \rangle \geq \langle y, x_i \rangle$ or equivalently if $y_0 \geq |y_i| \ \forall \ i \neq 0$. Thus

$$P_{c/0} = P\{y_0 \geq |y_i|, \ \forall \ i \neq 0 \} \text{ if symbol zero was sent} .$$

Suppose $y_0$ takes on a particular value $v \in [0, \infty)$, then since the noise components $n_i$ are iid

$$P_{c/0, y_0=v} = \prod_{i=1}^{N-1} P\{|n_i| \leq v\}$$
$$= [1 - 2Q(v/\sigma)]^{N-1} .$$

If $y_0 < 0$, then an incorrect decision is guaranteed if symbol zero was sent. (The reader should visualize the decision regions for this constellation). Thus

$$P_c = P_{c/0} = \int_0^\infty (\sqrt{2\pi\sigma^2})^{-1} e^{-\frac{v^2}{2\sigma^2}} [1 - 2Q(v/\sigma)]^{N-1} dv ,$$

yielding

$$P_c = 1 - \int_0^\infty (\sqrt{2\pi\sigma^2})^{-1} e^{-\frac{v^2}{2\sigma^2}} [1 - 2Q(v/\sigma)]^{N-1} dv .$$

This function can be evaluated numerically using a computer.

Figure 1.38: Pulse-Position Modulation (PPM).
Using the NNUB, which is slightly tighter than the union bound because the number of nearest neighbors is \( M - 2 \) for biorthogonal signaling,

\[
P_e \leq (M - 2) \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] = 2(N - 1) \cdot Q \left[ \sqrt{\frac{\mathcal{E}_x}{2\sigma^2}} \right].
\]

1.3.3.6 Circular Constellations - \( M \)-ary Phase Shift Keying

Examples of Phase Shift Keying appeared in Figures 1.22 and 1.30. In general, \( M \)-ary PSK places the data symbol vectors at equally spaced angles (or phases) around a circle of radius \( \sqrt{\mathcal{E}_x} \) in two dimensions. Only the phase of the signal changes with the transmitted message, while the amplitude of the signal envelope remains constant, thus the origin of the name.

PSK is often used on channels with nonlinear amplitude distortion where signals that include information content in the time varying amplitude would otherwise suffer performance degradation from nonlinear amplitude distortion. The minimum distance for \( M \)-ary PSK is given by

\[
d_{\min} = 2\sqrt{\mathcal{E}_x} \cdot \sin \left( \frac{\pi}{M} \right).
\]

The CFM is

\[
\zeta_x = 2 \sin^2 \left( \frac{\pi}{M} \right),
\]

which is inferior to block binary signaling for any constellation with \( M > 4 \). The NNUB on error probability is tight and equal to

\[
P_e < 2Q \left[ \frac{\sqrt{\mathcal{E}_x}}{\sigma} \sin \left( \frac{\pi}{M} \right) \right],
\]

for all \( M \).

1.3.4 Rectangular (and Hexagonal) Signal Constellations

This section studies some very common constellations for data transmission. These constellations use equally spaced points on translated one- or two-dimensional vector space known as a lattice. This study also introduces and uses some basic concepts, namely the previously defined SNR, shaping gain, the continuous approximation, and the peak-to-average power ratio. The constellations in this section are largely the foundation for all this text’s subsequent developments.

Subsection 1.3.4.1 studies pulse amplitude modulation (PAM), while Subsection 1.3.4.2 studies quadrature amplitude modulation (QAM). Subsection 1.3.4.3 discusses several measures of constellation performance.

1.3.4.0.1 The Continuous Approximation

For a geometrically uniform constellation, \( C \), the continuous approximation computes the constellation’s average energy by approximating a discrete energy sum with a continuous integral. A continuous uniform distribution approximates the constellation’s discrete probability distribution over a region defined by the constellation’s boundaries. In this constellation region, each symbol appears in the center of an identically shaped fundamental-volume region (or Voronoi Region), \( \mathcal{V}(\Lambda) \), where \( \Lambda \) is a lattice\(^{23} \) from which the points were selected. The volume of the Voronoi Region is \( V(\Lambda) \), so \( V(\Lambda) = |\mathcal{V}(\Lambda)| \). This decision-region view of the Voronoi region thus has a maximum number of nearest neighbors, or equivalently number of sides. The union of these Voronoi regions for the signal constellation’s points is the constellation’s Voronoi Boundary \( \mathcal{V}_x \triangleq \bigcup_{i=0}^{M-1} V_i(\Lambda) \) where \( V_i(\Lambda) \) corresponds to the \( i \)th symbol’s Voronoi region. The Voronoi Boundary envelops a volume \( |C| \cdot V(\Lambda) \) or \( M \cdot V(\Lambda) \). With the continuous approximation, the number of constellation points is viewed in this chapter as the number of messages \( |C| = M \) and so \( M \) is used here to avoid confusion in later chapters. The discrete distribution of the constellation is approximated by a uniform distribution over the Voronoi Boundary with continuous probability density \( p_x(u) = \frac{1}{|C| \cdot V(\Lambda)} \quad \forall \ u \in \mathcal{V}_x \). It may be

\(^{23}\)Or, more properly a coset thereof, see Appendix for more on lattices and cosets, but for now view a lattice as a grid of regularly spaced points is sufficient
in coded systems that $N > 2$ and thus the uniform distribution will be over that larger dimensionality where symbols are equally likely (uniform) - which does not necessarily imply uniform distribution in 1 or 2 dimensions, see Chapter 2.

**Definition 1.3.18 (Continuous Approximation)** The continuous approximation to a constellation’s average energy equals

$$
\mathcal{E}_x \approx \tilde{\mathcal{E}}_x = \int_{\mathcal{V}_x} \|u\|^2 \cdot \frac{1}{M \cdot V(\Lambda)} du ,
$$

where the $N$-dimensional integral covers the Voronoi Boundary $\mathcal{V}_x$ for the constellation $C$. The $M \cdot V(\Lambda)$ term is somewhat superfluous in that it is a constant, and thus really just a scale-factor for normalization, because the probability of the points in the distribution is uniform. It could be replaced by the volume of any continuous region over which the distribution is uniformly distributed. Indeed, the number of points $M$ is no longer relevant other than it helps “size” the region as a scale factor.

For large size signal sets with regular spacing between points, the error in using the continuous approximation is small, as several examples will demonstrate in this section.

For many regions, mathematicians have tabulated the squared energy, or equivalently the second moment of the region (scaled by the inverse volume $V^{-2/N}$). Problem 1.19 investigates a few simply such regions. The energy/second-moment and volume can be computed from basic geometric parameters. For instance for a circle (2D), the Area is $\pi r^2$ with radius $r$ and the second moment is $\frac{1}{2} \pi r^4$, making continuous approximation energy equal to $\frac{1}{2} \pi r^2$. The ratio of this to volume is then $\frac{1}{2}$ or equivalently the 2D volume (area) is twice the energy of a circle. The radius of the circle essentially grows as more points are packed into the circle so a larger constellation (presumably each point with some fixed $\mathcal{V}_x$) simply has a larger radius as more points are inserted, but the ratio of energy/volume will remain (with large $M$ or effectively continuous uniform distribution) $1/2$.

An interesting case would be the hypersphere of increasingly large dimensionality as $N \to \infty$, which has well-known limiting second moment for radius $r$, $\mathcal{E}_x = \pi \cdot e$. The volume of an $N$-dimensional sphere with radius $r$ is (when $N$ is even)

$$
V_N(r) = \frac{(\pi r^2)^{N/2}}{(\frac{N}{2}!)}. \tag{1.237}
$$

Then energy per dimension is

$$
\tilde{\mathcal{E}}_x = \frac{r^2}{N + 2}. \tag{1.238}
$$

The two dimensional energy total is $2 \cdot \tilde{\mathcal{E}}_x$. Energy is a squared quantity, so volume should also be related to a 2D squared quantity when compared with energy. A ratio of interest will be the quantity

$$
\frac{\tilde{\mathcal{E}}_x}{V^{2/N}} = \frac{r^2}{N + 2} \cdot \frac{(\frac{N}{2}!)^{N/2}}{(\pi r^2)} = \frac{(\frac{N}{2}!)^{N/2}}{\pi \cdot (N + 2)}. \tag{1.239}
$$

Stirling’s formula is that as $m \to \infty$, then $m! \to (m/e)^m$, so then the inverse of (1.239) becomes

$$
\lim_{N \to \infty} \frac{V^{2/N}}{\tilde{\mathcal{E}}_x} = 2\pi e . \tag{1.240}
$$
Pulse amplitude modulation, or amplitude shift keying (ASK), uses a one-dimensional constellation with \( M = 2^b \) symbols with \( b \) as a positive integer. Figure 1.39 illustrates the PAM constellation, which is a subset of lowest-energy points from a lattice 0 by \( d/2 \). The basis function can be any unit-energy function, but often \( \varphi_1(t) \) is
\[
\varphi_1(t) = \frac{1}{\sqrt{T}} \text{sinc} \left( \frac{t}{T} \right)
\] (1.241)
or another “Nyquist” pulse shape (see Chapter 3). The data-symbol amplitudes are
\[
\{x\} \in \pm \frac{d}{2}, \pm \frac{3d}{2}, \pm \frac{5d}{2}, \ldots, \pm \frac{(M-1)d}{2}
\] (1.242)
and all input levels are equally likely. The minimum distance between points in a PAM constellation abbreviates as
\[
d_{\text{min}} = d.
\] (1.243)
Both binary antipodal and “2B1Q” are examples of PAM signals.

PAM’s average energy is
\[
\mathcal{E}_x = \bar{\mathcal{E}}_x = \frac{1}{M} \left( \frac{2}{2} \right)^{M/2} \sum_{k=1}^{M/2} \left( \frac{2k - 1}{2} \right)^2 \cdot d^2
\] (1.244)
\[
= \frac{d^2}{2M} \sum_{k=1}^{M/2} (4k^2 - 4k + 1)
\] (1.245)
\[
= \frac{d^2}{2M} \cdot \left[ 4 \left( \frac{(M/2)^3}{3} + \frac{(M/2)^2}{2} + \frac{(M/2)}{6} \right) - 4 \left( \frac{(M/2)^2}{2} + \frac{(M/2)}{2} \right) + \frac{M}{2} \right]
\] (1.246)
\[
= \frac{d^2}{12} \cdot \left[ M^3 - M \right]
\] (1.247)
\[
= \frac{d^2}{12} \cdot [M^2 - 1].
\] (1.248)
The PAM minimum distance is a function of \( \mathcal{E}_x \) and \( M \):
\[
d = \sqrt{\frac{12\bar{\mathcal{E}}_x}{M^2 - 1}}.
\] (1.249)
Finally, given distance and average energy,
\[
\tilde{b} = \log_2 M = \frac{1}{2} \log \left( \frac{12 \bar{\mathcal{E}}_x}{d^2} + 1 \right).
\] (1.250)
Figure 1.39 shows that the decision region for an interior point of PAM extends over a length \( d \) interval centered on that point. The Voronoi Boundary of the constellation thus extends for an interval \([ -\frac{Md}{2}, \frac{Md}{2} ]\). The continuous approximation for PAM assumes a uniform distribution on this interval \([ -\frac{Md}{2}, \frac{Md}{2} ]\), and thus approximates the average energy of the constellation as

\[
E_x = \bar{E}_x \approx \int_{-M/2}^{M/2} \frac{x^2}{2(M/2)} \, dx = \frac{(M/2)^2}{3} = \frac{M^2 d^2}{12}. \tag{1.251}
\]

The continuous approximation for the average energy in (1.251) does not include (1.248)'s constant term \(-\frac{d^2}{12}\), which becomes negligible as \( M \) becomes large.

Since \( M = 2^b \), then \( M^2 = 4^b \), leaving alternative relations \((\bar{b} = b \text{ for } N = 1)\) for (1.248) and (1.249)

\[
E_x = \bar{E}_x = \frac{d^2}{12} \cdot [4^b - 1] = \frac{d^2}{12} \cdot [4^b - 1], \tag{1.252}
\]

and

\[
d = \sqrt{\frac{12E_x}{4^b - 1}}. \tag{1.253}
\]

The following recursion derives from increasing the number of bits, \( b = \bar{b} \), in a PAM constellation while maintaining constant minimum distance between signal points:

\[
\bar{E}_x(b + 1) = 4 \cdot \bar{E}_x(b) + \frac{d^2}{4}. \tag{1.254}
\]

Thus for moderately large \( b \), the required signal energy increases by a factor of 4 for each additional bit in the signal constellation. This corresponds to an increase of 6dB per bit, a measure commonly quoted by communication engineers as the required SNR increase for a transmission scheme to support an additional bit-per-dimension of information (presuming PAM is in use).

The PAM probability of correct symbol detection is

\[
P_c = \sum_{i=0}^{M-1} P_{c|i} \cdot p_x(i) \tag{1.255}
\]

\[
= \frac{M - 2}{M} \cdot \left( 1 - 2Q \left[ \frac{d_{\min}}{2\sigma} \right] \right) + \frac{2}{M} \cdot \left( 1 - Q \left[ \frac{d_{\min}}{2\sigma} \right] \right) \tag{1.256}
\]

\[
= 1 - \left( \frac{2M - 4 + 2}{M} \right) \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] \tag{1.257}
\]

\[
= 1 - 2 \left( 1 - \frac{1}{M} \right) \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] \tag{1.258}
\]

Thus, the PAM (symbol) error probability is

\[
P_e = \bar{P}_e = 2 \left( 1 - \frac{1}{2^b} \right) \cdot Q \left[ \frac{d_{\min}}{2\sigma} \right] < 2Q \left[ \frac{d_{\min}}{2\sigma} \right]. \tag{1.259}
\]

The average number of nearest neighbors for the constellation is \( 2(1 - 1/M) \); thus, the NNUB is exact for PAM. Thus

\[
P_e = 2 \left( 1 - \frac{1}{M} \right) \cdot Q \left( \sqrt{\frac{3}{M^2 - 1} \cdot SNR} \right) \tag{1.260}
\]
For $P_e = 10^{-6}$, $\frac{d}{2\sigma} \approx 4.75$ (13.5dB). Table 1.2 relates $b = \bar{b}$, $M$, $\frac{d}{2\sigma}$, the SNR, and the required increase in SNR (or equivalently in $\bar{E}_x$) to transmit an additional bit at an error probability $P_e = 10^{-6}$. Table 1.2 shows that for $b = \bar{b} > 2$, the approximation of 6dB per bit is very accurate.

Pulse amplitude constellations with $b > 2$ are typically known as 3B1O - three bits per octal signal (for 8 PAM) and 4B1H (4 bits per hexadecimal signal), but are rare in use with respect to the more popular quadrature amplitude modulation of Section 1.3.4.2.

### 1.3.4.2 Quadrature Amplitude Modulation (QAM)

QAM is a two-dimensional generalization of PAM. The two basis functions are usually

\[
\varphi_1(t) = \sqrt{\frac{2}{T}} \text{sinc} \left( \frac{t}{T} \right) \cos \omega_c t \ , \quad (1.261)
\]

\[
\varphi_2(t) = -\sqrt{\frac{2}{T}} \text{sinc} \left( \frac{t}{T} \right) \sin \omega_c t \ . \quad (1.262)
\]

The sinc($t/T$) term may be replaced by any Nyquist pulse shape as discussed in Chapter 3. The $\omega_c$ is a radian carrier frequency that is discussed for in Subsections 1.3.5 and 1.3.6; for now, $\omega_c \geq \pi/T$.

#### 1.3.4.2.1 The QAM Square Constellation

Figure 1.40 illustrates QAM Square Constellations. These constellations are the Cartesian products\(^{24}\) of 2-PAM with itself and 4-PAM with itself, respectively.

\(^{24}\)A Cartesian Product, a product of two sets, is the set of all ordered pairs of coordinates, the first coordinate taken from the first set in the Cartesian product, and the second coordinate taken from the second set in the Cartesian product.
Generally, square $M$-QAM constellations derive from the Cartesian product of two $\sqrt{M}$-PAM constellations. For $\bar{b}$ bits per dimension, the $M = 4^\bar{b}$ signal points are placed at the coordinates $\pm \frac{d}{2}, \pm \frac{3d}{2}, \pm \frac{5d}{2}, \ldots, \pm \frac{(\sqrt{M} - 1)d}{2}$ in each dimension. The average energy of square QAM constellations is easily computed as

$$E_{M-QAM} = E_{\mathbf{x}} = 2E_{\mathbf{x}} = \frac{1}{M} \sum_{i,j=1}^{\sqrt{M}} (x_i^2 + x_j^2)$$

(1.263)

$$= \frac{1}{M} \cdot \left[ \sqrt{M} \sum_{i=1}^{\sqrt{M}} x_i^2 + \sqrt{M} \sum_{j=1}^{\sqrt{M}} x_j^2 \right]$$

(1.264)

$$= 2 \cdot \frac{1}{\sqrt{M}} \cdot \sum_{i=1}^{\sqrt{M}} x_i^2$$

(1.265)

$$= 2E_{\sqrt{M}-PAM}$$

(1.266)

$$= d^2 \left( \frac{M-1}{6} \right)$$

(1.267)
Thus, the average energy per dimension of the $M$-QAM constellation

$$\bar{E}_x = d^2 \left( \frac{M - 1}{12} \right), \quad (1.268)$$

equals the average energy of the constituent $\sqrt{M}$-PAM constellation. The minimum distance $d_{\text{min}} = d$ can be computed from $\bar{E}_x$ (or $\tilde{E}_x$) and $M$ by

$$d = \sqrt{\frac{6\bar{E}_x}{M - 1}} = \sqrt{\frac{12\bar{E}_x}{M - 1}}. \quad (1.269)$$

Since $M = 4^b$, alternative relations for (1.268) and (1.269) in terms of the average bit rate $\bar{b}$ are

$$\bar{E}_x = \frac{\bar{E}_x}{2} = \frac{d^2}{12} \left[ 4^b - 1 \right], \quad (1.270)$$

and

$$d = \sqrt{\frac{12\bar{E}_x}{4^b - 1}}. \quad (1.271)$$

Finally,

$$b = \frac{1}{2} \log_2 \left( \frac{6\bar{E}_x}{d^2} + 1 \right) = \frac{1}{2} \log_2 \left( \frac{12\bar{E}_x}{d^2} + 1 \right), \quad (1.272)$$

the same as for a PAM constellation.

For large $M$, $\bar{E}_x \approx \frac{d^2}{12} M = \frac{d^2}{12} 4^b$, which is the same as that obtained by using the continuous approximation. The continuous approximation for two dimensional QAM uses a uniform constellation over the square defined by $[\pm \sqrt{M}, \pm \sqrt{M}]$,

$$E_x \approx \int_{-\sqrt{M}}^{\sqrt{M}} \int_{-\sqrt{M}}^{\sqrt{M}} \frac{x^2 + y^2}{4L^2} \, dx \, dy = 2 \left( \frac{\sqrt{M}}{2} \right)^2, \quad (1.273)$$

or $\frac{\sqrt{M}}{2} = \sqrt{1.5\bar{E}_x}$. Since the Voronoi region for each signal point in a QAM constellation has area $d^2$

$$M \approx \frac{4 \cdot (\frac{\sqrt{M}}{2})^2}{d^2} = \frac{6\bar{E}_x}{d^2} = \frac{12\bar{E}_x}{d^2}. \quad (1.274)$$

This result agrees with Equation 1.268 for large $M$. As the number of points increases, the energy-computation error caused by using the continuous approximation becomes negligible.

Increasing the number of bits, $b$, in a QAM constellation while maintaining constant minimum distance leads to the following recursion for average-energy increase:

$$E_x(b + 1) = 2 \cdot E_x(b) + \frac{d^2}{6}. \quad (1.275)$$

Asymptotically the average energy increases by 3dB for each added bit per two dimensional symbol.

The probability of error can be exactly computed for QAM by noting that the conditional probability of a correct decision falls into one of 3 categories:

1. corner points (4 points with only 2 nearest neighbors)

$$P_{c|\text{corner}} = \left( 1 - Q \left[ \frac{d}{2\sigma} \right] \right)^2 \quad (1.276)$$

2. inner points ($\sqrt{M} - 2)^2$ points with 4 nearest neighbors)

$$P_{c|\text{inner}} = \left( 1 - 2Q \left[ \frac{d}{2\sigma} \right] \right)^2 \quad (1.277)$$

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3. edge points \(4(\sqrt{M} - 2)\) points with 3 nearest neighbors

\[
P_{c|\text{edge}} = \left(1 - Q \left[\frac{d}{2\sigma}\right]\right) \left(1 - 2Q \left[\frac{d}{2\sigma}\right]\right).
\]

(1.278)

The probability of being correct is then (abbreviating \(Q \leftarrow Q \left[\frac{d}{2\sigma}\right]\))

\[
P_c = \sum_{i=0}^{M-1} P_{c/i} \pi_{\mathbf{x}}(i)
\]

(1.279)

\[
P_c = \frac{4}{M} (1 - Q)^2 + \frac{(\sqrt{M} - 2)^2}{M} (1 - 2Q)^2 + \frac{4(\sqrt{M} - 2)}{M} (1 - 2Q)(1 - Q)
\]

(1.280)

\[
= \frac{1}{M} \left[(4 - 8Q + 4Q^2) + (4\sqrt{M} - 8)(1 - 3Q + 2Q^2)\right]
\]

(1.281)

\[
+ (M - 4\sqrt{M} + 4)(1 - 4Q + 4Q^2)
\]

(1.282)

\[
= \frac{1}{M} \left[M + (4\sqrt{M} - 4M)Q + (4 - 8\sqrt{M} + 4M)Q^2\right]
\]

(1.283)

\[
= 1 + 4(\frac{1}{\sqrt{M}} - 1)Q + 4(\frac{1}{\sqrt{M}} - 1)^2Q^2
\]

(1.284)

Thus, the (symbol) error probability is

\[
P_e = 4 \left(1 - \frac{1}{\sqrt{M}}\right) \cdot Q \left[\frac{d}{2\sigma}\right] - 4 \left(1 - \frac{1}{\sqrt{M}}\right)^2 \cdot \left(Q \left[\frac{d}{2\sigma}\right]\right)^2 < 4 \left(1 - \frac{1}{\sqrt{M}}\right) \cdot Q \left[\frac{d}{2\sigma}\right].
\]

(1.285)

The average number of nearest neighbors for the constellation equals \(4(1 - 1/\sqrt{M})\), thus for QAM the NNUB is not exact, but usually tight. The corresponding normalized NNUB is

\[
P_e \leq 2 \left(1 - \frac{1}{2^b}\right) \cdot Q \left[\frac{d}{2\sigma}\right] = 2 \left(1 - \frac{1}{2^b}\right) \cdot Q \left[\sqrt{\frac{3}{M-1} \text{SNR}}\right],
\]

(1.286)

which equals the PAM result. For \(P_e = 10^{-6}\), one determines that \(\frac{d}{2\sigma} \approx 4.75\) (13.5dB). Table 1.3 relates \(\bar{b}, M, \frac{d}{2\sigma}\), the SNR, and the required increase in SNR (or equivalently in \(\bar{E}_x\)) to transmit an additional bit of information. As with PAM for average bit rates of \(\bar{b} > 2\), the approximation of 3dB per bit per two-dimensional additional for the average energy increase is accurate.

<table>
<thead>
<tr>
<th>(b = 2\bar{b})</th>
<th>(M)</th>
<th>(\frac{d}{2\sigma}) for (\bar{P}<em>e = 10^{-6} \approx 2Q \left[\frac{d</em>{\min}}{2\sigma}\right])</th>
<th>(\text{SNR} = \frac{\left(\frac{d}{2\sigma}\right)}{\left(\frac{d_{\min}}{2\sigma}\right)})</th>
<th>(\text{SNR increase} = \frac{M-1}{(M-1)-1}) dB/bit</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (= 4)</td>
<td>4</td>
<td>13.7dB</td>
<td>13.7dB</td>
<td>—</td>
</tr>
<tr>
<td>4 (= 16)</td>
<td>16</td>
<td>13.7dB</td>
<td>20.7dB</td>
<td>7.0dB</td>
</tr>
<tr>
<td>6 (= 64)</td>
<td>64</td>
<td>13.7dB</td>
<td>27.0dB</td>
<td>6.3dB</td>
</tr>
<tr>
<td>8 (= 256)</td>
<td>256</td>
<td>13.7dB</td>
<td>33.0dB</td>
<td>6.0dB</td>
</tr>
<tr>
<td>10 (= 1024)</td>
<td>1024</td>
<td>13.7dB</td>
<td>39.0dB</td>
<td>6.0dB</td>
</tr>
<tr>
<td>12 (= 4096)</td>
<td>4096</td>
<td>13.7dB</td>
<td>45.0dB</td>
<td>6.0dB</td>
</tr>
<tr>
<td>14 (= 16,384)</td>
<td>16,384</td>
<td>13.7dB</td>
<td>51.0dB</td>
<td>6.0dB</td>
</tr>
</tbody>
</table>

Table 1.3: QAM constellation energies.

The constellation figure of merit for square QAM is

\[
\xi_x = \frac{3}{M - 1} = \frac{3}{4^b - 1} = \frac{3}{2^b - 1}.
\]

(1.287)
When \( b \) is odd, it is possible to define a SQ QAM constellation by taking every other point from a \( b+1 \) SQ QAM constellation. (See Problem 1.14.)

Two examples illustrate the wide use of QAM transmission.

**EXAMPLE 1.3.7 (Cable Modem)** Cable modems use what was an existing cable broadcast-TV systems’ coaxial cables for two-way transmission (presuming the cable TV provider has sent personnel to the various unidirectional blocking points in the network and replaced them with so-called ”diplex” filters). Cable modem conventions (i.e., DOCSIS) use QAM in both directions of transmission. The downstream direction from cable TV end to customer is typically at a carrier frequency well above the used TV band, somewhere between 300 MHz and 500 MHz. The upstream direction is below 50 MHz, typically between 5 and 40 MHz. The symbol rate is typically \( 1/T=2\text{MHz} \) so the data rate is some multiple of 4 Mbps on any given carrier. Typically about 10 carriers can be used (so a multiple of 40 Mbps maximum) for a group of customers with consistent channel characteristics in an immediate neighborhood. Each group is thus shared leading to the famous ”cable” hogging problem when one customer uses all his neighbors’ bandwidth (Cable operators notoriously quote only the peak speed for the group when selling the service, which is misleading if multiple users simultaneously use the system.)

**EXAMPLE 1.3.8 (Satellite TV Broadcast)** Satellite television uses 4QAM in for broadcast transmission at one of 20 carrier frequencies between 12.2 GHz to 12.7 GHz from satellite to customer receiver for some suppliers and satellites. Corresponding carriers between 17.3 and 17.8 GHz are used to send the signals from the broadcaster to the satellite, again with QAM. The symbol rate is \( 1/T=19.151 \text{MHz} \), so the aggregate data rate is a multiple of 38.302 Mbps on any of the 20 carriers. This is sufficient to carry multiple TV stations per carrier/QAM signal. (Some stations watched by many, for instance sports, may get a larger allocation of bandwidth and carry a higher-quality image than others that are not heavily watched. An ultra-high-definition TV channel requires 20-30 Mbps if sent with full fidelity. Each carrier is transmitted in a 24 MHz transponder channel on the satellite – these 24 MHz channels were originally used to broadcast a single analog TV channel, modulated via FM unlike terrestrial analog broadcast television (which uses only 6 MHz for analog TV).
1.3.4.2.2 QAM Cross Constellations  The QAM cross constellation also allows for odd numbers of bits per symbol in QAM data transmission. To construct a QAM cross constellation with $b$ bits per symbol one augments a square QAM constellation for $b - 1$ bits per symbol by adding $2^{b-1}$ data symbols that extend the sides of the QAM square. The corners are excluded as shown in Figure 1.41.

![Figure 1.42: 32CR constellation.](image)

One computation of average energy of QAM cross constellations doubles the energy of the two large rectangles ($[2^{b-3} + 2^{b-1}] \times 2^{b-1}$) and then subtracts the energy of the inner square ($2^{b-1} \times 2^{b-1}$). The energy of the inner square is

$$E_{x(inner)} = \frac{d^2}{6} (2^{b-1} - 1). \quad (1.288)$$

The total sum of energies for all the data symbols in the inner-square-plus-two-side-rectangles is (looking only at one quadrant, and multiplying by 4 because of symmetry)

$$\mathcal{E} = \frac{d^2}{4} \sum_{k=1}^{2^{b-3}} \sum_{l=1}^{2^{b-5}} [(2k - 1)^2 + (2l - 1)^2] \quad (1.289)$$

$$= \frac{d^2}{4} \left[ 3 \cdot 2^{b-5} \left( \frac{2^{b-3} - 2^{b-1}}{6} \right) + 2^{b-3} \left( \frac{27 \cdot 2^{3b-9} - 3 \cdot 2^{b-3}}{6} \right) \right] \quad (1.290)$$

$$= \frac{d^2}{4} \left[ 2^{b-7} \left( \frac{2^{3b-3} - 2^{b-1}}{6} \right) + 2^{b-5} \left( 9 \cdot 2^{3b-9} - 2^{b-3} \right) \right] \quad (1.291)$$

$$= \frac{d^2}{4} \left[ 2^{2b-3} - 2^{b-2} + 9 \cdot 2^{b-5} - 2^{b-2} \right] \quad (1.292)$$

$$= \frac{d^2}{4} \left[ \frac{13}{32} 2^{2b} - 2^{b-1} \right] \quad (1.293)$$

Then

$$\mathcal{E}_x = \frac{2\mathcal{E} - 2^{b-1}E_{x(inner)}}{2^b} = \frac{d^2}{4} \left[ \frac{26}{32} 2^b + \frac{1}{3} 2^{b-2} + \frac{2}{3} \right] \quad (1.294)$$

$$= \frac{d^2}{4} \left( \frac{13}{16} - \frac{1}{6} \right) 2^b - \frac{2}{3} \quad (1.295)$$

$$= \frac{d^2}{4} \left[ \frac{31}{48} 2^b - \frac{2}{3} \right] = \frac{d^2}{6} \left[ \frac{31}{32} M - 1 \right] \quad (1.296)$$
The minimum distance \( d_{\text{min}} = d \) can be computed from \( E_{\mathcal{X}} \) (or \( \tilde{E}_{\mathcal{X}} \)) and \( M \) by

\[
d = \sqrt{\frac{6E_{\mathcal{X}}}{\frac{31}{32} M - 1}} = \sqrt{\frac{12E_{\mathcal{X}}}{\frac{31}{32} M - 1}} = \sqrt{\frac{12\tilde{E}_{\mathcal{X}}}{\frac{31}{32} 4^b - 1}}.
\] (1.297)

In (1.296), for large \( M \), \( E_{\mathcal{X}} \approx \frac{31}{192} M = \frac{31}{192} 4^b \), the same as the continuous approximation.

The following recursion derives from increasing the number of bits, \( b \), in a QAM cross constellation while maintaining constant minimum distance:

\[
E_{\mathcal{X}}(b + 1) = 2 \cdot E_{\mathcal{X}}(b) + \frac{d^2}{6}.
\] (1.298)

As with the square QAM constellation asymptotically the average energy increases by 3 dB for each added bit per two dimensional symbol.

The probability of error can be bounded for QAM Cross by noting that a lower bound on the conditional probability of a correct decision falls into one of two categories:

1. inner points \( \left\{ 2^b - 4 \left( 3 \cdot 2^{\frac{b-1}{2}} - 2 \cdot 2^{\frac{b-3}{2}} \right) \right\} = \left\{ 2^b - 4 \left( 2^{\frac{b+1}{2}} \right) \right\} \) with four nearest neighbors

\[
P_{\text{c/inner}} = \left( 1 - 2Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \right)^2
\] (1.299)

2. side points \( 4 \left( 3 \cdot 2^{\frac{b-1}{2}} - 2 \cdot 2^{\frac{b-3}{2}} \right) = 4 \left( 2^{\frac{b+1}{2}} \right) \) with three nearest neighbors. (This calculation is only a bound because some of the side points have fewer than three neighbors at distance \( d_{\text{min}} \))

\[
P_{\text{c/outer}} = \left( 1 - Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \right) \left( 1 - 2Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \right).
\] (1.300)

The probability of a correct decision is then, abbreviating \( Q = Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \),

\[
P_c \geq \frac{1}{M} \left[ 4 \left( 2^{\frac{b+1}{2}} \right) (1 - Q)(1 - 2Q) \right] + \frac{1}{M} \left[ \left( 2^b - 4 \left( 2^{\frac{b+1}{2}} \right) \right)^2 \right] \] (1.301)

\[
= \frac{1}{M} \left[ 4 \cdot 2^{\frac{b+1}{2}} (1 - 3Q + 2Q^2) + \left( 2^b - 2^{\frac{b+1}{2}} \right) (1 - 4Q + 4Q^2) \right]
\] (1.302)

\[
= 1 - \left[ -2^{\frac{b+1}{2}} + 4 \right] Q + \left[ 2^{\frac{b+1}{2}} - 2 \cdot 2^{\frac{b+3}{2}} + 4 \right] Q^2
\] (1.303)

Thus, the probability of symbol error is bounded by

\[
P_c \leq 4 \left( 1 - \frac{1}{\sqrt{2M}} \right) Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] - 4 \left( 1 - \sqrt{\frac{2}{M}} \right) \left( Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] \right)^2
\] (1.304)

\[
< 4 \left( 1 - \frac{1}{\sqrt{2M}} \right) Q \left[ \frac{d_{\text{min}}}{2\sigma} \right] < 4Q \left[ \frac{d_{\text{min}}}{2\sigma} \right].
\] (1.305)

The average number of nearest neighbors for the constellation is \( 4(1 - 1/\sqrt{2M}) \); thus the NNUB is again accurate. The normalized probability of error is

\[
P_{\text{c}} \leq 2 \left( 1 - \frac{1}{2^{b+5}} \right) Q \left[ \frac{d_{\text{min}}}{2\sigma} \right],
\] (1.306)

which agrees with the PAM result when one includes an additional bit in the constellation, or equivalently an extra .5 bit per dimension. To evaluate (1.307), Equation 1.297 relates that

\[
\left( \frac{d_{\text{min}}}{2\sigma} \right)^2 = \frac{3 \text{ SNR}}{\frac{31}{32} M - 1}
\] (1.307)
Table 1.4 lists the incremental energies and required SNR for QAM cross constellations in a manner similar to Table 1.3. There are also square constellations for odd numbers of bits that Problem 1.14 addresses.

<table>
<thead>
<tr>
<th>$b = 2b$</th>
<th>$M$</th>
<th>$2\frac{d}{\sigma}$ for $P_e = 10^{-6} \approx 2Q\left(\frac{d_{\min}}{2\sigma}\right)$</th>
<th>$\text{SNR} = \frac{(31/32) \cdot M - 1}{(31/32) \cdot (M - 1) - 1}$</th>
<th>$\text{SNR increase} = \frac{(31/32) \cdot M - 1}{(31/32) \cdot (M - 1) - 1}$</th>
<th>dB/bit</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>32</td>
<td>13.7dB</td>
<td>23.7 dB</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>13.7dB</td>
<td>29.8dB</td>
<td>6.1dB</td>
<td>3.05dB</td>
</tr>
<tr>
<td>9</td>
<td>512</td>
<td>13.7dB</td>
<td>35.8dB</td>
<td>6.0dB</td>
<td>3.0dB</td>
</tr>
<tr>
<td>11</td>
<td>2048</td>
<td>13.7dB</td>
<td>41.8dB</td>
<td>6.0dB</td>
<td>3.0dB</td>
</tr>
<tr>
<td>13</td>
<td>8192</td>
<td>13.7dB</td>
<td>47.8dB</td>
<td>6.0dB</td>
<td>3.0dB</td>
</tr>
<tr>
<td>15</td>
<td>32,768</td>
<td>13.7dB</td>
<td>53.8dB</td>
<td>6.0dB</td>
<td>3.0dB</td>
</tr>
</tbody>
</table>

Table 1.4: QAM Cross constellation energies.

1.3.4.2.3 Vestigial Sideband Modulation (VSB), CAP, and OQAM

From the perspective of performance and constellation design, there are many alternate basis function choices for QAM that are equivalent. These choices sometimes have value from the perspective of implementation considerations. They are all equivalent in terms of this section’s AWGN fundamentals when implemented for successive message transmission. In successive transmission, the basis functions must be orthogonal to one another for all integer-symbol-period time translations. Then successive samples at the demodulator output at integer multiples of $T$ will be independent; then also, the one-shot optimum receiver can be used repeatedly in succession to detect successive messages optimally on the AWGN (see Chapter 3 successive transmission degradation in the presence of “intersymbol interference” on band-limited AWGN channels.)

The PAM basis function always exhibits this desirable translation property on the AWGN, and so do the QAM basis functions as long as $\omega_c \geq \pi/T$. The QAM basis functions are not unique with respect to satisfaction of the translation property, with VSB/SSB, CAP, and OQAM all being variants:

VSB Vestigial sideband modulation (VSB) is an alternative modulation method that is equivalent to QAM. In QAM, typically the same unit-energy basis function ($\sqrt{1/T} \cdot \text{sinc}(t/T)$) is “double-sideband modulated” independently by the same carrier’s sine and cosine to generate the two QAM basis functions. In VSB, a double bandwidth sinc function and its Hilbert transform (see Section 1.3.5 for a discussion of Hilbert transforms) are “single-side-band modulated.”

$$\varphi_1(t) = \sqrt{\frac{1}{T}} \cdot \text{sinc}\left(\frac{2t}{T}\right) \cdot \cos \omega_c t$$

$$\varphi_2(t) = \sqrt{\frac{1}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right) \cdot \sin \left(\frac{\pi t}{T}\right) \cdot \sin \omega_c t$$

A natural symbol-rate choice for successive transmission with these two basis functions might appear to be $2/T$, twice the rate associated with QAM. However, these basis functions’ successive translations by integer multiples of $T/2$ are not orthogonal – that is $<\varphi_1(t), \varphi_j(t - kT/2)> \neq \delta_{ij}$; however, $<\varphi_i(t), \varphi_j(t - kT)> = \delta_{ij}$ for any integer $k$. Thus, the symbol rate for successive orthogonal transmissions needs to be $1/T$.

VSB designers often prefer to exploit the observation that $<\varphi_1(t), \varphi_2(t - kT/2)> = 0$ for all odd integers $k$ to implement the VSB transmission system as a time-varying one-dimensional modulation at rate $2/T$ dimensions per second. Thus, the modulator uses a different basis function on adjacent symbol periods, alternating between the two. The optimum receiver consists of two matched filters to the two basis functions, which have their outputs each sampled at rate $1/T$ (staggered relative to

25This simple description is actually single-side-band (SSB), a special case of VSB. VSB uses practical realizable functions instead of the unrealizable sinc functions that simplify fundamental developments here in Chapter 1.
one another by $T/2$). The detector interleaves these samples to form a single one-dimensional detected-symbol stream. Nonetheless, different VSB designers may call the VSB constellations by two-dimensional names: For instance, one may hear of 16 VSB or 64 VSB, which are equivalent to 16SQ QAM (or 4PAM) and 64SQ QAM (8PAM) respectively. VSB transmission was initially more convenient for upgrading existing analog systems that were already VSB (i.e., commercial broadcast television before transition to all-digital for instance) to digital systems that use the same bandwidths and carrier frequencies - that is where the carrier frequencies are not centered within the existing band. VSB otherwise has no fundamental performance advantages or differences from QAM.

**CAP** Carrierless Amplitude/Phase (CAP) transmission systems are also very similar to QAM. The basis functions of QAM are time-varying when $\omega_c$ is arbitrary – that is, the basis functions on subsequent transmissions may differ. CAP is a method that can eliminate this time variation for any choice of carrier-frequency, making the combined transmitter implementation appear “carrierless” and thus time-invariant. CAP has the same one-shot basis functions as QAM, but also has a time-varying encoder constellation when used for successive transmission of two-dimensional symbols. The time-varying CAP encoder implements a sequence of additional two-dimensional constellation rotations that are known and easily removed at the receiver after the demodulator and just before the detector. The time-varying encoder usually selects the sequence of rotations so that the initial carrier phase (argument of sines and cosines) is the same for each symbol period, regardless of the actual carrier frequency. Effectively, all carrier frequencies thus appear the same, hence the term “carrierless.” The sequence of rotations has an angle that increases linearly with time and can often be very easily implemented (and virtually omitted when differential encoding - see Subsection 1.3.6 - is implemented). See Section 2.4.

**OQAM** Offset QAM (OQAM) or “staggered” QAM uses the alternative basis functions

\begin{align}
\varphi_1(t) &= \sqrt{\frac{2}{T}} \cdot \text{sinc}\left(\frac{t}{T}\right) \cdot \cos\left(\frac{\pi t}{T}\right) \\
\varphi_2(t) &= -\sqrt{\frac{2}{T}} \cdot \text{sinc}\left(\frac{t - T/2}{T}\right) \cdot \sin\left(\frac{\pi t}{T}\right)
\end{align}

(1.311)

(1.312)

effectively “offseting” the two dimensions by $T/2$. For one-shot transmission, such offset has no effect (the receiver matched filters effectively re-align the two dimensions) and OQAM and QAM are the same. For successive transmission, the derivative (rate of change) of $x(t)$ is less for OQAM than for QAM, effectively reducing transmitted signals' spurious bandwidth when the sinc functions cannot be perfectly implemented. OQAM signals will never take the value $x(t) = 0$, while this value is instantaneously possible with QAM – thus nonlinear transmitter/receiver amplifiers are not as stressed by OQAM. There is otherwise no fundamental performance difference between OQAM and QAM.

**1.3.4.2.4 Forney’s Gap** The gap, $\Gamma$, is an approximation introduced by Forney for constellations with $\bar{b} \geq 1/2$ that is empirically evident in the PAM and QAM tables. Specifically, if one knows the SNR for an AWGN channel, the number of bits that can be transmitted with PAM or QAM according to

\[ \bar{b} = \frac{1}{2} \log_2 \left( 1 + \frac{\text{SNR}}{\Gamma} \right) \]  

(1.313)

At error rate $P_e = 10^{-6}$, the gap is 8.8 dB. For $P_e = 10^{-7}$, the gap is 9.5 dB. If the designer knows the SNR and the desired performance level ($P_e$) or equivalently the gap, then the number of bits per dimension (and thus the achievable data rate $R = b/T$) are immediately computed. Chapters 2, 8, and ?? will introduce more sophisticated encoder designs where the gap can be reduced, ultimately to 0 dB, enabling a highest possible data rate of $.5 \log_2(1 + \text{SNR})$, sometimes known as the AWGN’s “channel capacity.” QAM and PAM are thus about 9 dB away in terms of efficient use of SNR from ultimate limits.
1.3.4.3 Constellation Performance Measures

Having introduced many commonly used signal constellations for data transmission, several performance measures compare coded systems based on these constellations.

1.3.4.3.1 Coding Gain

Of fundamental importance to the comparison of two systems that transmit the same number of bits per dimension is the **coding gain**, which specifies the improvement of one constellation over another when used to transmit the same information.

\[
\gamma = \left( \frac{d_{\text{min}}^2(\mathbf{x}) / \mathcal{E}_x}{d_{\text{min}}^2(\mathbf{\bar{x}}) / \mathcal{E}_{\bar{x}}} \right) = \frac{\zeta_x}{\zeta_{\bar{x}}},
\]

(1.314)

where both constellations are used to transmit \( \bar{b} \) bits of information per dimension.

A coding gain of \( \gamma = 1 \) (0dB) implies that the two systems perform equally. A positive gain (in dB) means that the constellation with data symbols \( \mathbf{x} \) outperforms the constellation with data symbols \( \mathbf{\bar{x}} \). The coding gain effectively causes the \( \bar{E}_x \) to be the same in both systems through normalization to it in both the numerator and denominator of (1.314). Thus it is a fair comparison when with the same \( \bar{b} \) for both systems. An example compares the two constellations in Figures 1.30 and 1.32 and obtains

\[
\gamma = \frac{\zeta_x(8\text{AMPM})}{\zeta_x(8\text{PSK})} = \frac{2}{\sin^2(\pi/8)} \approx 1.37 \text{ (1.4dB)}.
\]

(1.315)

A **lattice** is a set of vectors in \( N \)-dimensional space that is closed under vector addition – that is, the sum of any two vectors is another vector in the set. A translation of a lattice produces a **coset** of the lattice. Most good signal constellations are chosen as subsets of cosets of lattices. The **fundamental volume** for a lattice measures the region around a point:

\[
\text{Definition 1.3.20 (Fundamental Volume)} \quad \text{The fundamental volume} \ V(\Lambda) \ \text{of a lattice} \ \Lambda \ \text{(from which a signal constellation is selected) is the volume of the decision region for any single point in the lattice. This decision region is the lattice’s previously defined Voronoi Region,} \ \mathcal{V}(\Lambda), \ \text{with volume} \ V(\Lambda) = |\mathcal{V}(\Lambda)|. \ \text{A lattice’s Voronoi Region,} \ \mathcal{V}(\Lambda), \ \text{differs from the constellation’s Voronoi Boundary,} \ \mathcal{V}_x, \ \text{with the latter being the union of} \ M \ \text{of the former} \ V(\Lambda). \ \mathcal{V}_x \ \text{may follow a different (“shaping”) lattice} \ \Lambda_s \ \text{that is not equal, to or even a scaled version of,} \ \Lambda \ (\text{which is then called the coding lattice}).
\]

For example, an \( M\)-QAM constellation as \( M \to \infty \) is a translated subset (coset) of the two-dimensional rectangular lattice \( Z^2 \), so \( M\)-QAM is a translation of \( Z^2 \) as \( M \to \infty \). Similarly as \( M \to \infty \), the \( M\)-PAM constellation becomes a coset of the one dimensional lattice \( Z \).

The coding gain, \( \gamma \) of one constellation based on \( x \) with lattice \( \Lambda \) and volume \( V(\Lambda) \) with respect to another constellation with \( \bar{x}, \bar{\Lambda}, \) and \( V(\bar{\Lambda}) \) can be rewritten as

\[
\gamma = \left( \frac{d_{\text{min}}^2(\mathbf{x}) / V^{2/N}(\Lambda)}{d_{\text{min}}^2(\mathbf{\bar{x}}) / V^{2/N}(\bar{\Lambda})} \right) \cdot \left( \frac{V^{2/N}(\Lambda)}{V^{2/N}(\bar{\Lambda})} \right) = \gamma_f + \gamma_s \quad (dB)
\]

(1.316)
The two quantities on the right in (1.317) are called the **fundamental gain** $\gamma_f$ and the **shaping gain** $\gamma_s$ respectively.

**Definition 1.3.21 (Fundamental Gain)** The fundamental gain $\gamma_f$ of a lattice, upon which a signal constellation is based, is

$$
\gamma_f \triangleq \frac{\left( x^2 \min \right)_{\Lambda}}{\left( \bar{V}^2/N(\Lambda) \right)}
$$

(1.318)

The fundamental gain measures the efficiency of the spacing of the points within a particular constellation per unit of fundamental volume surrounding each point.

**Definition 1.3.22 (Shaping Gain)** The shaping gain $\gamma_s$ of a signal constellation is defined as

$$
\gamma_s = \frac{\left( V^{2/N}(\Lambda) \right)_{\bar{x}}}{\left( \bar{V}^{2/N}(\Lambda) \right)_{\bar{x}}}
$$

(1.319)

The shaping gain measures the efficiency of the shape of the boundary of a particular constellation in relation to the average energy per dimension required for the constellation.

Using a continuous approximation, the designer can extend shaping gain to constellations with different numbers of points as

$$
\gamma_s = \left( \frac{V^{2/N}(\Lambda)}{\bar{V}^{2/N}(\Lambda)} \right)_{\bar{x}} \cdot 2^{\bar{b}(\bar{x})}
$$

(1.320)

When the same underlying lattice of points is used in both reference and constellation under evaluation, the shaping gain is the ratio of energies, effectively measuring how constellation’s outer boundary efficiency in terms of squeezing more points into the average constellation/symbol energy.

Using (1.240) and comparing the shaping gain of a hypersphere with large number of dimensions versus repeated PAM, or SQ QAM, use finds the limiting shaping gain of a hypersphere with respect to a hypercube is

$$
\gamma_s \to \frac{2\pi e}{\pi^2/2} = \frac{\pi e}{6} = 1.53 \text{ dB}
$$

(1.321)

a best possible shaping improvement.

### 1.3.4.3.2 Peak-to-Average Power Ratio (PAR)

For practical system design, the system’s peak power may also need to be limited. This constraint can manifest itself in several different ways. For example if the modulator uses a Digital-to-Analog Converter (or Analog-to-Digital Converter for the demodulator) with a finite number of bits (or finite dynamic range), then the signal peaks can not be arbitrarily large. In other systems the channel or modulator/demodulator may include amplifiers or repeaters that saturate at high peak signal voltages. Yet another way is in adjacent channels where crosstalk exists and a high peak on one channel can couple into the other channel, causing an impulsive noise hit and an unexpected error in the adjacent system. Thus, the Peak-to-Average Power Ratio (PAR) is a measure of immunity to these important types of effects.

The peak energy is:
**Definition 1.3.23 (Peak Energy)** The $N$-dimensional peak energy for any signal constellation is $\mathcal{E}_{\text{peak}}$.

$$\mathcal{E}_{\text{peak}} = \max_i \sum_{n=1}^{N} x_{in}^2.$$  

(1.322)

The peak energy of a constellation should be distinguished from the peak squared energy of a signal $x(t)$, which is $\max_{i,t} |x_i(t)|^2$. This later quantity is important in analog amplifier design or equivalently in however the filters $\varphi_n(t)$ are implemented.

The peak energy of a constellation concept allows precise definition of the PAR:

**Definition 1.3.24 (Peak-to-Average Power Ratio)** The $N$-dimensional Peak-to-Average Power Ratio, $\text{PAR}_x$, for $N$-dimensional Constellation is

$$\text{PAR}_x = \frac{\mathcal{E}_{\text{peak}}}{\mathcal{E}_x}$$  

(1.323)

For example 16SQ QAM has a PAR of 1.8 in two dimensions. For each of the one-dimensional 4-PAM constellations that constitute a 16SQ QAM constellation, the one-dimensional PAR is also 1.8. These two ratios need not be equal, however, in general. For instance, for 32CR, the two-dimensional PAR is $34/20 = 1.7$, while observation of a single dimension when 32CR is used gives a one-dimensional PAR of $25/(.75(5) + .25(25)) = 2.5$. Typically, the peak squared signal energy is inevitably yet higher in QAM constellations and depends on the choice of $\varphi(t)$.

### 1.3.4.4 Hexagonal Signal Constellations in 2 Dimensions

![Hexagonal lattice](image)

Figure 1.43: Hexagonal lattice.
The most dense packing of regularly spaced points in two dimensions is the hexagonal lattice shown in Figure 1.43. The volume (area) of the decision region for each point is

\[ V = 6 \left( \frac{1}{2} \right) \left( \frac{d}{2} \right) \left( \frac{d}{\sqrt{3}} \right) = \frac{d^2 \sqrt{3}}{2}. \]  
(1.324)

If the minimum distance between any two points is \( d \) in both constellations, then the fundamental gain of the hexagonal constellation with respect to the QAM constellation is

\[ \gamma_f = \frac{d^2 \sqrt{3}}{2} \frac{2}{\sqrt{3}} = 0.625 \text{dB}. \]  
(1.325)

The encoder/detector for constellations based on the hexagonal lattice may be more complex than those for QAM.

1.3.5 Baseband Modulation

Baseband modulation uses basis functions with most energy at low frequencies. The majority of modulation methods in Sections 1.2 - 1.3 so far are baseband, although a few (like PSK and QAM) use basis functions that have energy centered at or near a carrier or center frequency \( \omega_c = 2\pi f_c \). These latter passband modulation methods are useful in many applications where transmission occurs over a limited narrow bandwidth, typically centered at or near the carrier frequency of the passband modulation. Digital television transmission on Channel 2 in the US has carrier frequency 52 MHz and non-negligible energy only from 50 to 56 MHz. (Channels 3 through 60 typically are also 6 MHz wide using carrier frequencies of 52 + 6 MHz with \( i \) some positive integer.) Cellphones use carrier frequencies from 600 MHz to 70 GHz, but have nonzero energy over a narrow band that is typically from 1 MHz to 100 MHz wide. LTE transmission systems effectively combine many narrow carriers, each of width typically 15 kHz wide as addressed in Chapter 4. Digital satellite transmission uses QAM and carriers in the 12 and 17 GHz bands with transponder bandwidths of about 26 MHz. There are numerous other examples. Signal energy is present only in these narrow “passbands” and consequently subject to filtering. This section teaches a common analysis method for such systems without explicit need for the carrier frequency, nor its inclusion in the basis functions, nor even in the channel transfer function. This theory of passband system analysis allows a framework for later chapters’ important suboptimal receivers for both baseband and passband modulation.

![Figure 1.44: The filtered AWGN.](image)

Especially in passband transmission-system design, the channel is often band-limited so that the AWGN model of earlier sections adds a filter as in Figure 1.44. The filter \( h(t) \) represents the physical
channel's band-limiting effect, which may be caused by filters in the transmission path that create the passband channel or by natural finite-bandwidth constraints of transmission lines or wireless multiple-path connections. The filter $h(t)$ distorts on the transmitted modulated signal $x(t)$. Preferably, $x(t) * h(t) \approx x(t)$, but the designer may not be able to ensure small distortion. High frequencies are inevitably attenuated in all channels, but many channels also attenuate low frequencies. Furthermore, different frequencies may have different levels of attenuation in real channels. Whether modeling filters or actual physical effects, an imperfect channel impulse response $h(t)$ will affect transmission performance.

The new complex-baseband models that are here developed will apply to either the real baseband case (like PAM), where trivially all imaginary components are zero, and to the passband case (like QAM) where all imaginary components are not necessarily zero, allowing a single complex-symbol-vector theory of receiver processing in the remainder of this text.

1.3.5.1 Passband Representations and Terminology

A passband signal has energy concentrated in the vicinity of a frequency $\omega_c = 2\pi f_c$ in anticipation of transmission through a passband channel that only passes energy in this same frequency band. Thus, passband signals are designed for passband channels. Passband signals usually have been generated through multiplication of a “lowpass” signal by a sinusoid to move the energy away from low frequencies towards the frequency band around $\omega_c$. Such passband modulation is used on channels that do not pass DC or on channels that several signals simultaneously share in non-overlapping frequency bands (and thus have different carrier frequencies).

This subsection first investigates a number of equivalent representations of a passband signal, the most interesting of which is the baseband-equivalent signal in Subsection 1.3.5.1.1. The design replaces the original modulated passband signal with the baseband-equivalent signal in transmission analysis. Subsection 1.3.5.1.1’s objective is the generation of such an equivalent signal from the original signal. Subsection 1.3.5.1.2 studies the frequency content of baseband-equivalent signals, essentially showing that amplitude is doubled and translated to DC. Since all baseband signals center transmitted energy at DC, a common baseband-processing method can be applied, for any passband channel, as in Subsection 1.3.5.1.4. Figure 1.50 is a quick summary of this entire subsection.

1.3.5.1.1 Passband Signal Equivalents

The real-valued signal $x(t)$ is a passband signal when its nonzero Fourier transform is near $\omega_c$, as in Figure 1.45. Passband signals never have DC content, so $X(0) = 0$.

**Definition 1.3.25 (Carrier-Modulated Signal)** A carrier-modulated signal is any passband signal that can be written in the following form

$$x(t) = a(t) \cdot \cos(\omega_c t + \theta(t)),$$

where $a(t)$ is the modulated signal’s time-varying amplitude or envelope and $\theta(t)$ is its time-varying phase. $\omega_c$ is the carrier frequency (in radians/sec).

The carrier frequency $\omega_c$ is chosen sufficiently large compared with respect to the amplitude and phase variations of $a(t)$ so that the power spectral density does not have significant energy at $\omega = 0$. See Figure 1.45, wherein the spectrum of $X(\omega)$ is in the passband $\omega_{low} < |\omega| < \omega_{high}$. In digital communication, $x(t)$ is equivalently written in quadrature form using the trigonometric identity $\cos(u + v) = \cos(u) \cos(v) - \sin(u) \sin(v)$, leading to a quadrature decomposition:

**Definition 1.3.26 (Quadrature Decomposition)** The quadrature decomposition of a carrier modulated signal is

$$x(t) = x_I(t) \cdot \cos(\omega_c t) - x_Q(t) \cdot \sin(\omega_c t),$$

where $x_I(t)$ and $x_Q(t)$ are the in-phase and quadrature components, respectively.
where $x_I(t) = a(t) \cdot \cos \left( \theta(t) \right)$ is modulated signal’s time-varying inphase component,
and $x_Q(t) = a(t) \cdot \sin \left( \theta(t) \right)$ is its time-varying quadrature component.

Relationships determining $(a(t), \theta(t))$ from $(x_I(t), x_Q(t))$ are

$$a(t) = \sqrt{x_I^2(t) + x_Q^2(t)} \quad ,$$

(1.328)

and

$$\theta(t) = \tan^{-1} \left[ \frac{x_Q(t)}{x_I(t)} \right] \quad .$$

(1.329)

In (1.329), the inverse tangent applies with known individual polarities of both numerator and denominator, so there is no quadrant ambiguity in computing $\theta(t)$.

In passband processing and analysis, the objective is to eliminate explicit consideration of the carrier frequency $\omega_c$ and directly analyze systems using only the inphase and quadrature components. These inphase and quadrature components can be combined into a two-dimensional vector, or into an equivalent complex signal. By convention, a graph of a quadrature-modulated signal plots the inphase component along the real axis and the quadrature component along the imaginary axis as shown in Figure 1.46.

The resultant complex vector $x_{bb}(t)$ is the complex baseband-equivalent signal.

**Definition 1.3.27 (Baseband-Equivalent Signal)** The complex baseband-equivalent signal for $x(t)$ in (1.326) is

$$x_{bb}(t) \stackrel{\Delta}{=} x_I(t) + jx_Q(t) \quad ,$$

(1.330)

where $j = \sqrt{-1}$.
The baseband-equivalent signal expression no longer explicitly contains the carrier frequency \( \omega_c \). Another complex representation that explicitly uses \( \omega_c \) is the analytic\(^{26}\) equivalent signal for \( x(t) \):

\[
x_A(t) \triangleq x_{bb}(t) \cdot e^{j\omega_c t}.
\]  

**Definition 1.3.28 (Analytic-Equivalent Signal)** The analytic-equivalent signal for \( x(t) \) in (1.326) is

\[
x_A(t) \triangleq x_{bb}(t) \cdot e^{j\omega_c t}.
\]  

The original real-valued passband signal \( x(t) \) is the real part of the analytic equivalent signal:

\[
x(t) = \Re \{x_A(t)\}.
\]  

The Hilbert transform of \( x(t) \), denoted by \( \hat{x}(t) \), is the imaginary part of the analytic signal as

\[
\hat{x}(t) = 3 \{x_A(t)\}.
\]  

(See Appendix C for more details on the Hilbert transform and a proof of (1.333).) Finally, the inphase component \( x_I(t) \) and the quadrature component \( x_Q(t) \) can be expressed using the signal \( x(t) \) and its Hilbert transform \( \hat{x}(t) \) as (using \( x_{bb}(t) = x_I(t) + jx_Q(t) = x_A(t) \cdot e^{-j\omega_c t} \)):

\[
x_I(t) = x(t) \cdot \cos(\omega_c t) + \hat{x}(t) \cdot \sin(\omega_c t) \quad (1.334)
\]
\[
x_Q(t) = \hat{x}(t) \cdot \cos(\omega_c t) - x(t) \cdot \sin(\omega_c t) \quad (1.335)
\]

Thus, four equivalent forms for representing a real passband signal \( x(t) \) with carrier frequency \( \omega_c \) are:

1. magnitude, phase \( a(t), \theta(t) \)
2. inphase, quadrature \( x_I(t), x_Q(t) \)
3. complex baseband \( x_{bb}(t) \)
4. analytic \( x_A(t) \)

**EXAMPLE 1.3.9 (Translation between equivalent representations:)** A passband QAM signal is

\[
x(t) = \operatorname{sinc}(10^6t) \cdot \cos(2\pi10^7t) + 3 \cdot \operatorname{sinc}(10^6t) \cdot \sin(2\pi10^7t) \quad .
\]  

The carrier frequency is 10 MHz and the symbol period is 1 \( \mu s \). The inphase and quadrature components are

\[
x_I(t) = \operatorname{sinc}(10^6t) \quad (1.338)
\]
\[
x_Q(t) = -3 \cdot \operatorname{sinc}(10^6t) \quad (1.339)
\]

so

\[
x_{bb}(t) = (1 - 3j) \cdot \operatorname{sinc}(10^6t) \quad .
\]  

The amplitude and phase of the complex baseband signal are

\[
a(t) = \sqrt{10} \cdot \operatorname{sinc}(10^6t) \quad (1.341)
\]
\[
\theta(t) = \tan^{-1}\left[-\frac{3}{1}\right] = -71.6^\circ \quad .
\]  

Thus,

\[
x(t) = \sqrt{10} \cdot \operatorname{sinc}(10^6t) \cdot \cos(\omega_c t - 71.6^\circ) \quad .
\]  

Finally,

\[
x_A(t) = (1 - 3j) \cdot \operatorname{sinc}(10^6t) \cdot e^{2\pi10^7t} \quad .
\]  

Subsubsection 1.3.5.1.2 next considers the relationship of the Fourier transforms of \( x(t) \), \( x_{bb}(t) \), and \( x_A(t) \).
1.3.5.1.2 Frequency Spectrum of Analytic- and Baseband-Equivalent Signals

Using Equations (1.332) and (1.333) the analytic signal is represented as shown in Figure 1.47.

\[ x_A(t) = x(t) + j\tilde{x}(t) \]  

(1.345)

![Figure 1.47: Analytic signal composition.](image)

Taking the Fourier Transform of both sides of (1.345) yields\(^{27}\)

\[ X_A(\omega) = \begin{cases} 
1 + \text{sgn}(\omega) \cdot X(\omega) & \text{for } \omega > 0 \\
2 \cdot X(\omega) & \omega = 0 \\
0 & \omega < 0
\end{cases} \]  

(1.346)

(1.347)

The analytic equivalent signal, \( x_A(t) \), has nonzero value only for the positive frequencies of \( x(t) \) and is identically zero for negative frequencies. The real signal \( x(t) \)'s Fourier transform \( X(\omega) \) has two symmetry properties: The real part \( \Re\{X(\omega)\} \) is even in \( \omega \), while the imaginary part \( \Im\{X(\omega)\} \) is odd in \( \omega \). Knowledge of only the non-negative frequencies of \( X(\omega) \), such as are supplied by the analytic signal, is sufficient for reconstruction of \( X(\omega) \). This confirms that the analytic signal \( x_A(t) \) is truly “equivalent” to the original signal \( x(t) \).

Using Equation (1.331), the Fourier transform of the baseband equivalent signal is simply the Fourier transform of the analytic signal translated in frequency \( \omega_c \). Thus

\[ X_A(\omega) = X_{bb}(\omega - \omega_c) \]  

(1.348)

and

\[ X_{bb}(\omega) = X_A(\omega + \omega_c) \]  

(1.349)

Use of (1.331) and (1.332) allows reconstruction of the signal \( x(t) \) from the baseband equivalent signal \( x_{bb}(t) \) and the carrier frequency \( \omega_c \). The baseband equivalent signal, in general, may be complex-valued, and thus as shown in Figure 1.48 the spectrum of \( x_{bb}(t) \) may be asymmetric about the origin \( \omega = 0 \).

![Figure 1.48: Baseband signal spectrum.](image)

\(^{27}\)If \( \tilde{x}(t) \) is the Hilbert transform of \( x(t) \), then the Fourier transform of \( \tilde{x}(t) \) is \(-j\text{sgn}(\omega)X(\omega)\), where \( X(\omega) \) is the Fourier Transform of \( x(t) \), as shown in Appendix C.
EXAMPLE 1.3.10 (Continuing Example) Figure 1.49 shows the original, baseband, and analytic equivalent spectra of the signal

\[ x(t) = \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + 3\text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t) \]  

The doubling in amplitude of the two complex signals’ Fourier transforms occurs because all energy in these complex representations appears in a single positive-frequency band.
1.3.5.1.5 Equivalent representations of the channel response. Any of the four representations found for passband signals in the previous section apply to the impulse response or Fourier transform for the channel by using the same equations and substituting \( h \) for \( x \). For instance, linear time-invariant channels can be described by a real-valued impulse response \( h(t) \). For any \( h(t) \), the analytic-equivalent channel \( h_A(t) \) is

\[
h_A(t) \triangleq h(t) + j\tilde{h}(t) .
\]

Similarly in the frequency domain,

\[
H_A(\omega) = \{1 + \text{sgn}(\omega)\} \cdot H(\omega) .
\]

The baseband-equivalent channel is defined in the same manner as a baseband equivalent signal, except the carrier frequency \( \omega_c \) is set equal to that of the input, and output, signals.

**Definition 1.3.29 (Baseband Equivalent Channel (at carrier frequency \( \omega_c \)))**

_The baseband equivalent channel at any carrier frequency \( \omega_c \) is given by_

\[
h_{bb}(t) \triangleq h_A(t) \cdot e^{-j\omega_c t} .
\]

For valid application of the term “baseband equivalent”, the carrier frequency should be sufficiently large to guarantee that \( h_{bb}(t) \) has no significant energy content at frequencies \( |\omega| > \omega_c \), i.e., \( |H_{bb}(\omega)| = 0 \forall |\omega| > \omega_c \).

1.3.5.1.6 The equivalent views of channel input/output relations The frequency-domain representation of the passband system at the top of Figure 1.50 is,

\[
Y(\omega) = H(\omega) \cdot X(\omega) .
\]
Multiplying both sides of (1.354) by $1 + \text{sgn}(\omega)$ leads to (middle of Figure 1.50)

$$Y_A(\omega) = H(\omega) \cdot X_A(\omega)$$

$$\Rightarrow Y_A(\omega) = \left\{ H(\omega) \cdot \frac{1}{2} \cdot (1 + \text{sgn}(\omega)) \right\} \cdot X_A(\omega) = \left[ \frac{1}{2} \cdot H_A(\omega) \right] \cdot X_A(\omega),$$

where the second relationship follows by observing that since the input has nonzero spectra only for positive frequencies, only the channel filtering at those same positive frequencies (recalling that the factor $(1/2) \cdot [1 + \text{sgn}(\omega)]$ has interest). More importantly, since the linear time-invariant passband channel $h(t)$ only scales and phase shifts each frequency independently, the output $y(t)$ has its power spectral density concentrated in the same frequency region (or a smaller region if the channel zeroes a band) as the input $x(t)$. Shift of the output spectrum $y(t)$ down by $\omega_c$ yields

$$Y_{bb}(\omega) = Y_A(\omega + \omega_c) = \left[ \frac{1}{2} H_A(\omega + \omega_c) \right] X_A(\omega + \omega_c)$$

$$= \left[ \frac{1}{2} H_{bb}(\omega) \right] X_{bb}(\omega) \quad \text{(1.358)}$$

which appears at the bottom of Figure 1.50. This leads to the definition of the baseband equivalent system

**Definition 1.3.30 (Baseband Equivalent System)** The baseband equivalent system for a passband system described by $y(t) = x(t) \ast h(t)$, where $x(t)$ is a passband signal, is given by

$$y_{bb}(t) = \left( x_{bb}(t) \ast \frac{1}{2} h_{bb}(t) \right)$$

or

$$Y_{bb}(\omega) = H(\omega + \omega_c) \cdot X_{bb}(\omega).$$

Obtaining the baseband equivalent channel is easy! Simply slide the Fourier transform of the channel response down to DC. Because the channel may be asymmetric with respect to $\omega_c$, the baseband equivalent channel can be complex and usually is. The complexity of dealing with cosines, sines, and carrier frequencies is removed by the baseband-equivalent representation. Any channel with any carrier frequency can thus be represented in a common baseband framework, which will be convenient for many analyses in digital transmission. This is why baseband-equivalent channels dominate in their use in digital-transmission analysis. The baseband-equivalent input is convolved with the complex channel corresponding to $H(\omega + \omega_c)$ to get the baseband-equivalent output. A channel that is not passband, but rather initially real baseband, simply corresponds to the baseband equivalent input/output representation with all imaginary parts zeroed, and $H(\omega)$ used directly ($\omega_c = 0$).

The input/output relationships can thus be summarized as follows: For the passband signals and systems,

$$y(t) = x(t) \ast h(t)$$

$$Y(\omega) = X(\omega) \cdot H(\omega). \quad \text{(1.363)}$$

For the analytic-equivalent system,

$$y_A(t) = x_A(t) \ast \frac{1}{2} h_A(t)$$

$$Y_A(\omega) = X_A(\omega) \cdot H(\omega). \quad \text{(1.365)}$$
For the baseband equivalent system,
\[
y_{bb}(t) = x_{bb}(t) \ast \frac{1}{2} h_{bb}(t)
\]
(1.366)
\[
Y_{bb}(\omega) = X_{bb}(\omega) \cdot H(\omega + \omega_c).
\]
(1.367)

Any of these three equivalent relations (and \(\omega_c\)) fully describe the passband system.

**EXAMPLE 1.3.11 (Bandpass channel for previous bandpass signals)** A channel impulse response is \(h(t) = 2 \times 10^6 \cdot \text{sinc}(10^6t) \cdot \cos(2\pi 10^7t)\) corresponds to
\[
H(f) = \begin{cases} 
1 & \text{if } |f \pm 10^7| < 0.5 \times 10^6 \\
0 & \text{elsewhere}
\end{cases}
\]
(1.368)

Then
\[h_I(t) = 2 \times 10^6 \cdot \text{sinc}(10^6t)\]
and
\[h_Q(t) = 0,
\]
so that \(h_{bb}(t) = h_I(t)\) or
\[
H_{bb}(f) = \begin{cases} 
2 & \text{if } |f| \leq 500 \text{ kHz} \\
0 & \text{elsewhere}
\end{cases}
\]
(1.369)

Using the signal in Examples 1.3.9 and 1.3.10 as the channel input, the channel output is
\[
Y_{bb}(f) = H(f + f_c) \cdot X_{bb}(f) = 1 \cdot X_{bb}(f) \quad |f| < 500 \text{ kHz}
\]
(1.369)
or
\[
x_{bb}(t) \ast \frac{1}{2} h_{bb}(t) = x_{bb}(t) = (1 - 3j) \cdot \text{sinc}(10^6t) \quad .
\]
(1.370)

Zero quadrature channel or \(h_Q(t) = 0\) does not mean that no quadrature signal components are passed – it means that inphase components remain inphase components at the channel output, and quadrature components similarly then remain quadrature components at the channel output.

Appendix B extends the results of this section to passband random processes (this section only considered deterministic signals) used on passband deterministic channels. The next subsection considers the addition of random noise to the channel output.

**1.3.5.2 Baseband-Equivalent AWGN Channel**

This subsection investigates a passband filtered AWGN channel’s action on transmitted signals. Some results from Appendix B on passband random processes appear, and the reader may best read that appendix first before proceeding, although such reading is not completely necessary. Figure 1.51 summarizes a scaling factor used, explicitly or tacitly, by all developments based on passband processes. This scale factor is simply for analytical purposes and will make results consistent in all regards with those appearing earlier in this chapter.
The channel output $y(t)$ is processed by the combination of a scale factor $\frac{1}{\sqrt{2}}$ and a phase splitter to generate a baseband equivalent signal $\tilde{y}_{bb}(t)$. The tilde is often dropped in the literature and the splitter output is often called $y_{bb}(t)$ even though the scale factor is included. Generally speaking, the phase splitter adds a signal to $j$ times its own Hilbert transform. A random processes’ Hilbert transform has the same power as the original process (Appendix B). Thus, the phase splitter generally doubles power, somewhat arbitrarily. The scale factor $1/\sqrt{2}$ that precedes Figure 1.51’s phase splitter causes the power of $\tilde{y}_{bb}(t)$ and $y(t)$ to be the same. The power-scaling occurs equally for both the noise and signal, since both are present in the channel output $y(t)$. So, performance is not changed (no matter what the scale factor is), and the ratio of minimum distance to noise standard deviation is also unchanged by any scaling. Nonetheless, Figure 1.51’s particular scaling makes the ensuing analysis consistent with this chapter’s earlier results when complex signals are used.

For analysis, the scale factor can be “pushed back” through the channel and to the noise, and the Figure 1.51’s middle baseband-equivalent corresponds to Figure 1.51’s upper system with the explicit baseband-input-signal scaling. The scale factor then occurs separately in each of the output’s noise and signal components. Each of the next two subsections independently investigates this scaling and illustrates its consistency with previous results. Since there is a factor of $1/\sqrt{2}$ in both inphase and quadrature QAM basis functions, this factor then is already present conceptually in Figure 1.51’s bottom equivalent system, and thus the scaled noise $n_{bb}(t)/\sqrt{2}$ and the baseband channel $H(\omega + \omega_c)$ are the proper combination to represent the filtered AWGN with complex baseband input symbol ($x_1 + jx_2$) and the QAM basis-function component $\varphi(t)$.

### 1.3.5.2.1 Noise scaling in the baseband AWGN

Appendix D shows that a random process’ analytic-equivalent power spectral density is equal to four times the power spectral density of the original signal’s positive-frequency part, which tacitly implies a doubling of noise power.\(^{28}\)

Since the scaled WGN, $n_{bb}(t)/\sqrt{2}$, in Figure 1.51 has power spectral density.

\[
S_n(\omega) = \frac{N_0}{4},
\]

---

\(^{28}\)The autocorrelation of the analytic equivalent noise is $r_A(\tau) = 2(r_n(\tau) + j\tilde{r}_n(\tau))$. See Appendix B - FIX THIS - for more details.
the power spectrum of the analytic equivalent of \( n_{bb}(t)/\sqrt{2} \) is

\[
S_A(\omega) = \begin{cases} 
N_0 & \omega > 0 \\
\frac{N_0}{2} & \omega = 0 \\
0 & \omega < 0 
\end{cases}
\]  \hspace{1cm} (1.372)

The baseband equivalent noise has power spectrum that simply translates \( S_A(\omega) \) to baseband, or

\[
S_{bb}(\omega) = \begin{cases} 
N_0 & \omega > -\omega_c \\
\frac{N_0}{2} & \omega = -\omega_c \\
0 & \omega < -\omega_c 
\end{cases}
\]  \hspace{1cm} (1.373)

Strictly speaking, \( S_A(\omega) \) and \( S_{bb}(\omega) \) do not correspond to white noise. However, practical systems will always use a carrier frequency that is at least equal to the signal frequency, \( \omega_{\text{high}} \), that corresponds to the highest-frequency nonzero baseband signal component – that is, the design always modulates with a carrier frequency large enough to “get away” from DC. In this case, the baseband equivalent’s power spectrum appears as if it were “white” or flat at \( N_0 \) for all frequencies of practical interest. This baseband demodulated noise signal is complex AWGN with power spectral density \( N_0 \), and correspondingly power spectral density \( \frac{N_0}{2} \) for each real dimension.

Whenever the scaled phase-splitting arrangement of Figure 1.51 is used, this text defines baseband equivalent WGN as follows:

**Definition 1.3.31 (Baseband Equivalent WGN)** Baseband Equivalent White Gaussian Noise is a random process, \( \tilde{n}_{bb}(t) \), that is generated, essentially, through demodulation of the Passband AWGN in Figure 1.51. The complex random process’ autocorrelation, \( r_{bb}(\tau) \) is thus

\[
r_{bb}(\tau) = N_0 \cdot \delta(\tau) \hspace{1cm} ,
\]  \hspace{1cm} (1.374)

and the power spectral density is thus

\[
S_{bb}(f) = N_0 \hspace{1cm} .
\]  \hspace{1cm} (1.375)

From Appendix D, the baseband autocorrelation is then

\[
r_{bb}(\tau) = 2r_I(\tau) = 2r_Q(\tau) = N_0 \delta(\tau) \hspace{1cm} ,
\]  \hspace{1cm} (1.376)

so that the inphase and quadrature noises each have power spectral density \( \frac{N_0}{2} \) and are white noise signals. Further, from Appendix D and (1.376),

\[
r_{IQ}(\tau) = 0 \hspace{1cm} ,
\]  \hspace{1cm} (1.377)

that is, the inphase and quadrature noises are uncorrelated for all time lags with baseband equivalent WGN.

The complex baseband noise is two dimensional (two real dimensions), and the noise variance per dimension is thus \( \frac{N_0}{2} \), which is the reason for the scaling that was introduced in the definition of passband WGN. This scaling makes the noise variance per dimension the same as discussed earlier in this chapter.

**1.3.5.2.2 Scaling of the Signal** A brief review of the basis-function modulator of Chapter 1 will assist understanding of the effect of the scaling: The two normalized QAM passband functions for transmission on the one-shot AWGN are again

\[
\varphi_1(t) = \sqrt{2} \cdot \varphi(t) \cdot \cos(\omega_c t) \\
\varphi_2(t) = -\sqrt{2} \cdot \varphi(t) \cdot \sin(\omega_c t) 
\]  \hspace{1cm} (1.378)

92
where for practical reasons, $\omega_c$ is high enough. The modulated signal

$$x(t) = x_1 \cdot \varphi_1(t) + x_2 \cdot \varphi_2(t)$$

$$= \sqrt{2} \{ x_1 \cdot \varphi(t) \cdot \cos(\omega_c t) \} - \sqrt{2} \{ x_2 \cdot \varphi(t) \cdot \sin(\omega_c t) \} ,$$

has baseband equivalent signal

$$x_{bb}(t) = \sqrt{2} (x_1 + jx_2) \cdot \varphi(t) .$$

The scaling of Figure 1.51 removes the extra factor of $\sqrt{2}$ that arose through normalization of the modulated basis function. The bottom diagram in Figure 1.51 shows this removal explicitly so that the system appears as a complex baseband system with complex input

$$\tilde{x}_{bb}(t) = (x_1 + jx_2) \cdot \varphi(t) .$$

Equation (1.383) becomes

$$\tilde{x}_{bb}(t) = x_{bb} \cdot \varphi(t)$$

where

$$x_{bb} \triangleq (x_1 + jx_2) .$$

Equations (1.384) and (1.385) constitute a single-dimension complex baseband representation of the QAM modulator with (now normalized) basis function $\varphi(t)$ that is entirely consistent in all regards with the two-real-dimensional representation. The average energy of the complex signal constellation is

$$\mathcal{E}_{bb} = \mathcal{E}_x = 2\bar{\mathcal{E}}_x ,$$

which maintains the convention that a complex signal is equivalent to a two-dimensional real signal in defining $\bar{\mathcal{E}}_x$.

The tildes are necessary for those learning baseband analysis, but are dropped without comment throughout the literature. So one often sees a complex AWGN defined by Figure 1.52, where the tildes are dropped, but the scaling of noise and signal are consistent with the $1/\sqrt{2}$ in Figure 1.51.

Furthermore, this figure is often used to represent one-dimensional real systems where no passband modulation effects are of concern. In this case, the quadrature (imaginary) dimension is tacitly zeroed, while the real dimension carries the signal and has noise power spectral density $N_0/2$, entirely consistent with earlier developments.

### 1.3.5.3 Conversion to a baseband equivalent channel

This subsection applies baseband analysis to a QAM system where the channel and noise have been modeled as equivalent baseband. The system then looks like a PAM system except that inputs, outputs and internal quantities are all complex with the real dimension corresponding to the “cosine” modulated component and the imaginary dimension corresponding to the “sine” modulated component. After moving to a complex baseband equivalent, the effect of the carrier has been removed from all subsequent analysis.

---

29 One verifies that these functions are indeed normalized – if $\varphi(t)$ is normalized, as we assume – by investigating their power spectra under modulation.
1.3.5.3.1 Demodulators for the generation of the baseband equivalent  The actual generation of the baseband equivalent output signal can take one of the two equivalent forms in Figure 1.53. Figure 1.53(a) repeats the “phase-splitter” generation of the analytic equivalent, $y_A(t)$. The scale factor is absorbed into the definition of noise power spectral density and baseband input modulator for convenient analysis, as in the last subsection. The analytic signal $y_A(t)$ is demodulated by $e^{-j\omega_c t}$ to generate $y_{bb}(t)$. Figure 1.53(b) illustrates a more obvious form of generating $y_{bb}(t)$ that is sometimes used in practice. The structure in Figure 1.53(b) generates the inphase and quadrature components, $y_I(t)$ and $y_Q(t)$ by multiplying $y(t)$ by $2\cos(\omega_c(t))$ and $2\sin(\omega_c(t))$ in parallel. Then,

$$2\cos(\omega_c(t))y(t) = y_I(t)2\cos(\omega_c(t))^2 - y_Q(t)2\sin(\omega_c(t))\cos(\omega_c(t))$$

$$= y_I(t)(1 + \cos(2\omega_c t)) - y_Q(t)\sin(2\omega_c t))$$

and

$$2\sin(\omega_c(t))y(t) = y_I(t)2\cos(\omega_c(t))\sin(\omega_c(t)) - y_Q(t)2\sin(\omega_c(t))^2$$

$$= y_I(t)\sin(2\omega_c t) + y_Q(t)(\cos(2\omega_c t) - 1)$$

Lowpass filtering of $2\cos(\omega_c(t))y(t)$ and $2\sin(\omega_c(t))y(t)$ removes the signal artifacts centered at $2\omega_c$. In practice, it is usually easier to implement the two identical lowpass filters in Figure 1.53 (b) than the Hilbert filter in Figure 1.53 (a); so the implementation shown in Figure 1.53 (b) may be preferred in simple communication systems. In more sophisticated designs, especially those involving equalization (see Chapter 3), the implementation in Figure 1.53 (a) can have some practical performance advantages for the implementation of carrier-phase recovery systems (see Chapter 6).

![Figure 1.53: Complex demodulator.](image)

**EXAMPLE 1.3.12 (demodulation of a specific signal)** As in Example 1.3.9 from Section 1.3.5.1, a passband AWGN-channel output QAM signal is

$$z(t) = \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + 3 \cdot \text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t) + n(t) .$$

(1.387)
The carrier frequency is 10 MHz and the symbol period is 1 μs. Proceeding through the demodulator in Figure 1.53(a), $z(t)$ is the channel output signal that is input to the demodulator. The signal after scaling by $\frac{1}{\sqrt{2}}$ is

$$y(t) = \frac{z(t)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + \frac{3}{\sqrt{2}} \cdot \text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t) + \frac{n(t)}{\sqrt{2}},$$

so that effectively this choice of location for defining $y(t)$ allows the $1/\sqrt{2}$ scaling factor to be viewed as absorbed into the channel. The Hilbert transform of this scaled signal for the lower path in parallel in Figure 1.53(a) is

$$\hat{y}(t) = \frac{1}{\sqrt{2}} \cdot \text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t) - \frac{3}{\sqrt{2}} \cdot \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + \tilde{n}(t).$$

Multiplying the Hilbert transform by $j$ and adding to the upper unchanged component ($y(t)$) creates the analytic signal

$$y_A(t) = y(t) + j\hat{y}(t),$$

which after multiplication by the carrier-demodulating term $e^{-j\omega_c t}$ provides a baseband equivalent signal $y_{bb}(t) = e^{-j\omega_c t} \cdot y_A(t)$

$$y_{bb}(t) = \frac{1 - 3j}{1000 \cdot \sqrt{2}} \cdot 1000 \cdot \text{sinc}(10^6 t) + n_{bb}(t).$$

This baseband signal has a real component of $.001/\sqrt{2}$ and an imaginary component of $-.003/\sqrt{2}$ and the component of noise in each of these dimensions is $\frac{N_0}{2}$. These are indeed the components that were associated with this signal earlier in this chapter as

$$x(t) = \text{sinc}(10^6 t) \cdot \cos(2\pi 10^7 t) + 3 \cdot \text{sinc}(10^6 t) \cdot \sin(2\pi 10^7 t) + n(t) = x_1 \cdot \phi_1(t) + x_2 \cdot \phi_2(t)$$

where the normalized basis functions are

$$\phi_1(t) = \sqrt{2} \cdot 1000 \cdot \text{sinc}(10^6 t) \cdot \cos(\omega_c t)$$

$$\phi_2(t) = -\sqrt{2} \cdot 1000 \cdot \text{sinc}(10^6 t) \cdot \sin(\omega_c t).$$

If the baseband signal $y_{bb}(t)$ is now passed through the matched filter $1000 \cdot \text{sinc}(10^6 t)$ and sampled at time 0, the two components $.001/\sqrt{2}$ and $-.003/\sqrt{2}$ are obtained on the real and imaginary dimensions of the output. The relevant independent noise component in each of these dimensions is $\frac{N_0}{2}$, the power-spectral density of the original white noise, which is equal to its variance per dimension.

The subsequent subsection now adds a filtering channel with impulse response $h(t)$ to the system and investigates modeling of the filtered AWGN as a complex baseband equivalent for QAM transmission.

### 1.3.5.3.2 Generating the baseband equivalent: some examples

The transmission engineer is nominally presented with a variety of information about the transmission channel for which they must design a modem. Often in this author’s experience, this information is not in a convenient form initially, and a considerable amount of time and effort is spent understanding and modelling the channel. Part of this understanding involves generating the baseband-equivalent channel so that this tool’s analysis can be applied. The first example of a two-ray mathematical model for a channel may be easier for the student to follow mathematically because it starts with a very plausible and tractable mathematical model. The second example starts with information that must be converted to an acceptable response.

**EXAMPLE 1.3.13 (Two-ray wireless channel)** Wireless transmission sometimes is (over-simply) modeled by a two-ray model. A transmission path between transmit and receive antennas has then two paths between the antenna, one direct and one indirect. The latter path usually represents a reflection from a building, mountain, or other physical entity. The
second path is usually delayed (say by $\tau=1.1 \mu s$) and attenuated (say 90% of the amplitude of the first path) with respect to the first path. Let us then say that the channel impulse response is then

$$h(t) = g \cdot [\delta(t) - 0.9\delta(t-\tau)] ,$$

(1.395)

where $g$ is an attenuation factor that models the path loss and antenna losses. The Fourier transform is

$$H(f) = g \cdot [1 - 0.9e^{-j2\pi f\tau}] .$$

(1.396)

The noise will be white and a combination of a number of factors, natural and man-made with one-sided PSD -150 dBm/Hz. Wireless systems often use carrier frequencies between 800 and 900 MHz, so let us choose a carrier for QAM modulation at 852 MHz and further choose 4QAM transmission with a symbol rate of $1/T=1.0$ MHz. Thus, frequencies between 852-.5 = 851.5 MHz and 852+.5=852.5 MHz are of interest, corresponding to a baseband-equivalent channel with frequencies between -500 kHz and + 500 kHz.

The complex-channel model for this transmission is then

$$\frac{1}{2}h_{bb}(t) = g \cdot [\delta(t) - 0.9\delta(t-\tau)] \cdot e^{-j2\pi f_{c}t}$$

(1.397)

with Fourier transform

$$\frac{1}{2}H_{bb}(f) = H(f + f_{c}) = g \cdot \left[1 - 0.9 \cdot e^{-j2\pi(f+f_{c})\tau}\right] .$$

(1.398)

Figure 1.54 plots the original channel response from 850 MHz to 860 MHz, along with the complex channel for baseband modeling (with factor of 1/2 included) from -500 kHz to 500 kHz. The channel clearly has “notching” effects because of the possibility of the second path adding out-of-phase (with phase $\pi$) at some frequencies. The wider the bandwidth of a QAM transmission system, the more likely one (or more) of the “dips” is to occur in the transmission band of interest. Thus, this “multipath” distortion will lead to a non-flat or filtered-AWGN channel response (which means the techniques of Chapter 3 and later chapters are necessary for reliable recovery of messages). The baseband-equivalent (actually 1/2 amplitude is included in the plot) is clearly not symmetric about frequency zero, meaning
its baseband-equivalent impulse response is complex, as the formula above in (1.397) also implies. The real and imaginary parts of the baseband-equivalent response appear in Figure 1.55. The baseband-equivalent noise for the model introduced in this Section is still white and has $N_0 = -150$ dBm/Hz, or equivalently $N_0 = 10^{-18}$. For typical values of $g$ in well designed transmission systems, this will be a few orders of magnitude below the signal levels. It’s very simple in this case: Slide the Fourier transform in the band of interest down to DC, then set the complex noise level equal to the single-sided PSD.

In more sophisticated transmission design, the 2-ray model that easily led to a nice compact mathematical description is a rarity. More likely, the engineer will be given (or will have to measure themselves) the channel frequency attenuation in dB at several frequencies in the band of transmission, along with the measured delay of signals through the channel at each of these frequencies. The noise is also likely to have been measured within bands centered around each of the measured channel frequencies (or perhaps at other frequencies). The process of conversion to a complex baseband channel may be tedious, but follows the same steps as in the next example.

**Figure 1.55:** Real and imaginary parts of (scaled) baseband equivalent channel response for two-ray example.

**Figure 1.56:** Illustration of passband insertion loss and (1/2 times) baseband-equivalent transfer function for home-phone network example.

**EXAMPLE 1.3.14 (Telephone Line Channel)** Telephone lines today are sometimes used
for transmission of data within the home, and this example looks at a speed of 10 Mbps, using a transmission. The carrier frequency is 7.5 MHz and the symbol rate is 5 MHz for a 4 QAM signal. Telephone line attenuation versus frequency is often measured in terms of “insertion loss” in dB, a ratio of the voltage at the line output to the voltage at the same load point when the phone line is removed. For a well-matched system, it can be determined that this insertion loss is 6 dB above the transfer function from source to load, which is the desired function for digital transmission analysis. Figure 1.56 plots the insertion loss in dB for a 26-gauge phone line of length 300 meters. The baseband equivalent channel response is in the frequency range from 5 MHz to 10 MHz, which the designer “slides” so that 7.5 MHz now appears as DC, as also illustrated in Figure 1.56. The baseband characteristic has been increased by 6 dB to get the transfer function, and again the baseband complex channel shown includes the scale factor of 1/2. The designer would presumably be given (or has measured) the insertion loss at a sufficient number of frequencies between 5 and 10 MHz, stored those values in a file, and now is analyzing them with digital signal processing. To use common digital signal processing operations like the inverse Discrete Fourier Transform, the measured values will need to be equally spaced in frequency between 5 MHz and 10 MHz. Let us say here that 501 measurements with spacing 10 kHz have been so taken. These 501 values form the amplitudes (after conversion of dB back into linear-scale values) of the channel transfer function at the frequencies 5 MHz, 5.01 MHz, ... 10 MHz, or for baseband (increased by 6 dB to compute transfer function from insertion loss) equivalent from -2.5 MHz to 2.5 MHz.

The line is also characterized in terms of its delay at all these same frequencies, usually measured in microseconds and plotted in Figure 1.57. The index will be from \( n = 0, ..., 500 \) across the frequency band of interest. Since delay is negative the derivative of phase, to compute the phase angles for the baseband equivalent transfer function, the phase needs to be accumulated (with minus sign) from -2.5 MHz to each and every frequency of interest according to

\[
\angle H_{bb}(-2.5MHz + n \cdot .01MHz) = \theta_0 - \sum_{i=0}^{n} \text{Delay} \left[ H_{bb}(i) \right],
\]

where \( \theta_0 \) is an constant arbitrary phase reference that ultimately has no effect on transceiver performance, and thus usually taken to be 0. The baseband equivalent channel (scaled by 1/2) is then found by inverse DFT’ing (IFFT command in Matlab) the vector of values \( H_{bb}(n) \) \( n = 0, ..., 500 \). Because of the arbitrary phase, the time-domain response is usually
not centered and has nonzero components at the beginning and end of the response. Simple circular shift (already included in Figure 1.57) will provide a “centered” \( h_{lb}(t) \) sampled at the symbol rate (which can be made causal by simple reindexing of the time axis). The transmit psd of the 4QAM signal is about -57 dBm/Hz, so that the power is then about 10 dBm (or 10 milliwatts). To interpolate the baseband response to finer time-resolution than the symbol rate, a band wider than 5-10 MHz must be measured, translated to DC, and then inverse transformed.

Simple circular shift (already included in Figure 1.57) will provide a “centered” \( h_{lb}(t) \) sampled at the symbol rate (which can be made causal by simple reindexing of the time axis). The transmit psd of the 4QAM signal is about -57 dBm/Hz, so that the power is then about 10 dBm (or 10 milliwatts). To interpolate the baseband response to finer time-resolution than the symbol rate, a band wider than 5-10 MHz must be measured, translated to DC, and then inverse transformed.

An interesting effect in telephone-line transmission is that neighbors’ data signals can be “heard” through electromagnetic coupling between phone lines in phone cables “upstream.” It then can flow back into the home, and often is viewed as a contributor to noise. Thus, the noise is not “white,” and a simple model for the one-sided power spectral density of this noise has power spectral density:

\[
-187 + 15 \log_{10}(f) \text{ dBm/Hz}.
\]  

(1.400)

This power-spectral density can be computed with \( f \) values from 5 MHz to 10 MHz, and then translated to baseband to obtain the baseband-equivalent psd as in Figure 1.59.

To find the so-called “white-noise equivalent” in (see later in Section 1.3.7) for this complex
baseband-equivalent channel, the inverse noise PSD can be IFFT’d to the time-domain and factored using the roots command in MATLAB. Terms with roots of magnitude greater than 1 correspond to the minimum-phase factorization, said inverse can then be convolved with the channel \(5 \cdot h_{bb}(kT)\) to find Subsection 1.3.7’s white-noise equivalent channel samples at the symbol rate.

This last example seems like much tedious work, but it is perhaps simple compared to what communication engineers do. The example emphasizes how important it is for communications designers to know and well model their channel so that the theories and guidance learned from this text can be applied.

1.3.5.3.3 Complex generalization of inner products and analysis  Optimum demodulation theory for complex signals with baseband-equivalent WGN, or more generally any complex channel (see later sections), is essentially the same as that for real signals earlier in this chapter. All analysis and structure of detectors previously derived carries through with the following complex-arithmetic generalizations:

1. The inner product becomes

\[
\langle x, y \rangle = x^* y = \int_{-\infty}^{\infty} x^*(t) \cdot y(t) dt ,
\]

\(x^*\) means conjugate transpose of \(x\).

2. The matched filter is conjugated, that is \(\varphi(T - t) \rightarrow \varphi^*(T - t)\).

3. Energies of complex scalars are \(E_x = E\{|x(t)|^2\}\), or the expected magnitude of the complex scalar, and \(\bar{E}_x = E_x / 2\).

For the MIMO case, a superscript of * will mean conjugate transpose of the matrix or vector. Further the integral above in Equation (1.401) becomes a sum of \(L_x\) integrals, the matched filter becomes \(L_x\) parallel matched filters. Energy per dimension will be divided by the total number of real dimensions, as always.

1.3.6 Passband Analysis for QAM alternatives

Passband analysis directly applies to QAM modulation in a way that simply requires computing a channel’s baseband equivalent for convolution with the complex input \((x_1 + jx_2) \cdot \varphi(t)\). Some transmission systems instead may use one of Section 1.3.6’s three other implementations, VSB, CAP, or OQAM. This section addresses the specifics of how the passband analysis concepts discussed so far still apply to these other passband modulation types. In all cases, a complex-equivalent channel can be found easily from the given channel transfer function.

1.3.6.1 Passband VSB Analysis

This subsection starts with SSB (single-side-band) and then generalizes to VSB. With SSB, the transmitted signal has \(x_I(t)\) and \(x_Q(t)\) that are Hilbert transforms of one another and thus

\[
x(t) = x_I(t) \cdot \cos(\omega_c t) - \tilde{x}_I(t) \cdot \sin(\omega_c t) .
\]

Such a signal only exhibits nonzero energy content for frequencies exceeding the carrier frequency (and for frequencies below the negative of the carrier frequency). The SSB baseband-equivalent signal is also therefore analytic, for which we introduce the new notation

\[
x_{bb}(t) = x_{bb}(t) = x_I(t) + \tilde{x}_I(t) .
\]
The subscript of $Ab$ is intended to represent a new signal that is both analytic and baseband and used in SSB analysis. The baseband equivalent of a channel output is consistently

$$y_{Ab}(t) = x_{Ab}(t) \cdot \left( \frac{h_{Ab}(t)}{2} \right)$$  \hfill (1.404)
$$Y_{Ab}(\omega) = X_{Ab}(\omega) \cdot H(\omega + \omega_c) \quad \omega > 0 \quad .$$  \hfill (1.405)

The previous passband-channel analysis applies for any carrier frequency and not just one centered within the passband. Thus, baseband-equivalent analysis directly applies to SSB also and the “Ab” notation has just made explicit the carrier-frequency position on the band’s lower edge. However, the input construction is such that $x_Q(t)$ is no longer independent of $x_I(t)$. Generally speaking, with this SSB constraint, twice as many dimensions per second are transmitted within $x_I(t)$ for SSB than would be the case for QAM with the same bandwidth. However, QAM can independently use the quadrature dimension whereas for SSB this quadrature dimension is completely determined from the inphase dimension. The analysis for lower sideband (instead of the assumed upper sideband in this analysis) follows by simply negating the quadrature component and choosing the carrier frequency at the upper edge of the passband, then the baseband equivalent is nonzero for negative frequencies only, but still a special case of the general analysis of Sections 1.3.5.1 and 1.3.5.3.

VSB transmission is based on SSB transmission. In general, VSB systems have $x_I(t)$ and $x_Q(t)$ selected in such a way that they are almost Hilbert transforms of one another. A VSB system may be easier to implement in practice and is always based on an equivalent SSB signal. With this text’s nomenclature, a VSB signal’s baseband equivalent, $x_{Vb}(t)$, has “vestigial” symmetry about $f = 0$ – that is $X_{Vb}(f) + X_{Vb}(-f) = X_{Ab}(f) \forall f > 0$ where $X_{Ab}(f)$ is for the (analytic) SSB signal in (1.403) upon which the VSB signal is based. The VSB signal is based on a carrier frequency that is not at the passband edge. This carrier frequency is the point around which the passband signal exhibits vestigial symmetry. This frequency is again selected for the baseband equivalent representation of the channel,

$$Y_{Vb}(\omega) = X_{Vb}(\omega) \cdot H(\omega + \omega_c) \quad \omega > -\omega_c \quad .$$  \hfill (1.406)

Terrestrial digital television broadcast in the USA uses VSB transmission with carrier frequencies at the nominal TV carrier positions of $52 \text{ MHz} + i \cdot (6 \text{ MHz})$, effectively $\tilde{b} = 2$ (constellation is coded so it is called 64 VSB, where extra levels are redundant for coding, see Chapter 2) and a symbol rate of roughly $5 \text{ MHz}$, for a data rate of $20 \text{ Mbps}$. The signals thus have non-zero energy from about $1.5 \text{ MHz}$ below the carrier and to $3.5 \text{ MHz}$ above the carrier using vestigial transmit symmetry with respect to that carrier.

### 1.3.6.2 Passband CAP Analysis

Analysis of CAP (carrierless amplitude phase) modulation essentially relies on analytic signal and channel equivalents instead of baseband equivalents. A CAP signal is generated according the Werner’s observation that the sum (sequence) of analytic QAM signals ($x_k$ can be complex and represents the two-dimensional QAM symbol transmitted at symbol time instant $k$ and $\varphi(t)$ is the baseband equivalent modulating function):

$$x_{A}(t) = \sum_k x_k \cdot \varphi(t - kT) \cdot e^{j\omega_c t}$$  \hfill (1.407)
$$= \sum_k x_k \cdot \varphi(t - kT) \cdot e^{j\omega_c t} \cdot e^{-j\omega_c kT} \cdot e^{+j\omega_c kT}$$  \hfill (1.408)
$$= \sum_k (x_k \cdot e^{+j\omega_c kT}) \cdot \varphi(t - kT) \cdot e^{j\omega_c (t-kT)}$$  \hfill (1.409)
$$= \sum_k \tilde{x}_k \cdot \varphi_A(t - kT)$$  \hfill (1.410)

where the new quantities are defined as

$$\varphi_A(t) = \varphi(t) \cdot e^{j\omega_c t} \quad \text{and} \quad \tilde{x}_k = x_k \cdot e^{+j\omega_c kT}$$  \hfill (1.411)
Thus, a CAP system uses simple rotation of the encoder outputs to create a symbol-time-invariant realization of the subsequent modulation. Since the sequence of rotations is known, the receiver need only detect \( \tilde{x}_k \), and \( x_k \) can easily be determined by reverse rotations,

\[ x_k = \tilde{x}_k \cdot e^{-j\omega_k T}. \]

In practice, the rotations are ignored and the sequence \( \tilde{x}_k \) itself directly carries the information, noting that the rotations at each end simply undo each other and have no bearing on performance or functionality. They are necessary only for equivalence to a QAM signal.

The channel output of interest is then the analytic channel output so that

\[
g_A(t) = x_A(t) * \left( \frac{h_A(t)}{2} \right) \quad \text{(1.413)}
\]

\[
Y_A(\omega) = X_A(\omega) \cdot H(\omega) \quad \text{if } \omega > 0 \quad \text{(1.414)}
\]

With CAP then, only the analytic signals are of interest, and the complex channel that describes the method is the analytic equivalent channel that is found by zeroing the Fourier transform for negative frequencies (and for which the notation \( H_{CAP} \) is specific to analysis of CAP transmission over a channel with response generally noted by \( h(t) \)).

\[
H_{CAP}(\omega) = H(\omega) \cdot \frac{1}{2} (1 + sgn(\omega)) \quad \text{(1.415)}
\]

In practice, on a channel with narrow transmission band relative to the carrier frequency, intermediate-frequency (IF) demodulation is used to reduce (but not zero) the effective center frequency of the transmission band closer to DC. Then CAP is applied to the IF demodulated signal. The IF demodulation treats the transmission signals as if they were analog signals and can be considered outside the realm and interest of digital data transmission.

Nonetheless, after conversion to complex equivalent channels, both QAM and CAP receiver processing can be generally described by the processing of a complex channel output, and such a complex model is this section’s objective.

### 1.3.6.3 OQAM or “Staggered QAM”

The OQAM basis functions appear earlier in this subsection. One could rewrite this entire Subsection with \( \cos(\omega_c t) \) replaced by \( \text{sinc}(t/T) \cdot \cos(\omega_c t) \) and most importantly \( \sin(\omega_c t) \) replaced by \( \text{sinc}(t-T/2)/T \cdot \sin(\omega_c t) \) everywhere – and all results would still hold. However, there is an easier way to reuse what has already been derived: Now, with the reader’s understanding of passband signals, a transmission engineer can say that the essential difference between OQAM and QAM is that the quadrature component is delayed by one-half symbol period with respect to the inphase component in OQAM. The designer can analyze a new equivalent channel input with double the symbol rate and a time-varying encoder that alternates between a nonzero inphase component (with zero quadrature component) and a nonzero quadrature component (with zero inphase component). This new time-varying double-speed symbol sequence can then be applied to a conventional QAM modulator to generate the OQAM sequence. All results developed so far then apply to this new equivalent system running at twice the symbol rate. The energy per dimension \( \bar{E}_x \) will reduce by a factor of 2 if power is maintained constant.

Thus, to find the output of a channel with impulse response \( h(t) = \Re\{h_{bb}(t) \cdot e^{j\omega_c t}\} \) to an OQAM input, simply convolve the baseband equivalent of the continuous-time \( x(t) \) formed from the double-symbol-rate “interleaved” symbol sequence with \( [h_{bb}(t)]/2 \). Again a complex channel will have been constructed for analysis – the objective for this chapter.

The inphase and quadrature dimensions are not strictly independent since they have alternating zero values. This dependence or correlation effectively halves the bandwidth so that an OQAM system running with symbol rate \( 1/T \) and the basis functions in Section 1.6, even though analyzed as a QAM system running with interdependent symbols at rate \( 2/T \), occupies the same bandwidth as QAM.

\[^{30}\text{This } \tilde{x}_k \text{ is notation used specific to the CAP situation here, and is not intended to be equivalent to any other temporary uses of a tilde on a quantity elsewhere in this textbook.}\]
fact as the function $\varphi_i(t)$ is generalized (See Chapter 3) so that $\varphi(t) \neq \sqrt{1/T}\text{sinc}(t/T)$, then OQAM typically requires less bandwidth in terms of the inevitable “non-brick-wall” energy roll-off associated with practical filter design.

1.3.6.4 The difference between complex and baseband equivalent channels

In digital-transmission literature and field of application, most practicing transmission specialists always use a complex channel to describe any channel, baseband or passband. When all complex quantities are real, the baseband case is a special case of the more general complex channel. Typically, a practicing engineer will have to find their complex equivalent channel from the information provided, which is usually the magnitude/phase characteristics of the real channel (or their equivalents) over the passband. This process essentially involves sliding the passband down to DC and then inverse transforming the result to get a complex channel equivalent. No factors of two need be involved because the doubling implied in finding $h_A(t)$ is removed by convolution with $h_{bb}(t)/2$ a few steps later in determining a channel output pulse response by convolving the channel complex impulse response with the transmit filter. Thus, the factors of “2” cancel and the channel is just represented as a complex channel with convolution directly and no factors of 2 introduced and then deleted. These factors of two correspond only to equivalent passband signals that exist in the transmission system, but are not truly necessary for analysis. The extra factor of $1/\sqrt{2}$ is also often absorbed into the complex channel response and the subsequent AWGN just represented as having $\frac{N_0}{2}$ per dimension or as its passband power-spectral density. Since performance only depends on the ratio of signal power to noise, as long as this ratio is correct, scale factors on both signals are irrelevant from an analysis perspective. These factors of 2 may be crucial in determining the dynamic range of front-end quantities in an actual receiver design, but otherwise do not affect performance as long as the SNR is determined correctly.

Thus, the literature on transmission almost always uses complex signals to represent channels and no factors of 2 are included or need be. However, the engineer responsible for presenting the analysis will have correctly taken them into account when providing the SNR. Thus, when one is given a complex channel (and not told specifically that it is a baseband equivalent channel derived as shown in this Chapter), the designer can only assume that convolution should occur without any additional factors of two. This subsection attempted to address these factors directly to assist field engineers who may well encounter them in calibrating their designs.

1.3.6.4.1 The complex filter Another important distinction to mention is the complex filter. Once a receiver or transmitter has established a complex (two-dimensional) signal and is using complex filters to process that signal, there is no factor of 2 involved. Any factor of two would only be necessary if both of the convolved quantities did correspond as in this subsection to passband signals, and the designer desired to find the equivalent passband signals and convolve them. If all signals are complex, then there is no need for passband equivalences and convolution proceeds correctly for any internal complex filters without any factors of two involved. Chapter 3 uses such filters within a receiver and does return to passband discussion, so convolution proceeds directly without factors of 2 and correctly represents the receiver’s internal filtering of complex signals.

1.3.7 Additive Self-Correlated Noise

In practice, additive noise is often Gaussian, but its power spectral density is not flat. Engineers often call this type of noise “self-correlated” or “colored”. The noise remains independent of the message signals but correlates with itself from time instant to time instant. The origins of colored noise are many. Receiver filtering effects, noise generated by other communications systems (“crosstalk”), and electromagnetic interference are all sources of self-correlated noise. A narrow-band transmission of a radio signal that somehow becomes noise for an unintended channel is another common example of self-correlated noise and called “RF” noise (RF is an acronym for “radio frequency”).

Self-correlated Gaussian noise can significantly alter the detector’s performance with respect to a detector designed for white Gaussian noise. This section investigates the optimum detector for colored
noise and also considers the performance loss when using a suboptimum detector designed for AWGN is used with Additive Correlated Gaussian Noise (ACGN).

This study is facilitated by first investigating the filtered “one-shot” AWGN channel in Subsection 1.3.7.1. Subsection 1.3.7.2 then finds the optimum detector for additive self-correlated Gaussian noise, by adding a whitening filter that transforms the self-correlated noise channel into a filtered AWGN channel. Subsection 1.3.7.2.2 studies the vector channel, for which (for some unspecified reason) the noise has not been whitened and describes the optimum detector given this vector channel. Finally, Subsection 1.3.7.3 studies the degradation that occurs when the noise correlation properties are not known for design of the receiver, and an optimum receiver for the AWGN is used instead.

1.3.7.1 The Filtered (One-Shot) AWGN Channel

![Figure 1.60: Filtered AWGN channel.](image)

Figure 1.60 illustrates the filtered AWGN channel. The modulated signal $x(t)$ undergoes filtering by $h(t)$ before the addition of the white Gaussian noise. When $h(t) \neq \delta(t)$, the filtered signal set $\{\tilde{x}_i(t)\}$ may differ from the transmitted signal set $\{x_i(t)\}$. This transformation may change the error probability as well as the structure of the optimal detector. This subsection still considers only one use of the channel with $M$ possible messages for transmission. Transmission over this type of channel can incur a significant penalty from intersymbol interference between successively transmitted data symbols. In the “one-shot” case, however, analysis need not consider this intersymbol interference. Intersymbol interference is considered in Chapters 3, 4, and 5.

For any channel input signal $x_i(t)$, the corresponding filtered output equals $\tilde{x}_i(t) = h(t) \ast x_i(t)$. Decomposing $x_i(t)$ by an orthogonal basis set, $\tilde{x}_i(t)$ becomes

\[
\tilde{x}_i(t) = h(t) \ast x_i(t) \quad (1.416)
\]

\[
= h(t) \ast \sum_{n=1}^{N} x_{in} \cdot \varphi_n(t) \quad (1.417)
\]

\[
= \sum_{n=1}^{N} x_{in} \{h(t) \ast \varphi_n(t)\} \quad (1.418)
\]

\[
= \sum_{n=1}^{N} x_{in} \cdot \phi_n(t) , \quad (1.419)
\]

where

\[
\phi_n(t) \triangleq h(t) \ast \varphi_n(t) . \quad (1.420)
\]

Note that:
The set of $N$ functions $\{\phi_n(t)\}_{n=1,\ldots,N}$ is not necessarily orthonormal.

For the channel to convey any and all constellations of $M$ messages for the signal set $\{x_i(t)\}$, the basis set $\{\phi_n(t)\}$ must be linearly independent.

The first observation can be easily proven by finding a counterexample, an exercise for the interested reader. The second observation emphasizes that if some dimensionality is lost by filtering, signals in the original signal set that differed only along the lost dimension(s) would appear identical at the channel output. For example consider the two signals $\tilde{x}_k(t)$ and $\tilde{x}_j(t)$.

$$\tilde{x}_k(t) - \tilde{x}_j(t) = \sum_{n=1}^{N} (x_{kn} - x_{jn}) \cdot \phi_n(t) = 0 ,$$  \hspace{1cm} (1.421)

If the set $\{\phi_n(t)\}$ is linearly independent then the sum in (1.421) must be nonzero: a contradiction to (1.421). If this set of vectors is linearly dependent, then (1.421) can be satisfied, resulting in the possibility of ambiguous transmitted signals. Failure to meet the linear independence condition could mandate a redesign of the modulated signal set or a rate reduction (decrease of $M$). The dimensionality loss and ensuing redesign of $\{x_i(t)\}_{i=0:M-1}$ is studied in Chapters 4 and 5. This chapter assumes such dimensionality loss does not occur.

If the set $\{\phi_n(t)\}$ is linearly independent, then the Gram-Schmidt procedure in Appendix A generates an orthonormal set of $N$ basis functions $\{\psi_n(t)\}_{n=1,\ldots,N}$ from $\{\phi_n(t)\}_{n=1,\ldots,N}$. A new signal constellation $\{\tilde{x}_i\}_{i=0:M-1}$ can be computed from the filtered signal set $\{\tilde{x}_i(t)\}$ using the basis set $\{\psi_n(t)\}$.

$$\tilde{x}_{in} = \int_{-\infty}^{\infty} \tilde{x}_i(t) \cdot \psi_n(t) dt = \langle \tilde{x}_i(t), \psi_n(t) \rangle .$$  \hspace{1cm} (1.422)

Using the previous analysis for AWGN, a tight upper bound on message error probability is still given by

$$P_e \leq N_e Q\left[\frac{d_{\text{min}}}{2\sigma}\right] ,$$  \hspace{1cm} (1.423)

where $d_{\text{min}}$ is the minimum Euclidean distance between any two points in the filtered signal constellation $\{\tilde{x}_i\}_{i=0:M-1}$. The matched filter implementation of the demodulator/detector does not need to compute $\{\psi_n(t)\}_{n=1,\ldots,N}$ for the signal detector as shown in Figure 1.61. (For reference the reader can reexamine the detector for the unfiltered constellation in Figure 1.24).
In filtered AWGN analysis, the transmitted average energy $E_{\text{x}}$ is still measured at the channel input. Thus, while $E_{\text{x}}$ can be computed, its physical significance can differ from that of $E_{\text{x}}$. If, as is often the case, the energy constraint is at the input to the channel, then the comparison of various signaling alternatives, as performed earlier in this chapter could change depending on the specific filter $h(t)$.

### 1.3.7.2 Optimum Detection in the Presence of Self-Correlated Noise

The Additive Self-Correlated Gaussian Noise (ACGN) channel is illustrated in Figure 1.62. The only change with respect to Figure 1.19 is that the autocorrelation function of the additive noise $r_n(\tau)$, need not equal $\frac{N_0}{2} \cdot \delta(\tau)$. Simplification of the ensuing development defines and uses a normalized noise autocorrelation function

$$\bar{r}_n(\tau) \triangleq \frac{r_n(\tau)}{\frac{N_0}{2}} .$$

(1.424)
The power spectral density of the unnormalized noise is then

\[ S_n(f) = \frac{N_0}{2} \cdot \tilde{S}_n(f), \quad (1.425) \]

where \( \tilde{S}_n(f) \) is the Fourier Transform of \( \tilde{r}_n(\tau) \).

### 1.3.7.2.1 The Whitening filter

The whitening-filter analysis of the ACGN channel “whitens” the colored noise with a whitening filter \( g(t) \), and then uses the previous section’s filtered-AWGN results where the filter \( h(t) = g(t) \). To ensure no loss of information when filtering the noisy received signal by \( g(t) \), the filter \( g(t) \) should be invertible. By the reversibility theorem, the receiver can use an optimal detector for this newly generated filtered AWGN without performance loss. Actually, the condition on invertibility of \( g(t) \) is sufficient but not necessary. For a particular signal set, a necessary condition is that the filter be invertible over that signal set. For the filter to be invertible on any possible signal set, \( g(t) \) must necessarily be invertible. This subtle point is often overlooked by most works on this subject.

For \( g(t) \) to whiten the noise,

\[ [\tilde{S}_n(f)]^{-1} = |G(f)|^2. \quad (1.426) \]

In general many filters \( G(f) \), may satisfy Equation (1.426) but only some of the filters shall possess realizable inverses (the particular choice is the so-called minimum-phase choice that has all poles and zeros in the left-half plane, or on the \( s = j\omega \) axis with multiplicity 1 in that “marginally realizable” case - recognizing of course that is white noise so no whitening filter is needed, or the filter is trivially the dirac delta function \( \delta(t) \)).

To ensure the existence of a realizable inverse \( S_n(f) \) must satisfy the **Paley-Wiener Criterion**.

**Theorem 1.3.6 (Paley-Wiener Criterion)** If

\[ \int_{-\infty}^{\infty} \frac{|\ln S_n(f)|}{1 + f^2} df < \infty, \quad (1.427) \]

then there exists a \( G(f) \) satisfying (1.426) with a realizable inverse. (Thus the filter \( g(t) \) is a 1-to-1 mapping).

If the Paley-Wiener criterion were violated by a noise signal, then it is possible to design transmission systems with infinite data rate (that is when \( S_n(f) = 0 \) over a given bandwidth) or to design transmission systems for each band over which Paley-Wiener is satisfied (that is the bands where noise is essentially of finite energy). A full development of Paley Wiener is deferred until Appendix A of Chapter 3. This subsection’s analysis always assumes Equation (1.427) is satisfied.\(^{31}\) With a 1-to-1 \( g(t) \) that satisfies (1.426), the ACGN channel converts into an equivalent filtered white Gaussian noise channel as shown in Figure 1.60 replacing \( h(t) \) with \( g(t) \). The performance analysis of ACGN is identical to that derived for the filtered AWGN channel in Subsection 1.3.7.1. A further refinement handles the filtered ACGN channel by whitening the noise and then analyzing the filtered AWGN with \( h(t) \) replaced by \( h(t) \ast g(t) \).

A process sometimes called “analytic\(^{32}\) continuation” of \( \tilde{S}_n(s) \) determines an invertible \( g(t) \):

\[ \tilde{S}_n(s) = \tilde{S}_n\left(f = \frac{s}{2\pi j}\right), \quad (1.428) \]

where \( \tilde{S}_n(s) \) can be canonically (and uniquely) factored into causal (and causally invertible) and anticausal (and anticausally invertible) parts as

\[ \tilde{S}_n(s) = \tilde{S}_n^+(s) \cdot \tilde{S}_n^-(s), \quad (1.429) \]

\(^{31}\)Chapters 4 and 5 also expand to the correct form of transmission that should be used when (1.427) is not satisfied.

\(^{32}\)This text does not need the concept of analytic functions that is developed in most prerequisite systems courses, where an analytic function is one for which the function and all derivatives are absolutely integrable over some domain associated with the analytic definition - the convergence region for Lapace or Z/D-transforms. In particular, Chapter 2 will use the term “analytic” in another related context to describe a signal added to \( j \) times its Hilbert Transform.
where
\[ \tilde{S}_n^+(s) = \tilde{S}_n^-(s) \]  
(1.430)

If \( \tilde{S}_n(s) \) is rational, then \( \tilde{S}_n^+(s) \) is “minimum phase,” i.e. all poles and zeros of \( \tilde{S}_n^+(s) \) are in the left half plane. The filter \( g(t) \) is then given by
\[ g(t) = \mathcal{L}^{-1} \left\{ \frac{1}{\tilde{S}_n(s)} \right\} , \]
(1.431)

where \( \mathcal{L}^{-1} \) is the inverse Laplace Transform. The matched filter \( g(-t) \) is given by
\[ g(-t) = \mathcal{L}^{-1} \left\{ \frac{1}{\tilde{S}_n(-s)} \right\} . \]
(1.432)

\( g(-t) \) is anticausal and cannot be realized. Practical receivers instead realize \( g(T - t) \), where \( T \) is sufficiently large to ensure causality.

In general \( g(t) \) may be difficult to implement by this method; however, the next subsection considers a discrete equivalent of whitening that is more straightforward to implement in practice. When the noise is complex, Equation (1.430) generalizes to
\[ \tilde{S}_n^+(s) = [\tilde{S}_n^-(s^*)]^* \]  
(1.433)

Whitening of the MIMO channel will typically consist of an inverted matrix-square-root constant matrix to remove any spatial correlation between noises of different parallel channels, then followed by a scalar further whitening in time-frequency as in this subsection. (Practically, the same noise is hitting the different antennas or wires and this is spatially removed by the square root with the remaining core noise then whitened. More generally, this whitening can be handled in the practical discrete-time cases as shown in Chapters 4 and 5.)

1.3.7.2.2 The Vector Self-Correlated Gaussian Noise Channel This subsection considers a discrete equivalent of the ACGN
\[ y = x + n \]  
(1.434)

where the autocorrelation matrix of the noise vector \( n \) is
\[ E[n \cdot n^*] = \begin{bmatrix} R_n & \tilde{R}_n \end{bmatrix} \sigma^2 . \]
(1.435)

Both \( R_n \) and \( \tilde{R}_n \) are positive definite matrices. This discrete ACGN channel can often be substituted for the continuous ACGN channel. MIMO applies here directly with the correlation of noise now introducing a dependency of sorts between the parallel channels. All analysis proceeds identically, MIMO or not. The discrete noise vector can be “whitened”, transforming \( \tilde{R}_n \) into an identity matrix. The discrete equivalent to whitening \( g(t) \) by \( g(t) \) is a matrix multiplication. The \( N \times N \) whitening matrix in the discrete case corresponds to the whitening filter \( g(t) \) in the continuous case.

Cholesky factorization determines the invertible whitening transformation according (see Appendix A of Chapter 3):
\[ \tilde{R}_n = \tilde{R}^{1/2} \cdot \tilde{R}^{*1/2} , \]
(1.436)

where \( \tilde{R}^{1/2} \) is lower triangular and \( \tilde{R}^{*1/2} \) is upper triangular. These matrices constitute the matrix equivalent of a “square root”, and both matrices are invertible. Noting the definitions,
\[ \tilde{R}^{-1/2} \triangleq \left[ \tilde{R}^{1/2} \right]^{-1} , \]
(1.437)
\[ \tilde{R}^{-*1/2} \triangleq \left[ \tilde{R}^{*1/2} \right]^{-1} , \]
(1.438)

then to whiten \( n \), the receiver passes \( y \) through the matrix multiply \( \tilde{R}^{-1/2} \),
\[ \tilde{y} \triangleq \tilde{R}^{-1/2} \cdot y = \tilde{R}^{-1/2} \cdot x + \tilde{R}^{-1/2} \cdot n = \tilde{x} + \tilde{n} . \]
(1.439)
The autocorrelation matrix for $\tilde{n}$ is

$$E[\tilde{n} \cdot \tilde{n}^*] = \bar{R}^{-/2} E[nn^*] \bar{R}^{s/2} = \bar{R}^{-/2} \left( \bar{R}^{1/2} \bar{R}^{s/2} \cdot \sigma^2 \right) \bar{R}^{-s/2} = \sigma^2 \cdot I. \quad (1.440)$$

Thus, the covariance matrix of the transformed noise $\tilde{n}$ is the same as the covariance matrix of the AWGN vector. By the theorem of reversibility, no information is lost in such a transformation.

**EXAMPLE 1.3.15 (QPSK with correlated noise)** For the example shown in Figure 1.22 suppose that the noise is colored with correlation matrix

$$R_n = \sigma^2 \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \quad (1.441)$$

Then

$$\bar{R}^{1/2} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad (1.442)$$

and

$$\bar{R}^{s/2} = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (1.443)$$

From (1.442),

$$\bar{R}^{-/2} = \begin{bmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix} \quad (1.444)$$

and

$$\bar{R}^{-s/2} = \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix}. \quad (1.445)$$

The signal constellation after the whitening filter becomes

$$\tilde{x}_0 = \bar{R}^{-/2} x_0 = \begin{bmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{2} - 1 \end{bmatrix}, \quad (1.446)$$

and similarly $\tilde{x}_2 = \begin{bmatrix} -1 & (\sqrt{2} + 1) \end{bmatrix}'$, $\tilde{x}_1 = \begin{bmatrix} 1 & (\sqrt{2} - 1) \end{bmatrix}'$, and $\tilde{x}_3 = \begin{bmatrix} -1 & (\sqrt{2} - 1) \end{bmatrix}'$.

This new constellation forms a parallelogram in two dimensions, where the minimum distance
is now along the shorter diagonal (between \(\tilde{x}_1\) and \(\tilde{x}_3\)), rather than along the sides and \(d_{\min} = 2.164 > 2\). This new constellation appears in Figure 1.63. Thus, the optimum detector for this channel with self-correlated Gaussian noise has larger minimum distance than for the white noise case, illustrating the important fact that having correlated noise is sometimes advantageous.

The example shows that correlated noise may lead to improved performance measured with respect to the same channel and signal constellation with white noise of the same average energy. Nevertheless, the noise autocorrelation matrix is often not known in implementation, or it may vary from channel use to channel use. Then, the detector is designed as if white noise were present anyway, and there is a performance loss with respect to the optimum detector. The next subsection deals with the calculation of this performance loss.

1.3.7.3 Performance of Suboptimal Detection with Self-Correlated Noise

A detector designed for the AWGN channel is obviously suboptimum for the ACGN channel, but is often used anyway, as the correlation properties of the noise may be hard to know in the design stage. In this case, the detector performance will be reduced with respect to optimum.

Computation of the amount by which performance is reduced uses the error-event vectors

\[
\epsilon_{ij} \Delta = \frac{x_i - x_j}{\|x_i - x_j\|} .
\] (1.447)

The noise vector’s component along an error-event vector is \(\langle n, \epsilon_{ij} \rangle\). The noise variance along this vector is

\[
\sigma_{ij}^2 \triangleq E\{\langle n, \epsilon_{ij} \rangle^2\} .
\]

Then, the NNUB becomes

\[
P_e \leq N_e Q \left[ \min_{i \neq j} \left\{ \frac{\|x_i - x_j\|^2}{2\sigma_{ij}} \right\} \right] .
\] (1.448)

For Example 1.3.15, the worst case argument of the Q-function in (1.448) is \(1/\sigma\), which represents a factor of \((2.164/2)^2 = .7\) dB loss with respect to optimum. This loss varies with rotation of the signal set, but not translation. If the signal constellation in Example 1.3.15 were rotated by 45°, as in Figure 1.27, then the increase in noise variance is \((1 + \sqrt{1/2})/\sqrt{2} = 2.3\) dB, but \(d_{\min}\) remains at 2 for this sub-optimum detector, so performance is 3 dB inferior than the optimum detector for the unrotated constellation. However, the optimum receiver for the rotated case would also have changed to have 3 dB worse performance for this rotation, so in this case the optimum rotated and sub-optimum rotated receiver have the same performance.
1.4 Finite-Field Channels

to be added
Chapter 1 Exercises

1.1 Our First Constellation - 10 pts

a. Show that the following two basis functions are orthonormal. (2 pts)

\[ \phi_1(t) = \begin{cases} \sqrt{2} \cos(2\pi t) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ \phi_2(t) = \begin{cases} \sqrt{2} \sin(2\pi t) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

b. Consider the following modulated waveforms.

\[ x_0(t) = \begin{cases} \sqrt{2} \cos(2\pi t) + \sin(2\pi t) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_1(t) = \begin{cases} \sqrt{2} \cos(2\pi t) + 3\sin(2\pi t) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_2(t) = \begin{cases} \sqrt{2} (3\cos(2\pi t) + \sin(2\pi t)) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_3(t) = \begin{cases} \sqrt{2} (3\cos(2\pi t) + 3\sin(2\pi t)) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_4(t) = \begin{cases} \sqrt{2} \cos(2\pi t) - \sin(2\pi t) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_5(t) = \begin{cases} \sqrt{2} (3\cos(2\pi t) - 3\sin(2\pi t)) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_6(t) = \begin{cases} \sqrt{2} (3\cos(2\pi t) - \sin(2\pi t)) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_7(t) = \begin{cases} \sqrt{2} (3\cos(2\pi t) - 3\sin(2\pi t)) & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_{i+8}(t) = -x_i(t) \] for \( i = 0, \ldots, 7 \)

d. Let

\[ y_i(t) = x_i(t) + 4\phi_3(t) \]

where

\[ \phi_3(t) = \begin{cases} 1 & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \]

Compute \( \mathcal{E}_y \) for the case where all signals are equally likely. (2 pts)
1.2 Inner Products - 10 pts
Consider the following signals:

\[ x_0(t) = \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{\pi}{6}\right) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_1(t) = \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{5\pi}{6}\right) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \]

\[ x_2(t) = \begin{cases} \frac{2}{\sqrt{T}} \cos\left(\frac{2\pi t}{T} + \frac{3\pi}{2}\right) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \]

a. Find a set of orthonormal basis functions for this signal set. Show that they are orthonormal.
    
    \text{Hint: Use the identity for } \cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b). \quad (4 \text{ pts})

b. Find the data symbols corresponding to the signals above for the basis functions you found in (a). \quad (3 \text{ pts})

c. Find the following inner products: \quad (3 \text{ pts})
   
   (i) \( <x_0(t), x_0(t)> \)
   (ii) \( <x_0(t), x_1(t)> \)
   (iii) \( <x_0(t), x_2(t)> \)

1.3 Multiple sets of basis functions - 5 pts
Consider the following two orthonormal basis functions:

\[ \varphi_1(t) = \begin{cases} 1/3 & \text{if } t \in [0, 2.25] \\ 0 & \text{otherwise} \end{cases} \]

\[ \varphi_2(t) = \begin{cases} 1/3 & \text{if } t \in [6.75, 9] \\ 0 & \text{otherwise} \end{cases} \]

Figure 1.64: Basis functions.

a. Use the basis functions given in Figure 1.64 to find the modulated waveforms \( u(t) \) and \( v(t) \) given the data symbols \( \mathbf{u} = [1 \ 1] \) and \( \mathbf{v} = [2 \ 1] \). It is sufficient to draw \( u(t) \) and \( v(t) \). \quad (2 \text{ pts})

b. For the same \( u(t) \) and \( v(t) \), a different set of two orthonormal basis functions is employed for which \( \mathbf{u} = [\sqrt{2} \ 0] \) produces \( u(t) \). Draw the new basis functions and find the \( \mathbf{v} \) that produces \( v(t) \). \quad (3 \text{ pts})

1.4 Minimal orthonormalization with MATLAB 5 pts
Each column of the matrix \( \Lambda \) given below is a data symbol that is used to construct its corresponding modulated waveform from the set of orthonormal basis functions \( \{\phi_1(t), \phi_2(t), \ldots, \phi_6(t)\} \). The set of modulated waveforms described by the columns of \( \Lambda \) can be represented with a smaller number of basis functions.
The transmitted signals \(a_i(t)\) are represented (with a superscript of * meaning matrix or vector transpose) as

\[
a_i(t) = a_i^* \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_6(t) \end{bmatrix}
\]

(1.451)

\[A(t) = A^* \varphi(t) \] (1.452)

Thus, each row of \(A(t)\) is a possible transmitted signal.

a. Use MATLAB to find an orthonormal basis for the columns of \(A\). Record the matrix of basis vectors.

The MATLAB commands `help` and `orth` will be useful. In particular, execution of \(Q = \text{orth}(A)\) in matlab produces a \(6 \times 3\) orthogonal matrix \(Q\) such that \(Q^*Q = I\) and \(A^* = [A^*Q]Q^*\). The columns of \(Q\) can be thought of as a new basis – thus try writing \(A(t)\) and interpreting to get a new set of basis functions and description of the 8 possible transmit waveforms. The Matlab command of `help orth` will give a summary of the `orth` command. To enter the matrix \(B\) in matlab (for example) shown below, simply type \(B=[1 \ 2; \ 3 \ 4]\); (2 pts)

\[B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\] (1.453)

b. How many basis functions are actually needed to represent our signal set? What are the new basis functions in terms of \(\{\phi_1(t), \phi_2(t), \ldots, \phi_6(t)\}\)? (2 pts)

c. Find the new matrix \(\hat{A}\) which gives the data symbol representation for the original modulated waveforms using the smaller set of basis functions found in (b). \(\hat{A}\) will have 8 columns, one for each data symbol. The number of rows in \(\hat{A}\) will be the number of basis functions you found in (b). (1 pts)

1.5 Decision rules for binary channels - 10 pts

a. Figure 1.65’s Binary Symmetric Channel (BSC) has binary (0 or 1) inputs and outputs. It outputs each bit correctly with probability \(1 - p\) and incorrectly with probability \(p\). Assume 0 and 1 are equally likely inputs. State the MAP and ML decision rules for the BSC when \(p < \frac{1}{2}\). How are the decision rules different when \(p > \frac{1}{2}\)? (5 pts)

b. Figure 1.66’s Binary Erasure Channel (BEC) has binary inputs as with the BSC. However there are three possible outputs. Given an input of 0, the output is 0 with probability \(1 - p_1\) and 2 with probability \(p_1\). Given an input of 1, the output is 1 with probability \(1 - p_2\) and 2 with probability \(p_2\). Assume 0 and 1 are equally likely inputs. State the MAP and ML decision rules for the BEC when \(p_1 < p_2 < \frac{1}{2}\). How are the decision rules different when \(p_2 < p_1 < \frac{1}{2}\)? (5 pts)

1.6 Minimax [Wesel] - 5 pts

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This exercise considers a 1-dimensional vector channel

\[ y = x + n, \]

where \( x = \pm 1, \) and \( n \) is Gaussian noise with \( \sigma^2 = 1. \) The Maximum-Likelihood (ML) Receiver that is
minimax, has decision regions:

\[ D_{ML, 1} = [0, \infty) \]

and

\[ D_{ML, -1} = (-\infty, 0) \]

So if \( y \) is in \( D_{ML, 1} \) then an ML receiver decodes \( y \) as \(+1\); and \( y \) in \( D_{ML, -1} \) decodes as \(-1\).

This exercise considers another receiver, \( R, \) where the decision regions are:

\[ D_{R, 1} = \left[ \frac{1}{2}, \infty \right) \]

and

\[ D_{R, -1} = \left( -\infty, \frac{1}{2} \right) \]
a. Find $P_{e,ML}$ and $P_{e,R}$ as a function of $p_X(1) = p$ for values of $p$ in the interval $[0, 1]$. On the same graph, plot $P_{e,ML}$ vs. $p$ and $P_{e,R}$ vs. $p$. (2 pts)

b. Find $\max_p P_{e,ML}$ and $\max_p P_{e,R}$. Are your results consistent with the Minimax Theorem? (2 pts)

c. For what value of $p$ is $D_R$ the MAP decision rule? (1 pt)

Note: For this problem you will need to use the $Q(\cdot)$ function discussed in Appendix B. Here are some relevant values of $Q(\cdot)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$Q(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.3085</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1587</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0668</td>
</tr>
</tbody>
</table>

1.7 Irrelevancy/Decision Regions. (From Wozencraft and Jacobs) - 7 pts

a. Consider the channel in Figure 1.67 where $x$, $n_1$, and $n_2$ are independent binary random variables. All the additions shown below are modulo two. (Equivalently, the additions may be considered xor’s.)

\[
\begin{align*}
\text{x} & \rightarrow \quad \text{n}_1 \quad \rightarrow \quad \text{+} \quad \rightarrow \quad \text{y}_1 \\
\text{x} & \rightarrow \quad \text{+} \quad \rightarrow \quad \text{y}_2 \\
\text{n}_1 + \text{n}_2 & \rightarrow \quad \text{+} \quad \rightarrow \quad \text{y}_3
\end{align*}
\]

Figure 1.67: 1st Channel for Irrelevancy/Decision Regions.

- Given only $y_1$, is $y_3$ relevant? (1 pt)
- Given $y_1$ and $y_2$, is $y_3$ relevant? (1 pt)

For the rest of the problem, consider the second channel in Figure 1.68. One of the two signals $x_0 = -1$ or $x_1 = 1$ is transmitted over this channel. The noise random variables $n_1$ and $n_2$ are statistically independent of the transmitted signal $x$ and of each other. Their density functions are,

\[ p_{n_1}(n) = p_{n_2}(n) = \frac{1}{2} e^{-|n|} \quad (1.454) \]

b. Given $y_1$ only, is $y_2$ relevant? (1 pt)
c. Prove that the optimum decision regions for equally likely messages are shown in Figure 1.69, (3 pts)

![Diagram](image1.png)

Figure 1.69: regions2

d. A receiver chooses $x_1$ if and only if $(y_1 + y_2) > 0$. Is this receiver optimum for equally likely messages? What is the probability of error? (Hint: $P_e = P\{y_1 + y_2 > 0/x = -1\} \cdot p_x(-1) + P\{y_1 + y_2/x = 1\} \cdot p_x(1)$ and use symmetry. Recall the probability density function of the sum of 2 random variables is the convolution of their individual probability density functions) (4 pts)

e. Prove that the optimum decision regions are modified as indicated in Figure 1.70 when $Pr\{X = x_1\} > 1/2$. (2 pts)
1.8 Optimum Receiver. (From Wozencraft and Jacobs) - 6 pts

Suppose one of M equiprobable signals \( x_i(t) \), \( i = 0, \ldots, M - 1 \) is to be transmitted during a period of time \( T \) over an AWGN channel. Moreover, each signal is identical to all others in the subinterval \( [t_1, t_2] \), where \( 0 < t_1 < t_2 < T \).

a. Show that the optimum receiver may ignore the subinterval \( [t_1, t_2] \). (2 pts)

b. Equivalently, show that if \( x_0, \ldots, x_{M-1} \) all have the same projection in one dimension, then this dimension may be ignored. (2 pts)

c. Does this result necessarily hold true if the noise is Gaussian but not white? Explain. (2 pts)

1.9 Receiver Noise (use MATLAB for all necessary calculations - courtesy S. Li, 2005.) - 13 pts

Each column of \( A \) given below is a data symbol that is used to construct its corresponding modulated waveform from a set of orthonormal basis functions (assume all messages are equally likely):

\[
\Phi(t) = \begin{bmatrix}
\phi_1(t) & \phi_2(t) & \phi_3(t) & \phi_4(t) & \phi_5(t) & \phi_6(t)
\end{bmatrix}.
\]

The matrix \( A \) is given by

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
5 & 6 & 7 & 8 & 5 & 6 & 7 & 8
\end{bmatrix}
\] (1.455)

so that

\[
x(t) = \Phi(t)A = [x_0(t) \ x_1(t) \ \ldots \ x_7(t)]
\] (1.456)

A noise vector \( n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{bmatrix} \) is added to the symbol vector \( x \), such that
\[ y(t) = \Phi(t) \cdot (x + n) \]

where \( n_1 \ldots n_6 \) are independent, with \( n_k = \pm 1 \) with equal probability.

The transmitted waveform \( y(t) \) is demodulated using an ML detector. This problem examines the signal-to-noise ratio of the demodulated vector \( y = x + n \) with \( \sigma^2 \triangleq E(n^2) \)

a. Find \( \bar{E}_x \), \( \sigma^2 \), and SNR, \( \bar{E}_x / \sigma^2 \) if all messages are equally likely. (2 pts)

b. Find the minimal number of basis vectors and new matrix \( \hat{A} \) as in Problem 1.4, and calculate the new \( \varepsilon_x \), \( \sigma^2 \), and SNR. (4 points)

c. Let the new vector be \( \tilde{y} = \tilde{x} + \tilde{n} \), and discuss if the conversion from \( y \) to \( \tilde{y} \) is invariant (namely, if \( P_e \) is affected by the conversion matrix). Compare the detectors for parts a and b. (1 point)

d. Compare \( \bar{b} \), \( \varepsilon_x \) with the previous system. Is the new system superior? Why or why not? (2 pts)

e. The new system now has three unused dimensions, and the source would like to send 8 more messages by constructing a big matrix \( \bar{A} \), as follows:

\[
\bar{A} = \begin{bmatrix}
\hat{A} & 0 \\
0 & \hat{A}
\end{bmatrix}
\]

Compare \( \bar{b} \), \( \varepsilon_x \) with the original 6-dimensional system, and the 3-dimensional system in b). (4 pts)

1.10 Tilt - 10 pts

Consider the signal set shown in Figure 1.71 with an AWGN channel and let \( \sigma^2 = 0.1 \).

![Figure 1.71: A Signal Constellation](image)

a. Does \( P_e \) depend on \( L \) and \( \theta \)? (1 pt)

b. Find the nearest neighbor union bound on \( P_e \) for the ML detector assuming \( p_x(i) = \frac{1}{9} \ \forall i \). (2 pts)

c. Find \( P_e \) exactly using the assumptions of the previous part. How close was the NNUB? (5 pts)

d. Suppose there is a minimum energy constraint on the signal constellation. How would this problem’s constellation be altered without changing the \( P_e \)? How does \( \theta \) affect the constellation energy? (2 pts)
1.11 Parseval - 5 pts Consider binary signaling on an AWGN $\sigma^2 = 0.04$ with ML detection for the following signal set. (Hint: consider various ways of computing $d_{\text{min}}$)

$$x_0(t) = \text{sinc}^2(t)$$
$$x_1(t) = \sqrt{2} \cdot \text{sinc}^2(t) \cdot \cos(4\pi t)$$

Determine the exact $P_e$ assuming that the two input signals are equally likely. (5 pts)

1.12 Disk storage channel - 10 pts

Binary data storage with a thin-film disk can be approximated by an input-dependent additive white Gaussian noise channel where the noise $n$ has a variance dependent on the transmitted (stored) input. The noise has the following input dependent density:

$$p(n) = \begin{cases} 
\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{n^2}{2\sigma_1^2}} & \text{if } x = 1 \\
\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{n^2}{2\sigma_0^2}} & \text{if } x = 0
\end{cases}$$

and $\sigma_1^2 = 31\sigma_0^2$. The channel inputs are equally-likely.

a. For either input, the output can take on any real value. On the same graph, plot the two possible output probability density functions (pdf’s). i.e. Plot the output pdf for $x = 0$ and the output pdf for $x = 1$. Indicate (qualitatively) the decision regions on your graph. (2 pts)

b. Determine the optimal receiver in terms of $\sigma_1$ and $\sigma_0$. (3 pts)

c. Find $\sigma_0^2$ and $\sigma_1^2$ if the SNR is 15 dB. SNR is defined as $\frac{\mathbb{E}[x_1^2]}{\mathbb{E}[n_1^2 + n_2^2]} = \frac{1}{\sigma_0^2 + \sigma_1^2}$. (1 pt)

d. Determine $P_e$ when SNR = 15 dB. (3 pts)

e. What happens as $\frac{\sigma_1^2}{\sigma_0^2} \rightarrow 0$? You may restrict your attention to the physically reasonable case where $\sigma_1$ is a fixed finite value and $\sigma_0 \rightarrow 0$. (1 pt)

1.13 Rotation with correlated noise - 7 pts

A two dimensional vector channel $y = x + n$ has correlated gaussian noise (that is the noise is not white and so not independent in each dimension) such that $\mathbb{E}[n_1] = \mathbb{E}[n_2] = 0$, $\mathbb{E}[n_1^2] = \mathbb{E}[n_2^2] = 0.1$, and $\mathbb{E}[n_1n_2] = 0.05$. $n_1$ is along the horizontal axis, and $n_2$ is along the vertical axis.

a. Suppose the transmitter uses the constellation in Figure 1.72 with $\theta = 45^\circ$ and $d = \sqrt{2}$, (i.e. $x_1 = (1, 1)$ and $x_2 = (-1, 1)$). Find the mean and mean square values of the noise projected on the line connecting the two constellation points. This value more generally is a function of $\theta$ when noise is not white. (2 pts)

b. The noise projected on the line in the previous part is Gaussian. Find $P_e$ for the ML detector. Assume the detector is designed for uncorrelated noise. (2 pts)

c. Fixing $d = \sqrt{2}$, find $\theta$ to minimize the ML detector $P_e$ and give the corresponding $P_e$. You may continue to assume that the receiver is designed for uncorrelated noise. (2 pts)

d. Could your detector in part a be improved by taking advantage of the fact that the noise is correlated? (1 pt)

1.14 Hybrid QAM - 10 pts

Consider the 64 QAM constellation with $d=2$ (see Figure 1.73.): The 32 hybrid QAM ($\times$) is obtained by taking one of two points of the constellation. This problem investigates the properties of such a constellation. Assume all points are equally likely and the channel is an AWGN.
a. Compute the energy $E_x$ of the 64 QAM and the 32 hybrid QAM constellations. (2 pts)

b. Find the NNUB for the probability of error for the 64 QAM and 32 hybrid QAM constellations. (3 pts)

c. What is $d_{min}$ for a 32 Cross QAM constellation having the same energy? (1 pt)

d. Find the NNUB for the probability of error for the 32 Cross QAM constellation. Compare with the 32 hybrid QAM constellation. Which one performs better? Why? (2 pts)

e. Compute the figure of merit for both 32 QAM constellations. Is your result consistent with the one of (d)? (2 pts)

1.15 Ternary Amplitude Modulation - 9 pts

Consider the general case of the 3-D TAM constellation for which the data symbols are,

$$(x_l, x_m, x_n) = \left(\frac{d}{2}(2l - 1 - M^\frac{1}{3}), \frac{d}{2}(2m - 1 - M^\frac{1}{3}), \frac{d}{2}(2n - 1 - M^\frac{1}{3})\right)$$

with $l = 1, 2, \ldots M^\frac{1}{3}, m = 1, 2, \ldots M^\frac{1}{3}, n = 1, 2, \ldots M^\frac{1}{3}$. Assume that $M^\frac{1}{3}$ is an even integer.
a. Show that the energy of this constellation is \(2\) pts

\[
E_x = \frac{1}{M} \left[ 3M^\frac{3}{2} \sum_{l=1}^{M^\frac{1}{2}} x_l^2 \right]
\]  
(1.457)

b. Now show that \(3\) pts

\[
E_x = \frac{d^2}{4} (M^{\frac{3}{2}} - 1)
\]

c. Find the NNUB \(P_e\) and \(P_e\) for an AWGN channel with variance \(\sigma^2\). \(3\) pts

d. Find \(b\) and \(\bar{b}\). \(1\) pt

e. Find \(E_x\) and the energy per bit \(E_b\). \(1\) pt

f. For an equal number of bits per dimension \(b = \bar{b}\), find the constellation figure of merit for PAM, QAM and TAM constellations with appropriate sizes of \(M\). Compare your results. \(2\) pts

1.16 Equivalency of rectangular-lattice constellations - \(9\) pts

Consider an AWGN system with a SNR = \(\frac{E_x}{\sigma^2}\) of \(22\) dB, a target probability of error \(P_e = 10^{-6}\), and a symbol rate \(\frac{1}{T} = 8\) KHz. The transmit power is \(20\) dBm.

a. Find the maximum data rate \(R = \frac{b}{T}\) that can be transmitted for \(2\) pts total this sub part

(i) PAM \((\frac{1}{2}\) pt)

(ii) QAM \((\frac{1}{2}\) pt)

(iii) TAM (1 pt) - see Problem 1.15

b. What is the NNUB normalized probability of error \(P_e\) for the systems used in (a). \(2\) pts

c. The remainder of this problem only considers QAM systems. Suppose that the desired data rate is \(40\) Kbps. What is the new transmit power needed to maintain the same probability of error? (The SNR is no longer \(22\) dB.) \(2\) pts

d. With a yet newer SNR of \(28\) dB, what is the highest data rate that can be reliably sent at the same probability of error \(10^{-6}\)? \(1\) pt

1.17 Frequency separation in FSK. (Adapted from Wozencraft & Jacobs) - \(5\) pts

Consider the following two signals used in a Frequency Shift Key communications system over an AWGN channel.

\[
x_0(t) = \begin{cases} \sqrt{\frac{2E_x}{T}} \cdot \cos(2\pi f_0(t)) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases}
\]

\[
x_1(t) = \begin{cases} \sqrt{\frac{2E_x}{T}} \cdot \cos(2\pi (f_0 + \Delta)t) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases}
\]

\(T = 100\mu s\) \(f_0 = 10^5\)Hz \(\sigma^2 = 0.01\) \(E_x = 0.32\)

a. Find \(P_e\) if \(\Delta = 10^4\). \(2\) pts

b. Find the smallest \(|\Delta|\) such that the same \(P_e\) found in part (a) is maintained. What type of constellation is this? \(3\) pts
1.18 Pattern Recognition - 8 pts

In this problem a simple pattern recognition scheme, based on optimum detectors is investigated. The patterns considered consist of a square divided into four smaller squares, as shown in Figure 1.74.

Each square may have two possible intensities, black or white. The class of patterns studied will consist of those having two black squares, and two white squares. For example, some of these patterns are as shown in Figure 1.75,

Each pattern can be encoded into a vector $x = [x_1 \ x_2 \ x_3 \ x_4]$ where each component indicates the ‘intensity’ of a small square according to the following rule,

Black square $\iff x_i = 1$
White square $\iff x_i = -1$

For a given pattern, a set of four sensors take measurements at the center of each small square and outputs $y = [y_1 \ y_2 \ y_3 \ y_4],

y = x + n \quad (1.458)$

Where $n = [n_1 \ n_2 \ n_3 \ n_4]$ is thermal noise (White Gaussian Noise) introduced by the sensors. The goal of the problem is to minimize the probability of error for this particular case of pattern recognition.

a. What is the total number of possible patterns ? (1 pt)
b. Write the optimum decision rule for deciding which pattern is being observed. Draw the corresponding signal detector. Assume each pattern is equally likely. (3 pts)
c. Find the union bound for the probability of error $P_e$. (2 pts)
d. Assuming that nearest neighbours are at minimum distance, find the NNUB for the probability of error $P_e$. (2 pts)

1.19 Shaping Gain - 8 pts

Find the shaping gain for the following two dimensional voronoi regions (decision regions) relative to the square voronoi region. Do this using the continuous approximation for a continuous uniform distribution of energy through the region.

a. equilateral triangle (2 pts)

b. regular hexagon (2 pts)

c. circle (2 pts)

d. Compare these different regions gains and explain the values qualitatively. (2 pts)

HINT: The following geometric identities may be helpful:

<table>
<thead>
<tr>
<th></th>
<th>equilateral triangle</th>
<th>circle</th>
<th>regular hexagon</th>
<th>square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>$\sqrt{3}/4 a^2$</td>
<td>$\pi r^2$</td>
<td>$3\sqrt{3}/8 a^2$</td>
<td>$d^2$</td>
</tr>
<tr>
<td>2nd Moment</td>
<td>$1/18 a^4$</td>
<td>$1/8 \pi r^4$</td>
<td>$5\sqrt{3}/16 a^4$</td>
<td>$1/6 d^4$</td>
</tr>
</tbody>
</table>

1.20 Recognize the Constellation (From Wozencraft and Jacobs) - 5 pts

On an additive white Gaussian noise channel, determine $P_e$ for the following signal set with ML detection. The answer will be in terms of $\sigma^2$.

(Hint: Plot the signals and then the signal vectors.)

\[
\begin{align*}
  x_1(t) &= \begin{cases} 
    1 & \text{if } t \in [0, 1] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_2(t) &= \begin{cases} 
    1 & \text{if } t \in [1, 2] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_3(t) &= \begin{cases} 
    1 & \text{if } t \in [0, 2] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_4(t) &= \begin{cases} 
    1 & \text{if } t \in [2, 3] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_5(t) &= \begin{cases} 
    1 & \text{if } t \in [0, 1] \\
    1 & \text{if } t \in [2, 3] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_6(t) &= \begin{cases} 
    1 & \text{if } t \in [1, 3] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_7(t) &= \begin{cases} 
    1 & \text{if } t \in [0, 3] \\
    0 & \text{otherwise}
  \end{cases} \\
  x_8(t) &= 0
\end{align*}
\]

1.21 Comparing bounds - 6 pts

Consider the following signal constellation in use on an AWGN channel.

\[
\begin{align*}
  x_0 &= (-1, -1) \\
  x_1 &= (1, -1) \\
  x_2 &= (-1, 1) \\
  x_3 &= (1, 1) \\
  x_4 &= (0, 3)
\end{align*}
\]

Leave answers for parts a and b in terms of $\sigma$. 

125
a. Find the union bound on $P_e$ for the ML detector on this signal constellation. (2 pts)
b. Find the Nearest Neighbor Union Bound on $P_e$ for the ML detector on this signal constellation. (2 pts)
c. Let the SNR = 14 dB and determine a numerical value for $P_e$ using the NNUB. (2 pts)

1.22 Basic QAM Design - 8 pts

Either square or cross QAM can be used on an AWGN channel with SNR = 30.2 dB and symbol rate $1/T = 10^6$.

a. Select a QAM constellation and specify a corresponding integer number of bits per symbol, $b$, for a modem with the highest data rate such that $P_e < 10^{-6}$. (3 pts)
b. Compute the data rate for part a. (1 pt)
c. Repeat part a if $P_e < 2 \times 10^{-7}$ is the new probability of error constraint. (3 pts)
d. Compute the data rate for part c. (1 pt)

1.23 Basic Detection - One shot or Two? - 10 pts

A 2B1Q signal with $d = 2$ is sent two times in immediate succession through an AWGN channel with transmit filter $p(t)$, which is a scaled version of the basis function. All other symbol times, a symbol value of zero is sent. The symbol period for one of the 2B1Q transmissions is $T = 1$, and the transmit filter is $p(t) = 1$ for $0 < t < 2$ and $p(t) = 0$ elsewhere. At both symbol periods, any one of the 4 messages is equally likely, and the two successive messages are independent. The WGN has power spectral density $N_0 = 5$.

a. Draw an optimum (ML) basis detector and enumerate a signal constellation. (Hint: use basis functions.) (3 pts)
b. Find $d_{\text{min}}$. (2 pts)
c. Compute $\tilde{N}_e$ counting only those neighbors that are $d_{\text{min}}$ away. (2 pts)
d. Approximate $P_e$ for your detector. (3 pts)

1.24 Discrete Memoryless Channel - 10 pts

Given a channel with $y|x$ as shown in Figure 1.76: ($y \in \{0, 1, 2\}$ and $x \in \{0, 1, 2\}$)

![Discrete Memoryless Channel](image)

Figure 1.76: Discrete Memoryless Channel
Let \( p_1 = 0.05 \)

a. For \( p_x(i) = 1/3 \), find the optimum detection rule. (3 pts)

b. Find \( P_e \) for part a. (3 pts)

c. Find \( P_e \) for the MAP detector if \( p_x(0) = p_x(1) = 1/6 \) and \( p_x(2) = 2/3 \). (4 pts)

### 1.25 Detection with Uniform Noise - 9 pts

A one-dimensional additive noise channel, \( y = x + n \), has uniform noise distribution

\[
p_n(v) = \begin{cases} 
\frac{1}{L} & |v| \leq \frac{L}{2} \\
0 & |v| > \frac{L}{2}
\end{cases}
\]

where \( L/2 \) is the maximum noise magnitude. The input \( x \) has binary antipodal constellation with equally likely input values \( x = \pm 1 \). The noise is independent of \( x \).

a. Design an optimum detector (showing decision regions is sufficient.) (2 pts)

b. For what value of \( L \) is \( P_e < 10^{-6} \) ? (1 pt)

c. Find the SNR (function of \( L \)). (2 pts)

d. Find the minimum SNR that ensures error-free transmission. (2 pts)

e. Repeat part d if 4-level PAM is used instead. (2 pts.)

### 1.26 Can you design or just use formulae? 8 pts

32 CR QAM modulation is used for transmission on an AWGN with \( \frac{N_0}{2} = 0.001 \). The symbol rate is \( 1/T = 400 kHz \).

a. Find the data rate \( R \). (1 pt)

b. What SNR is required for \( P_e < 10^{-7} \) ? (ignore \( N_e \)). (2 pts)

c. In actual transmitter design, the analog filter rarely is normalized and has some gain/attenuation, unlike a basis function. Thus, the average power in the constellation is calibrated to the actual power measured at the analog input to the channel. Suppose \( E_x = 1 \) corresponds to 0 dBm (1 milliwatt), then what is the power of the signals entering the transmission channel for the 32CR in this problem with \( P_e < 10^{-7} \) ? (1 pt)

d. The engineer under stress. Without increasing transmit power or changing \( \frac{N_0}{2} = 0.001 \), design a QAM system that achieves the same \( P_e \) at 3.2 Mbps on this same AWGN. (4 pts)

### 1.27 QAM Design - 10 pts

A QAM system with symbol rate \( 1/T=10 \) MHz operates on an AWGN channel. The SNR is 24.5 dB and a \( P_e < 10^{-6} \) is desired.

a. Find the largest constellation with integer \( b \) for which \( P_e < 10^{-6} \). (2 pts)

b. What is the data rate for your design in part a? (2 pts)

c. How much more transmit power is required (with fixed symbol rate at 10 MHz) in dB for the data rate to be increased to 60 Mbps? \( P_e < 10^{-6} \) (2 pts)

d. With SNR = 24 dB, an reduced-rate alternative mode is enabled to accommodate up to 9 dB margin or temporary increases in the white noise amplitude. What is the data rate in this alternative 9dB-margin mode at the same \( P_e < 10^{-6} \) ? (2 pts)

e. What is the largest QAM (with integer \( b \)) data rate that can be achieved with the same power, \( E_x/T \), as in part d, but with \( 1/T \) possibly altered? (2 pts)
1.28 Basic Detection 12 pts
A vector equivalent to a channel leads to the one-dimensional real system with \( y = x + n \) where \( n \) is exponentially distributed with probability density function
\[
p_n(u) = \frac{1}{\sigma \sqrt{2}} e^{-\frac{\sqrt{2}|u|}{\sigma}} \quad \text{for all } u
\] with zero mean and variance \( \sigma^2 \). This system uses binary antipodal signaling (with equally likely inputs) with distance \( d \) between the points. We define a function
\[
\tilde{Q}(x) = \begin{cases} 
\int_{x}^{\infty} \frac{1}{\sqrt{2}} e^{-\frac{\sqrt{2}u}{\sigma}} du = \frac{1}{2} \cdot e^{-\frac{\sqrt{2}|x|}{\sigma}} & \text{for } x \geq 0 \\
1 - \int_{|x|}^{\infty} \frac{1}{\sqrt{2}} e^{-\frac{\sqrt{2}u}{\sigma}} du = 1 - \frac{1}{2} \cdot e^{-\frac{\sqrt{2}|x|}{\sigma}} & \text{for } x \leq 0
\end{cases}
\] (1.460)
a. Find the values \( \tilde{Q}(-\infty), \tilde{Q}(0), \tilde{Q}(\infty), \tilde{Q}(\sqrt{10}) \). (2 pts)
b. For what \( x \) is \( \tilde{Q}(x) = 10^{-6} \)? (1 pt)
c. Find an expression for the probability of symbol error \( P_e \) in terms of \( d, \sigma \), and the function \( \tilde{Q} \). (2 pts)
d. Defining the SNR as \( \text{SNR} = \frac{\bar{E}_x}{\sigma^2} \), find a new expression for \( P_e \) in terms of \( \tilde{Q} \) and this SNR. (2 pts)
e. Find a general expression relating \( P_e \) to SNR, \( M \), and \( \tilde{Q} \) for PAM transmission. (2 pts)
f. What SNR is required for transmission at \( b = 1, 2, \) and \( 3 \) when \( P_e = 10^{-6} \)? (2 pts)
g. Would you prefer Gaussian or exponential noise if you had a choice? (1 pt)

1.29 QAM Design - 8 pts
QAM transmission is to be used on an AWGN channel with SNR=27.5 dB at a symbol rate of \( 1/T = 5 \text{ MHz} \) used throughout this problem. You’ve been hired to design the transmission system. The desired probability of symbol error is \( \bar{P}_e \leq 10^{-6} \).

a. (2 pts) List two basis functions that you would use for modulation.
b. (2 pts) Estimate the highest bit rate, \( \bar{b} \), and data rate, \( R \), that can be achieved with QAM with your design.
c. (1 pt) What signal constellation are you using?
d. (3 pts) By about how much (in dB) would \( \bar{E}_x \) need to be increased to have 5 Mbps more data rate at the same probability of error? Does your answer change for \( \bar{E}_x \) or for \( P_x \)?

1.30 7HEX Constellation - 10 pts
QAM transmission is used on an AWGN channel with \( \frac{N_0}{T} = .01 \). The transmitted signal constellation points for the QAM signal are given by \( \left[ \pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2} \right], \left[ 0, 0 \right], \text{and } \left[ 0, \pm 1 \right] \), with each constellation point equally likely.

a. (1 pt) Find \( M \) (message-set size) and \( \bar{E}_x \) (energy per dimension) for this constellation.
b. (2 pts) Draw the constellation with decision regions indicated for an ML detector.
c. (2 pts) Find \( N_e \) and \( d_{\text{min}} \) for this constellation.
d. (2 pts) Compute a NNUB value for \( \bar{P}_e \) for the ML detector of part b.
e. (1 pt) Determine \( \bar{b} \) for this constellation (value may be non-integer).
f. (2 pts) For the same $\bar{b}$ as part e, how much better in decibels is the constellation of this problem than SQ QAM?

1.31 Radial QAM Constellation - 7 pts
The QAM “radial” constellation in Figure 1.77 is used for transmission on an AWGN with $\sigma^2 = .05$. All constellation points are equally likely.

a. (2 pts) Find $E_x$ and $\bar{E}_x$ for this constellation.
b. (3 pts) Find $\bar{b}$, $d_{\text{min}}$, and $N_e$ for this constellation.
c. (2 pts) Find $P_e$ and $\bar{P}_e$ with the NNUB for an ML detector with this constellation.

1.32 A concatenated QAM Constellation - 15 pts
A set of 4 orthogonal basis functions $\{\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t)\}$ uses the following constellation in both the first 2 dimensions and again in the 2nd two dimensions: The constellation points are restricted in that an E (“even”) point may only follow an E point, and an O (“odd”) point can only follow an O point. For instance, the 4-dimensional point [+1 + 1 -1 -1] is permitted to occur, but the point [+1 +1 -1 +1] cannot occur.

a. (2 pts) Enumerate all $M$ points as ordered-4-tuples.
b. (3 pts) Find $b$, $\bar{b}$, and the number of bits/Hz or bps/Hz.
c. (1 pt) Find $E_x$ and $\bar{E}_x$ (energy per dimension) for this constellation.
d. (2 pts) Find $d_{\min}$ for this constellation.
e. (2 pts) Find $N_e$ and $\bar{N}_e$ for this constellation (you may elect to include only points at minimum distance in computing nearest neighbors).
f. (2 pts) Find $P_e$ and $\bar{P}_e$ for this constellation using the NNUB if used on an AWGN with $\sigma^2 = 0.1$.
g. (3 pts) Compare this 4-dimensional constellation fairly (which requires increasing the number of points in the constellation to 6 to get the same data rate). 4QAM.

1.33 Noise DAC - 15 pts

A random variable $x_1$ takes the 2 values ±1 with equal probability independently of a second random variable $x_2$ that takes the values ±2 also with equal probability. The two random variables are summed to $x = x_1 + x_2$, and $x$ can only be observed after zero-mean Gaussian noise of variance $\sigma^2 = .1$ is added, that is $y = x + n$ is observed where $n$ is the noise.

a. (1 pt) What are the values that the discrete random variable $x$ takes, and what are their probabilities? (1 pt)
b. (1 pt) What are the means and variances of $x$ and $y$?
c. (2 pts) What is the lowest probability of error in detecting $x$ given only an observation of $y$? Draw corresponding decision regions.
d. (1 pt) Relate the value of $x$ with a table to the values of $x_1$ and $x_2$. Explain why this is called a “noisy DAC” channel.
e. (1 pt) What is the (approximate) lowest probability of error in detecting $x_1$ given only an observation of $y$?
f. (1 pt) What is the (approximate) lowest probability of error in detecting $x_2$ given only an observation of $y$?
g. Suppose additional binary independent random variables are added so that the two bipolar values for $x_u$ are ±$2^{u-1}$, $u = 1, ..., U$. Which $x_u$ has lowest probability of error for any AWG noise, and what is that $P_e$? (1 pt)
h. For $U = 2$, what is the lowest probability of error in detecting $x_1$ given an observation of $y$ and a correct observation of $x_2$? (1 pt)
i. For $U = 2$, what is the lowest probability of error in detecting $x_2$ given an observation of $y$ and a correct observation of $x_1$? (1 pt)
j. What is the lowest probability of error in any of parts e through i if $\sigma^2 = 0$? What does this mean in terms of the DAC? (1 pt)
k. Derive a general expression for the probability of error for all bits $u = 1, ..., U$ where $x = x_1 + x_2 + ... + x_U$ in AWGN with variance $\sigma^2$ for part g? (2 pts)

1.34 Honeycomb QAM - 15 pts

The QAM constellation in Figure 1.79 is used for transmission on an AWGN with symbol rate 10MHz and a carrier frequency of 100 MHz.

Each of the solid constellation symbol possibilities is at the center of a perfect hexagon (all sides are equal) and the distance to any of the closest sides of the hexagon is $\frac{d}{2}$. The 6 empty points represent a possible message also, but each is used only every 6 symbol instants, so that for instance, the point
labelled 0 is a potential message only on symbol instants that are integer multiples of 6. The 1 point can only be transmitted on symbol instants that are integer multiples of 6 plus one, the 2 point only on symbol instants that are integer multiples of 6 plus two, and so on. At any symbol instant, any of the points possible on that symbol are equally likely.

a. What is the number of messages that can be possibly transmitted on any single symbol? What are $b$ and $\bar{b}$? (3 pts)
b. What is the data rate? (1 pt)
c. Draw the decision boundaries for time 0 of a ML receiver. (2 pts)
d. What is $d_{\text{min}}$? (1 pt)
e. What are $E_x$ and $\bar{E}_x$ for this constellation in terms of $d$? (3 pts)
f. What is the number of average number nearest neighbors? (1 pt)
g. Determine the NNUB expression that tightly upper bounds $\bar{P}_e$ for this constellation in terms of SNR. (2 pts)
h. Compare this constellation fairly to Cross QAM transmission. (1 pt)
i. Describe an equivalent ML receiver that uses time-invariant decision boundaries and a constant decision device with a simple preprocessor to the decision device. (1 pt).

1.35 Baseband Equivalents - 18 pts
A channel with additive white Gaussian noise has the channel shown in Figure 1.80 with unit gain and no phase distortion up to 50 MHz. The power spectral density of the noise is -103 dBm/Hz. The transmit power for a QAM modulator is 0 dBm = $E_x T$. The initial symbol rate is 1 MHz.

a. Suggest two ideal basis functions that use the lowest possible frequencies for this channel. (2 pts)
b. What is the SNR? (2 pts)
c. What is the data rate $R$ if $\bar{P}_e \leq 10^{-7}$? (2 pts)
d. What is the constellation used for your answer in part c? (1 pt)
e. Draw the modulator, and specify input bits, the message \( m \) and the mapping into the in-phase component \( x_I(t) \) and the quadrature component \( x_Q(t) \). (3 pts)
f. Draw a simple demodulator. (3 pts)
g. What is the highest data rate for \( \bar{P}_e \leq 10^{-7} \) using QAM or PAM and any symbol rate and/or carrier frequency? (3 pts)
h. What is the highest data rate for QAM/PAM if this channel had no filter and was purely an AWGN? (2 pts)

1.36 Comparison - 18 pts
Two passband transmission systems each use a transmit power \( (\frac{E_x}{T}) \) of 0 dBm to transmit 16 Mbps with a carrier frequency of \( f_c = 10^7 \) Hz over an AWGN characterized by \( \Delta f = -90 \text{dBm/Hz} \). System 1 uses SSB with \( M = 4 \) (looks like 4 PAM) constellation, while System 2 uses 16 QAM. The basis functions are:

System 1: \[ \varphi_1(t) = \frac{1}{\sqrt{T_1}} \cdot \text{sinc} \left( \frac{t}{T_1} \right) \cdot \cos (2\pi f_c t) - \frac{1}{\sqrt{T_1}} \cdot \text{sinc} \left( \frac{t}{T_1} \right) \cdot \sin (2\pi f_c t) \]

System 2: \[ \phi_1(t) = \frac{1}{\sqrt{T_2}} \cdot \text{sinc} \left( \frac{t}{T_2} \right) \cdot \cos (2\pi f_c t) \]
\[ \phi_2(t) = \frac{1}{\sqrt{T_2}} \cdot \text{sinc} \left( \frac{t}{T_2} \right) \cdot \sin (2\pi f_c t) . \]

(1.461)
a. Complete Table 1.5 below (13 pts).
b. Let represent the total postive bandwidth allowed for transmission and suppose \( W \leq 4 \text{ MHz} \). Which system should be used? (1 pt).
c. Suppose \( T \leq 125 \text{ ns} \). Which system should be used? (1 pt)
d. With no restrictions on \( W \) nor \( T \) (but maintaining constant transmit power of 0 dBm) and a gap of \( \Gamma = 8.8 \text{ dB} \), what is the highest data rate that can be achieved if only QAM designs are allowed? (3 pts)

1.37 Passband Representations - 9 pts
Consider the following passband waveform:
\[ x(t) = \text{sinc}^2(t) \cdot (1 + A \sin(4\pi t)) \cdot \cos(\omega_c t + \frac{\pi}{4}) , \]
where \( \omega_c >> 4\pi \).

Hint: It may be convenient in working this problem to use the identity \( \cos(a + b) = \cos a \cos b - \sin a \sin b \) to rewrite \( x(t) \) in inphase and quadrature, and to realize/define \( \text{sinc}^2(t) \) equal to a more general pulse shaping function \( p(t) \), recognizing that this particular choice of \( p(t) \) has a fourier transform that is well known and easily sketched.
Table 1.5: Table for Problem 1.92.

<table>
<thead>
<tr>
<th>Quantity desired</th>
<th>System 1</th>
<th>System 2</th>
<th>points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbol rate</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>N</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>( \bar{b} )</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>( E_x )</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>( \bar{E}_x )</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>SNR</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>( P_e )</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>( \bar{P}_e )</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

The table above represents the quantities and their corresponding points for Problem 1.92.

1. Sketch (roughly) \( \text{Re}[X(\omega)] \) and \( \text{Im}[X(\omega)] \). (2 pts)

2. Find \( x_{bb}(t) \), the baseband equivalent of \( x(t) \). Sketch (roughly) \( X_{bb}(\omega) \). (3 pts)

3. Find the \( x_{A}(t) \) analytic equivalent of \( x(t) \). (2 pts)

4. Find the Hilbert Transform of \( x(t) \). (2 pts)

1.38 A Two-Tap Channel - 8 pts

The two equally likely baseband signals, \( \tilde{x}_{bb,1}(t) \) and \( \tilde{x}_{bb,2}(t) \) illustrated in the following figure are used to transmit a binary sequence over a channel. The use of the scaling phase splitter in Figure 1.51 is assumed. Note that the two signals do not seem to be of the form \( (x_1 + jx_2)\phi(t) \) directly. However, this form can be applied if one views each of these two signals as a succession of four "one-shot" inputs to the channel, each of which can be construed as of the form \( (x_1 + jx_2)\phi(t - iT/4), \ i = 0, 1, 2, 3 \). This view is not necessary, however, to work this problem. Equivalently, an easy representation is a four-dimensional symbol vector.

The baseband equivalent channel impulse response is

\[
h_{bb}(t) = 4\delta(t) - 2\delta(t - T)
\]

The transmission rate is \( R = \frac{1}{2T} \) bits per second to avoid overlay of successive transmissions.

a. Sketch the two possible baseband-equivalent noise-free received waveforms. (2 pts)

b. Compute the squared distance between the two baseband possible signals as the integral of the squared difference between the two signals at the channel input. Compute the same distance after filtering by the channel impulse response. (2 pts)

c. Determine \( P_e \) for transmission of the two corresponding messages where \( n_{bb}(t) \) has autocorrelation \( r_{bb}(t) = N_0 \cdot \delta(t) \) with \( N_0 = \frac{E_x}{20} \) where \( E_x \) is the average energy before filtering by the channel response. Compare this to the situation where the channel simply passes \( x_{bb}(t) \) with no distortion and gain \( A \), where the channel would then have \( h_{bb}(t) = 2\delta(t) \). (4 pts)
1.39 A Bandpass Channel. (from Proakis - 10 pts)

The input $x(t)$ to a bandpass filter is

$$x(t) = u(t) \cdot \cos(w_c t)$$

where

$$u(t) = \begin{cases} 
A & \text{if } t \in [0, T] \\
0 & \text{otherwise}
\end{cases}$$

Please assume that $w_c$ is sufficiently high that $x(t)$ has only a negligible amount of energy near DC.

a. (3 pts) Determine the output $y(t) = g(t) * x(t)$ of a bandpass filter for all $t \geq 0$ if the impulse response of the filter is,

$$g(t) = \begin{cases} 
2 T e^{-t/T} \cos(w_c t) & \text{if } t \in [0, T] \\
0 & \text{otherwise}
\end{cases}$$

b. Sketch the equivalent lowpass output of the filter if it is passed through the scaling phase-splitter, $\tilde{y}_{bb}(t) = ?$ (1 pt)

c. Assume the baseband equivalent noise at the output of the scaling phase splitter has variance $N_0$ and that there are two dimensions. What is the SNR of the output? (3 pts)

d. For what value of $A$ is the $\text{SNR_{channel output}} = 13$ dB if the power spectral density of the channel’s AWGN ($\frac{N_0}{2}$) is -30 dBm/Hz and $1/T = 1000$ Hz? Repeat for -100 dBm/Hz and 1 MHz respectively. (3 pts)

1.40 Passband Equivalent System - 5 pts

A baseband-equivalent waveform ($w_c > 2\pi$)

$$\tilde{x}_{bb}(t) = (x_1 + jx_2) \cdot \text{sinc}(t)$$

is convolved with the complex filter

$$w_1(t) = \delta(t) - j\delta(t - 1)$$

a. (1 pt) Find

$$y(t) = w_1(t) * \tilde{x}_{bb}(t) \ .$$
b. (1 pt) Suppose \( y(t) \) is convolved with a second complex filter

\[
w_2(t) = 2j \cdot \text{sinc}(t)
\]

to get the complex filtered signal

\[
z(t) = w_2(t) \ast y(t)
\]
\[
= w_2(t) \ast w_1(t) \ast \tilde{x}_{bb}(t)
\]
\[
= w(t) \ast \tilde{x}_{bb}(t)
\]

so that \( z(t) \) is complex, yet corresponds to some passband signal. First find the complex signal \( z(t) \). Note that \( \text{sinc}(t) \ast \text{sinc}(t-k) = \text{sinc}(t-k) \), when \( k \) is an integer.

c. (3 pts) Let us define an analytic complex signal \( z_A(t) = z(t) \cdot e^{j\omega_ct} \) with real part:

\[
\tilde{z}(t) = \Re \{ z(t) \cdot e^{j\omega_ct} \} = \tilde{w}(t) \ast \tilde{x}(t)
\]

that is the result of some real filter \( \tilde{w}(t) \) acting on the transmitted passband signal \( \tilde{x}(t) = \Re \{\tilde{x}_{bb}(t)e^{j\omega_ct}\} \) (when convolved with the passband \( \tilde{x}(t) \) will produce \( \tilde{z}(t) \)). Show that

\[
\tilde{w}(t) = 4 \cdot \text{sinc}(t-1) \cdot \cos(\omega_c t) - 4 \cdot \text{sinc}(t) \cdot \sin(\omega_c t)
\]

and that correspondingly.

\[
\tilde{w}_{bb}(t) = 4 \cdot [\text{sinc}(t-1) - j\text{sinc}(t)]
\]

Hint: Use baseband calculations.

1.41 Matlab demodulator - 10 pts

This problem uses three matlab files: \texttt{x.mat}, \texttt{plt fft.m}, and \texttt{plt cplx.m}. These files are available at the web-site

\texttt{http://web.stanford.edu/group/cioffi/ee379a/;\;\;\;\;}.\;

This problem’s objective is to demodulate three symbols of a passband 4 QAM signal. The baseband basis function is a windowed sinc function. The sampling rate that provided the digital received signal was 1000 Hz. The received signal that this problem uses is the real part of the analytic signal. Usually the signal will have been convolved with a channel response and had noise added, but this problem ignores noise (so it is zero).

a. First, download the three needed files. Execute \texttt{load x.mat} which will create a 747 point vector \( x \) which contains 3 symbol periods of received signal. (Each symbol period is 249 samples). The function \texttt{plt cplx} plots the real and imaginary parts of a complex vector. It takes two arguments, the vector and the plot title. Execute \texttt{plt cplx(x, ’received signal’)} to produce a plot (and provide your plot). The received signal should be real. While the constellation points are not readily evident, the three symbols should be fairly evident. (1 pt)

b. Now, execute \texttt{plt fft(x, ’received signal’)} to plot an FFT of the received signal. Provide the plot. Neglect powers that are 50 dB smaller than other signals and report the frequency range to which this signal is bandlimited (2 pts)

c. Now use the function \texttt{hilbert()} provided by MATLAB to recover the analytic signal \( x_A \). You might want to execute \texttt{help hilbert} to get started. Use \texttt{plt fft} to plot the FFT of \( x_A \) (and provide the plot). How is this signal different from \( x \)? The discussion may again neglect signals 50 dB below peak signals. (2 pts)

d. The carrier frequency is 250 Hz. As mentioned before, the sampling frequency is 1000 Hz. Show that the discrete time radian carrier frequency is \( \frac{\pi}{2} \) radians/sec. (1 pt)
e. (4 pts) Demodulate \( x.A \) to create the baseband signal \( x.bb \) by executing the following command:

\[
x.bb = x.A .* \exp (-j * 0.5 * \pi * [0:746]);
\]

Plot both the FFT and complex time sequence as before (and provide the plots). In what range of frequencies is \( x.bb \) non-negligible? Why? By examining the complex time sequence plot, decode the received signal. The complex constellation points have been labeled as shown below. Correct selection of the correct constellation point for each symbol need only consider the sign of the real and imaginary scale factors multiplying the windowed sinc pulse.

![constellation plot](image)

**Figure 1.82: 4-QAM constellation with message labels 3, 5, 7 & 9**

### 1.42 Baseband Analysis - 4 pts

Consider the two baseband equivalent signals, \( \tilde{x}_{bb,1}(t) \) and \( \tilde{x}_{bb,2}(t) \).

\[
\tilde{x}_{bb,1}(t) = \begin{cases} 
A (1 + j) & \text{if } t \in [0, T] \\
0 & \text{otherwise}
\end{cases}
\]

\[
\tilde{x}_{bb,2}(t) = \begin{cases} 
A (1 - j) & \text{if } t \in [0, \frac{3}{4} T] \\
-A (1 - j) & \text{if } t \in (\frac{3}{4} T, T] \\
0 & \text{otherwise}
\end{cases}
\]

These signals can be used to transmit a binary signal set.

\[
p_{x_{bb}}(1) = p_{x_{bb}}(2) = \frac{1}{2}
\]

The transmitted signals are corrupted by AWGN having a baseband equivalent representation corresponding to the scaled phase splitter of Figure 1.51, \( \tilde{n}_{bb}(t) \), with an autocorrelation function

\[
r_{\tilde{n}_{bb}}(\tau) = E [\tilde{n}_{bb}(t)\tilde{n}_{bb}(t + \tau)] = N_0 \delta(t)
\]

a. Find \( E_x \). (2 pts)

b. Find \( P_e \) as a function of \( A, T \) and \( N_0 \). (1 pt)

c. Find \( \frac{A^2}{N_0} \) in terms of \( T \) if SNR=12.5 dB. Compute the probability of error \( P_e \) (a numerical value is required). (1 pt)

### 1.43 Twisted pairs - 3 pts

Figure 1.83 shows the magnitude(in dB) of the insertion losses (which is 6 dB more than the transfer function \( H(f) \)) for several lengths of two-types of twisted pair. Suppose the signals are passband, with frequencies ranging from 6MHz to 12 MHz. You want to analyse this transmission system in baseband. Assume a carrier frequency of \( f = 9 \) MHz. Assume the receiver uses a scaling phase splitter.

a. Draw the frequency responses of the complex channels to which you would apply the complex modulator input \( \tilde{x}_{bb}(t) \), corresponding to the scaling in Figure 1.51. (1 pt)
b. Compute the noise power spectral density (two-sided) of the WGN that you would add to each of your complex channel outputs to model transmission if the one-sided power spectral density of the AWGN noise on the channel is given as -140 dBm/Hz. (2 pts)

1.44 Signal Transformation Practice - 4 pts
Find the Hilbert transform of:

\[ x(t) = \text{sinc}\left(\frac{t}{T}\right) \cos(\omega_c t + \frac{\pi}{4}) \]

where \( \omega_c \geq \frac{\pi}{T} \).

1.45 Baseband Analysis with Parseval’s Help - 5 pts
The two baseband equivalent signals at the modulator output using the scaling in Figure 1.51 for binary transmission:

\[ \tilde{x}_0(t) = \frac{1}{\sqrt{T}} \text{sinc}\left(\frac{t}{T}\right) + j\sqrt{2} \frac{t}{T} \cos\left(\frac{\pi t}{T}\right) \cdot \text{sinc}(t/T) \]
\[ \tilde{x}_1(t) = j\sqrt{2} \frac{t}{T} \sin\left(\frac{\pi t}{T}\right) \cdot \text{sinc}(t/T) \]

are transmitted over an AWGN with \( \frac{N_0}{2} = .02 \).

a. Find \( \tilde{X}_0(f) \), \( \tilde{X}_1(f) \), and \( \tilde{X}_0(f) - \tilde{X}_1(f) \). (3 pts)

b. Determine \( P_e \). (2 pts)

1.46 Phase Distortion Only - 8 pts
A passband channel has complex baseband equivalent impulse response

\[ h_{bb}(t) = (1 + j)\delta(t) \]

A 4 QAM (QPSK) input with the constellation labeling below in Figure 1.84 is input to this channel. WGN is added at the output of this channel with power spectral density \( \frac{N_0}{2} = .04 \).

a. Calculate and draw the constellation for the corresponding four signal constellation points at the output of this channel and a scaling demodulator. Call these points \( \tilde{y}_0 \), ..., \( \tilde{y}_3 \) where the subscripts correspond to the subscripts on the channel input. (4 pts)

b. Write a quadrature decomposition for \( \tilde{y}_0 \)'s corresponding passband modulated signal. (2 pts)
c. Sketch the baseband AWGN power spectral density. (2 pts)

1.47 64 Single Sideband (SSB) - (18 pts)

Let \( m(t) \) be an 8-PAM signal with

\[
m(t) = \sum_k x_k \cdot p(t - kT)
\]

where \( x_k = \pm 1, \pm 3, \pm 5, \pm 7 \) and \( p(t) = \text{sinc}(2t/T) \). Also let \( \omega_c >> 2\pi/T \) while forming the single-sideband (SSB) modulated signal

\[
x(t) = m(t) \cos(\omega_c t) - \tilde{m}(t) \sin(\omega_c t)
\]

for transmission over an AWGN with \( N_0^2 \) as the power spectral density.

a. Find an analytic signal \( m_A(t) \) that is equivalent to \( x(t) \). Also find \( x_A(t) \) in terms of \( m_A(t) \). (3 pts)

b. Find the Hilbert transform of \( p(t) \), and rewrite your answer to part a without using \( m(t) \). (3 pts)

c. Write an expression for \( \tilde{m}(t) \) in terms of \( x_k \) that uses your result from part b. Evaluate this \( m(t) \) and \( \tilde{m}(t) \) in terms of integer multiple of \( T/2 \) sampling instants, \( n(T/2) \). What does this tell you about the pure phase changing of the Hilbert Transform and any interference at the channel output between successive transmitted symbols \( x_k \)? (2 pts)

d. What is the noise sample variance for the corresponding channel output samples of \( m(t) \)? Find the SNR for this sequence \( m(nT/2) \). Compare this SNR with that of 64SQ QAM on the same change with symbol rate \( 1/T \) and the same energy per dimension. (2 pts)

e. Find the single basis function for this SSB transmission system. Was straightforward estimation in part d of the symbol values from \( m(nT/2) \) an ML detector? Why or why not? Now compare again to 64 QAM. (4 pts)

f. Draw a block diagram of a ML receiver for this SSB transmission system using \( \tilde{m}(t) \) also. (4 pts)

g. Why is this signal called 64 VSB or equivalently 64 SSB (and not 8 SSB)? (1 pt)

h. For what \( N_0^2 \) is \( P_e < 10^{-6} \)? (answer in terms of \( T \)) (1 pt)

1.48 Complex Channel - Duobinary 1 + D - 19 pts

A complex channel, derived through the scaling of Figure 1.51, has binary inputs \( \tilde{x}_{ib}(t) \) in Figure 1.85 below. Let the passband filter be \( h_{ib}(t) = \delta(t) \) and the SNR = \( \frac{\bar{E}_x}{\sigma^2} = 10 \text{ dB} \) and the signal is considered two dimensional (one real and one imaginary dimension) for computation of \( \bar{E}_x \).
a. Write $\tilde{x}_{bb,1}(t)$ and $\tilde{x}_{bb,2}(t)$ in the form of 4 successive transmissions with symbol rate $1/T$. Each symbol must be of the form $(x_1 + jx_2) \cdot \varphi(t - kT)$, $k = 0, 1, 2, 3$. Thus, you must find $x_1, x_2,$ and $\varphi$. (4 pts)

b. Find $d_{\min}$ with $T = 1$. (2 pts)

c. Find $P_e$ for an ML detector with $T = 1$. (2 pts)

d. For the remainder of this problem, let $\frac{1}{T} h_{bb}(t) = \delta(t) + \delta(t - 1)$ and $T = 1$. Find the baseband-equivalents channel outputs prior to addition of baseband noise, $\tilde{n}_{bb}(t)$. (4 pts)

e. Find $d_{\min}$. (2 pts)

f. Find $P_e$ with an ML detector, assuming all inputs are equally likely. (2 pts)

g. Has the distortion introduced by the new $h_{bb}(t)$ improved or degraded this system? Why? (3 pts)

**1.49 Linear Frequency Decrease in Baseband 11 pts**

Figure 1.86 shows the Fourier transform, $H(f)$, of a bandlimited channel’s impulse response, $h(t)$. The input to the channel is

$$x(t) = \sqrt{2} \left\{ \sum_k a_k \cdot \varphi(t - kT) \right\} \cdot \cos(\omega_c t) - \left\{ \sum_k b_k \cdot \varphi(t - kT) \right\} \cdot \sin(\omega_c t) \right\} . \quad (1.462)$$

This channel is used for passband transmission with QAM and has AWGN with (2-sided) power spectral density $\sigma^2 = .01$. 

![Figure 1.86: Figure for “baseband equivalents”](image)

a. Draw $H_{bb}(f)$. (1 pt)

b. Find $h_{bb}(t)$. (2 pts)

c. Find the input $\tilde{x}_{bb}(t)$ as per Figure 1.51. (2 pts)
d. Find the complex channel model, including the channel complex impulse response and a numerical value for the noise power spectral density corresponding to the complex input you found in part c. The channel output should be the $\tilde{y}_{bb}(t)$ of Figure 1.51. (2 pts)

e. Find the analytic equivalent $h_A(t)$. (2 pts)

f. Write a simple expression for the Hilbert transform of $h(t)$. $\hat{h}(t) =$? (2 pts)

1.50 Linear Frequency Decrease in Passband - 10 pts

The Fourier transform of the impulse response of a channel is shown in Figure 1.87. The power spectral density of the additive Gaussian noise at the output of the channel is shown in Figure 1.88.

You are given the following integrals to avoid any need for doing integration in this problem (i.e., you can plug the formulae)

$$\int x^2 e^{bx} \, dx = \frac{e^{bx}}{b^2} \left( b^2 x^2 - 2bx + 2 \right) ; \quad \int xe^{bx} \, dx = \frac{e^{bx}}{b^2} \left( bx - 1 \right) ; \quad \int \frac{e^{bx}}{b} \, dx = \frac{e^{bx}}{b} \quad . \tag{1.463}$$

Throughout this problem, please use the scaling QAM demodulator of Figure 1.53.

a. Draw the Fourier transform of the complex channel, $\frac{1}{T}H_{bb}(f)$, that is used in this text to model the channel. (2 pts)

b. Find the power spectral density per real dimension of the noise in the complex baseband-equivalent channel that results from the scaled demodulator. (1 pt)
c. Find the complex-equivalent pulse response (time-domain) of the channel in part a if the transmitter uses the basis functions of QAM with $\varphi(t) = \frac{1}{\sqrt{T}} \cdot \text{sinc} \frac{t}{T}$. Interpret if $a \cdot \sqrt{T} = 1$ (4 pts)

d. If $\sigma^2 = 1$, and the transmitter sends 16 SQ QAM with constellation points $\begin{bmatrix} \pm 3 \\ \pm 1 \end{bmatrix}$, what is the lowest upper bound on the best possible SNR for a symbol-by-symbol detector on this channel? (3 pts)

1.51 Mini-Design - 12 pts
An AWGN with SNR=22 dB has baseband channel transfer function (there is no energy gain or loss in the channel):

$$H(f) = \begin{cases} 1 & |f| < 500 \text{ kHz} \\ 0 & |f| \geq 500 \text{ kHz} \end{cases}$$

(1.464)

a. Find an integer $\tilde{b}$ for QAM transmission with $P_e < 10^{-6}$ and compare with value found by the “gap approximation.” (2 pts)

b. Find the largest possible symbol rate and corresponding data rate. (2 pts)

c. Draw the corresponding signal constellation and label your points with bits. (2 pts) - hint, don’t concern yourself with clever bit labelings, just do it.

d. Find $P_b$ and $N_b$ for your design in part b. (3 pts)

e. Repart part b for PAM transmission on this channel and explain the difference in data rates. (3 pts)

1.52 Offset Carrier - 12 pts
For the AWGN channel with transfer function shown in Figure 1.89, a transmitted signal cannot exceed 1 mW (0 dBm) and the power spectral density is also limited according to $S_x(f) \leq -83 \text{ dBm/Hz}(\text{two-sided psd})$. The two-sided noise power spectral density is $\sigma^2 = -98 \text{ dBm/Hz}$. The carrier frequency is $f_c = 100 \text{ MHz}$ for QAM transmission. The probability of error is $P_e = 10^{-6}$.

![Figure 1.89: Channel Response.](image)

a. Find the baseband channel model, $\frac{1}{2} H_{bb}(f)$, for the scaled demodulator of Chapter 2. (2 pts)

b. Find the largest symbol rate that can be used with the 100 MHz carrier frequency? (1 pts)

c. What is the maximum signal power at the channel output with QAM? (2 pts)

d. What QAM data rate can be achieved with the symbol rate of part b? (2 pts)

e. Change the carrier frequency to a value that allows the best QAM data rate. (2 pts)

f. What is the new data rate for your answer in part e? (3 pts)
1.53 Complex Channel and Design - Midterm 2003 - 15 pts

Let \( x_k \) represent the successive independent transmitted symbols of a QAM constellation that can have only an integer number of bits for each symbol and for which each message is equally likely. Also let the pulse response of a filtered AWGN channel be \( p(t) = \sqrt{\frac{1}{T}} \cdot \text{sinc} \left( \frac{t}{T} \right) \) where \( \frac{1}{T} = 5 \text{ Mhz} \) is the symbol rate of the QAM transmission. The carrier frequency is 100 MHz. The transmitted signal has \( \bar{E}_x = 1.2 \) and the AWGN psd is \( \sigma^2 = .01 \). An expression for the modulated signal is

\[
x(t) = \Re \left\{ \sum_k x_k \cdot p(t - kT) \cdot e^{j\omega_c t} \right\}.
\]

(1.465)

Define \( g(t) = p(t) \cdot e^{j\omega_c t} \).

a. Find \( x_A(t) \) and \( x_{bb}(t) \). (2 pts)

b. Use the gap approximation to determine the number of bits per dimension, data rate, and the number of bits/second/Hz that can be transmitted on this channel with \( P_e \leq 10^{-6} \). (3 pts)

c. Suppose \( A_k = x_k \cdot e^{j\omega_c kT} \) is the actual message symbol sequence of interest for the rest of this problem. How do your answers in part b change (if at all)? Why? (2 pts)

d. Find the analytic equivalent of the channel output, \( y_A(t) \) in terms of only the message sequence \( A_k \) and \( g(t) \) without any direct use of the carrier frequency. (2 pts)

e. Draw an optimum (MAP) detector for \( A_k \) that does not use the carrier frequency directly. (3 pts)

f. Augment your answer in part e with a simple rotation that provides the MAP detector for \( x_k \). (1 pt)

g. This approach is used in some communications systems where the symbol rate and carrier frequency can be co-generated from the same oscillator, hence the knowledge of the symbol rate in the receiver tacitly implies then also knowing the carrier frequency. This is why the carrier was eliminated in the receiver of part e. Suggest a receiver implementation problem with this approach in general to replace QAM systems that would be used in transmission with symbol rates of up to 10 MHz and carriers above 1 GHz. (Hint - what does \( g(t) \) look like?) (2 pts)

1.54 Partial Response Class IV as a Passband Channel - 15 pts

A 16 QAM constellation is used to transmit a message over a filtered AWGN with \( \text{SNR} = \frac{\bar{E}_x}{\sigma^2} = 20 \text{ dB} \). The QAM symbol is given by\( \sqrt{\frac{2}{T}} \cdot [x_1 \cdot f(t) + x_2 \tilde{f}(t)] \) where \( f(t) = \text{sinc} \left( \frac{t}{T} \right) \). The real channel filter in has the shape shown in Figure 1.90 and is zero outside the frequency band of \((-1/T, 1/T)\).

\[
|H(f)| = \begin{cases} 
1 - e^{-j2\pi fT} & |f| \leq \frac{1}{T} \\
0 & \text{elsewhere}
\end{cases}
\]

Figure 1.90: Channel Response.

a. Show \( f(t) = \text{sinc} \left( \frac{t}{T} \right) \cdot \cos \left( \frac{\pi t}{T} \right) \) using frequency-domain arguments. Then find the Hilbert Transform \( \tilde{f}(t) \). (2 pts)
b. Find a quadrature representation for \( x(t) \) and the appropriate carrier frequency \( f_c \). (2 pts)

c. Find \( x_{bb}(t) \) and \( x_A(t) \). (2 pts)

d. Find \( h(t) \), \( h_{bb}(t) \), and \( h_A(t) \). **Hint:** multiplication of a transfer function by the ideal lowpass filter is the same as convolving with a sinc function, which may be useful in converting the obvious frequency-domain answers to the time domain. (2 pts)

e. For the scaled phase splitter of Chapter 2, find the output \( y_{bb}(t) \). (2 pts)

f. Design (draw) an optimum receiver for this single transmitted message using only one sampling device, one delay element, one matched filter, and one adder. Draw the decision regions for a single complex value that ultimately emanates from your receiver. (3 pts)

g. Calculate \( P_e \) for this optimum receiver. (2 pts)

1.55 **Vestigially Symmetric Channel Output - 12 pts**

\[ x_k \sqrt{2 \left( 5 \times 10^{-7} \right)} \quad Y(f) \quad x_k \sqrt{2 \left( 5 \times 10^{-7} \right)} \]

\(-15\quad -10\quad -5\quad 0\quad 5\quad 10\quad 15\quad f (\text{MHz})\]

Figure 1.91: Channel Response.

The Fourier Transform \( Y(f) \) of the output after QAM modulation for symbol \( x_k \) and channel and receiver filtering is shown in Figure 1.91. Assume that the receiver filter is of unit norm (unit energy) but is before the scaling phase-splitting operation. The AWGN of this channel (before receiver filtering) has power spectral density of \( \frac{N_0}{2} = 1 \), and let \( \bar{E}_x = 1 \).

a. Find \( \tilde{y}_{bb}(t) \) and its Fourier Transform \( \tilde{Y}_{bb}(f) \) after the scaling phase splitter. (2 pts)

b. Find the corresponding power spectral density of the baseband-equivalent noise \( \tilde{S}_{bb}(f) \) (2 pts)

c. Find the highest symbol rate for which there is no ISI. (2 pts)

d. Find the data rate if \( P_e \leq 10^{-6} \) for the symbol rate of part c. (4 pts)

1.56 **Baseband Equivalents - Midterm 2008 - 11 pts**

Figure 1.92 shows the Fourier Transform \( H(f) \) of a bandlimited channel’s impulse response \( h(t) \). The input to the channel is

\[ x(t) = \sqrt{2} \cdot \left\{ \left[ \sum_k a_k \cdot \varphi(t - kT) \right] \cdot \cos(2\pi f_c t) - \left[ \sum_k b_k \cdot \varphi(t - kT) \right] \cdot \sin(2\pi f_c t) \right\} \]

The channel is used for passband transmission with QAM and has AWGN with (2-sided) power spectral density \( \frac{N_0}{2} = 0.01 \).
a. Draw $H_{bb}(f)$. (1 pt)

b. Find $h_{bb}(t)$. (2 pts)

c. Find the input $\tilde{x}_{bb}(t)$ in terms of $\varphi(t)$. (1 pt)

d. For all that follows, let $\varphi(t) = \sqrt{\frac{2}{T}} \cdot \text{sinc} \left( \frac{2t}{T} \right)$. Show the complex channel model, including the channel complex impulse response and a numerical value for the noise power spectral density corresponding to the input you found in part c. Find the channel output $\tilde{y}_{bb}(t)$. (2 pts)

e. Find the analytic equivalent $h_{A}(t)$. (2 pts)

f. Write a simple expression for the Hilbert transform of $h(t)$, $\tilde{h}(t) = ?$ (2 pts)

g. Find $\tilde{y}(t)$. (1 pt)
Appendix A

Gram-Schmidt Orthonormalization Procedure

This appendix illustrates the construction of a set of orthonormal basis functions \( \varphi_n(t) \) from a set of modulated waveforms \( \{x_i(t), \ i = 0, \ldots, M - 1\} \). The process for doing so, and achieving minimal dimensionality is called **Gram-Schmidt Orthonormalization**.

**Step 1:**

Find a signal in the set of modulated waveforms with nonzero energy and call it \( x_0(t) \). Let

\[
\varphi_1(t) = \frac{x_0(t)}{\sqrt{\mathcal{E}_{x_0}}},
\]

where \( \mathcal{E}_{x} = \int_{-\infty}^{\infty} [x(t)]^2 dt \). Then \( x_0 = [\sqrt{\mathcal{E}_{x_0}} \ 0 \ldots 0] \).

**Step i for \( i = 2, \ldots, M \):**

- Compute \( x_{i-1,n} \) for \( n = 1, \ldots, i - 1 \) \( (x_{i-1,n} = \int_{-\infty}^{\infty} x_{i-1}(t) \varphi_n(t) dt) \).
- Compute

\[
\theta_i(t) = x_{i-1}(t) - \sum_{n=1}^{i-1} x_{i-1,n} \varphi_n(t)
\]

(A.2)

- if \( \theta_i(t) = 0 \), then \( \varphi_i(t) = 0 \), skip to step \( i + 1 \).
- If \( \theta_i(t) \neq 0 \), compute

\[
\varphi_i(t) = \frac{\theta_i(t)}{\sqrt{\mathcal{E}_{\theta_i}}} ,
\]

where \( \mathcal{E}_{\theta_i} = \int_{-\infty}^{\infty} [\theta_i(t)]^2 dt \). Then \( x_{i-1} = [x_{i-1,1} \ldots x_{i-1,i-1} \sqrt{\mathcal{E}_{\theta_i}} \ 0 \ldots 0]' \).

**Final Step:**

Delete all components, \( n \), for which \( \varphi_n(t) = 0 \) to achieve minimum dimensional basis function set, and reorder indices appropriately.
Appendix B

The Q Function

The Q Function is used to evaluate probability of error in digital communication - It is the integral of a zero-mean unit-variance Gaussian random variable from some specified argument to $\infty$:

**Definition B.0.1 (Q Function)**

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du
\]  

(B.1)

The integral cannot be evaluated in closed form for arbitrary $x$. Instead, see Figures B.1 and B.2 for a graph of the function that can be used to get numerical values. Note the argument is in dB ($20\log_{10}(x)$).

Note $Q(-x) = 1 - Q(x)$, so we need only plot $Q(x)$ for positive arguments.

We state without proof the following bounds

\[
(1 - \frac{1}{x^2}) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi x^2}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi x^2}}
\]

(B.2)

The upper bound in (B.2) is easily seen to be a very close approximation for $x \geq 3$.

Computation of the probability that a Gaussian random variable $u$ with mean $m$ and variance $\sigma^2$ exceeds some value $d$ then uses the Q-function as follows:

\[
P\{u \geq d\} = Q\left(\frac{d - m}{\sigma}\right)
\]

(B.3)

The Q-function appears in Figures B.3, B.1, and B.2 for very low SNR (-10 to 0 dB), low SNR (0 to 10 dB), and high SNR (10 to 16 dB) using a very accurate approximation (less than 1% error) formula from the recent book by Leon-Garcia:

\[
Q(x) \approx \left[\frac{\pi^{-1} x + \frac{1}{\pi} \sqrt{x^2 + 2\pi}}{\sqrt{2\pi}}\right] e^{-\frac{x^2}{2}}.
\]

(B.4)

For the mathematician at heart, $Q(x) = \frac{1}{2} \cdot \text{erfc}(x/\sqrt{2})$, where erfc is known as the complimentary error function by mathematicians.
Figure B.1: Low SNR Q-Function Values

Figure B.2: High SNR Q-Function Values
Figure B.3: Very Low SNR Q-Function Values
Appendix C

Hilbert Transform

The Hilbert transform is a linear operator that shifts the phase of a sinusoid by 90°:

**Definition C.0.1 (Hilbert Transform)** The Hilbert Transform of $x(t)$ is denoted $\hat{x}(t)$ and is given by

$$\hat{x}(t) = h(t) * x(t) = \int_{-\infty}^{\infty} \frac{x(u)}{\pi(t-u)} \, du ,$$

where

$$h(t) = \begin{cases} \frac{1}{\pi t} & t \neq 0 \\ 0 & t = 0 \end{cases} .$$

The Fourier Transform of $h(t)$ is

$$H(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} \frac{-j\sin \omega t}{\pi t} \, dt$$

$$= \int_{-\infty}^{\infty} \frac{-j\sin \pi u}{\pi u} \, \text{sgn}(\omega) \, du$$

$$= -j \left[ \int_{-\infty}^{\infty} \text{sinc}(u) \, du \right] \text{sgn}(\omega)$$

$$= -j \pi \text{sgn}(\omega) .$$

Equation (C.7) shows that a frequency component at a positive frequency is shifted in phase by $-90^\circ$, while a component at a negative frequency is shifted by $+90^\circ$. Summarizing

$$\hat{X}(\omega) = -j \text{sgn}(\omega) X(\omega) .$$

Since $|H(\omega)| = 1 \forall \omega \neq 0$, then $|X(\omega)| = |\hat{X}(\omega)|$, assuming $X(0) = 0$. This text only considers passband signals with no energy present at DC ($\omega = 0$). Thus, the Hilbert Transform only affects the phase and not the magnitude of a passband signal.

C.1 Examples

Let

$$x(t) = \cos(\omega_c t) = \frac{1}{2} (e^{j\omega_c t} + e^{-j\omega_c t}) ,$$

then

$$\hat{x}(t) = \frac{1}{2} (-je^{j\omega_c t} + je^{-j\omega_c t}) = \frac{1}{2j} (e^{j\omega_c t} - e^{-j\omega_c t}) = \sin(\omega_c t) .$$
Let
\[ x(t) = \sin(\omega_c t) = \frac{1}{2j} (e^{j\omega_c t} - e^{-j\omega_c t}) \quad , \]  
then
\[ \hat{x}(t) = \frac{1}{2j} (-je^{j\omega_c t} - je^{-j\omega_c t}) = -\frac{1}{2} (e^{j\omega_c t} + e^{-j\omega_c t}) = -\cos(\omega_c t) \quad . \]  
\[ (C.11) \]
\[ (C.12) \]
Note
\[ \hat{\hat{x}}(t) = h(t) * h(t) * x(t) = -x(t) \quad . \]  
\[ (C.13) \]
since \((-j\text{sgn}(\omega))^2 = -1 \forall \omega \neq 0\). A correct interpretation of the Hilbert transform is that every sinusoidal component is passed with the same amplitude, but with its phase reduced by 90 degrees.

C.2 Inverse Hilbert

The inverse Hilbert Transform is easily specified in the frequency domain as
\[ \mathcal{H}^{-1}(\omega) = j\text{sgn}(\omega) \quad , \]
\[ (C.14) \]
or then
\[ \hat{h}^{-1}(t) = -\hat{h}(t) = \begin{cases} -\frac{1}{\pi t} & t \neq 0 \\ 0 & t = 0 \end{cases} \quad . \]
\[ (C.15) \]
C.3 Hilbert Transform of Passband Signals

Given a passband signal \( x(t) \), form the quadrature decomposition
\[ x(t) = x_I(t) \cos(\omega_c t) - x_Q(t) \sin(\omega_c t) \quad \]  
\[ (C.16) \]
and transform \( x(t) \) into the frequency domain
\[ X(\omega) = \frac{1}{2} [X_I(\omega + \omega_c) + X_I(\omega - \omega_c)] - \frac{1}{2j} \left[ X_Q(\omega - \omega_c) - X_Q(\omega + \omega_c) \right] \quad . \]
\[ (C.17) \]
Equation C.17 shows that if \( X(\omega) = 0 \forall |\omega| > 2\omega_c \) then \( X_I(\omega) = 0 \) and \( X_Q(\omega) = 0 \forall |\omega| > \omega_c \).\footnote{Recall \( x_I(t) \) and \( x_Q(t) \) are real signals.} Using this fact the Hilbert transform \( \hat{X}(\omega) \) is given by
\[ \hat{X}(\omega) = \frac{j}{2} [X_I(\omega + \omega_c) - X_I(\omega - \omega_c)] + \frac{1}{2} [X_Q(\omega - \omega_c) + X_Q(\omega + \omega_c)] \quad . \]
\[ (C.18) \]
The inverse Fourier Transform of \( \hat{X}(\omega) \) then yields
\[ \hat{x}(t) = x_I(t) \sin(\omega_c t) + x_Q(t) \cos(\omega_c t) = \Im \{x_A(t)\} \quad , \]
\[ (C.19) \]
where \( \Im \) denotes the imaginary part.
Appendix D

Passband Processes

This appendix investigates properties of the correlation functions for a WSS passband random process \( x(t) \) in its several representations. For a brief introduction to the definitions of random processes see Appendix C.

D.1 Hilbert Transform

Let \( x(t) \) be a WSS real-valued random process and \( \tilde{x}(t) = h(t) * x(t) \) be its Hilbert transform. By Equation (E.8), the autocorrelation of \( \tilde{x} \) is

\[
\begin{align*}
r_{\tilde{x}}(\tau) &= h(\tau) * h^*(\tau) * r_x(\tau) = r_x(\tau).
\end{align*}
\]

(D.1)

Since \( |H(\omega)|^2 = 1 \quad \forall \; \omega \neq 0 \), and assuming \( S_X(0) = 0 \), then

\[
S_{\tilde{x}}(\omega) = S_x(\omega).
\]

(D.2)

Thus, a WSS random process and its Hilbert Transform have the same autocorrelation function and the same power spectral density.

By Equation (E.13)

\[
\begin{align*}
r_{\tilde{x},x}(\tau) &= h(\tau) * r_x(\tau) = \tilde{r}_x(\tau) \quad \text{(D.3)} \\
r_{x,\tilde{x}}(\tau) &= h^*(\tau) * r_x(\tau) = -h(\tau) * r_x(\tau) = -\tilde{r}_x(\tau) \quad \text{(D.4)}.
\end{align*}
\]

The cross correlation between the random process \( x(t) \) and its Hilbert transform \( \tilde{x}(t) \) is the Hilbert transform of the autocorrelation function of the random process \( x(t) \).

Note also that

\[
\begin{align*}
r_{\tilde{x},x}(\tau) &= h(\tau) * r_x(\tau) = h^*(\tau) * r_x(-\tau) = r_{x,\tilde{x}}(-\tau) = r_{\tilde{x},x}(-\tau) \quad \text{(D.5)}.
\end{align*}
\]

By using Equations (D.3), (D.4) and (D.5),

\[
r_{\tilde{x},x}(\tau) = \tilde{r}_x(\tau) = -r_{x,\tilde{x}}(\tau) = -r_{\tilde{x},x}(-\tau) \quad \text{(D.6)}.
\]

Equation (D.6) implies that \( r_{\tilde{x},x}(\tau) \) is an odd function, and thus

\[
r_{\tilde{x},x}(0) = 0.
\]

(D.7)

That is, a real-valued random process and its Hilbert Transform are uncorrelated at any particular point in time.
D.2 Quadrature Decomposition

The quadrature decomposition for any real-valued WSS passband random process and its Hilbert transform is

\[ x(t) = x_I(t) \cos \omega_c t - x_Q(t) \sin \omega_c t \]  \hspace{1cm} (D.8)
\[ \hat{x}(t) = x_I(t) \sin \omega_c t + x_Q(t) \cos \omega_c t. \]  \hspace{1cm} (D.9)

The baseband equivalent complex-valued random process is

\[ x_{bb}(t) = x_I(t) + jx_Q(t) \]  \hspace{1cm} (D.10)

and the analytic equivalent complex-valued random process is

\[ x_A(t) = x(t) + j\hat{x}(t) = x_{bb}(t)e^{j\omega_c t}. \]  \hspace{1cm} (D.11)

The original random process can be recovered as

\[ x(t) = \Re \{ x_A(t) \}. \]  \hspace{1cm} (D.12)

The autocorrelation of \( x_A(t) \) is

\[ r_A(\tau) = E \{ x_A(t)x_A^*(t - \tau) \} \]  \hspace{1cm} (D.13)
\[ = 2 \left( r_x(\tau) + j\hat{r}_x(\tau) \right). \]  \hspace{1cm} (D.14)

The right hand side of Equation (D.14) is twice the analytic equivalent of the autocorrelation function \( r_x(\tau) \). The power spectral density is

\[ S_A(\omega) = 4 \cdot S_x(\omega) \quad \omega > 0. \]  \hspace{1cm} (D.15)

The functions in the quadrature decomposition of \( x(t) \) also have autocorrelation functions:

\[ r_I(\tau) \overset{\Delta}{=} E \{ x_I(t)x_I^*(t - \tau) \} \]  \hspace{1cm} (D.16)
\[ r_Q(\tau) \overset{\Delta}{=} E \{ x_Q(t)x_Q^*(t - \tau) \} \]  \hspace{1cm} (D.17)
\[ r_{IQ}(\tau) \overset{\Delta}{=} E \{ x_I(t)x_Q^*(t - \tau) \} \]  \hspace{1cm} (D.18)

One determines

\[ r_x(\tau) = E \{ x(t)x^*(t - \tau) \} \]  \hspace{1cm} (D.19)
\[ = r_I(\tau) \cos \omega_c t \cos \omega_c (t - \tau) \]
\[ - r_{IQ}(\tau) \cos \omega_c t \sin \omega_c (t - \tau) \]
\[ - r_Q(\tau) \sin \omega_c t \cos \omega_c (t - \tau) \]
\[ + r_{IQ}(\tau) \sin \omega_c t \sin \omega_c (t - \tau). \]

Standard trigonometric identities simplify (D.19) to

\[ r_x(\tau) = \frac{1}{2} \left[ r_I(\tau) + r_Q(\tau) \right] \cos \omega_c \tau \]
\[ + \frac{1}{2} \left[ r_{IQ}(\tau) - r_{QI}(\tau) \right] \sin \omega_c \tau \]
\[ - \frac{1}{2} \left[ r_Q(\tau) - r_I(\tau) \right] \cos \omega_c (2t - \tau) \]
\[ - \frac{1}{2} \left[ r_{IQ}(\tau) + r_{QI}(\tau) \right] \sin \omega_c (2t - \tau). \]

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Strictly speaking, most modulated waveforms are WS cyclostationary with period $T_c = \frac{2\pi}{\omega_c}$, i.e. $E[x(t)x^*(t - \tau)] = r_x(t, t - \tau) = r_x(t + T_c, t + T_c - \tau)$ (See Appendix C). For cyclostationary random processes a time-averaged autocorrelation function of one variable $\tau$ can be defined by $r_x(\tau) = 1/T_c \int_{T_c/2}^{T_c/2} r_x(t, t - \tau) dt$, and this new time-averaged autocorrelation function will satisfy the properties derived thus far in this section. The next set of properties require that the random process to be WSS, not WS cyclostationary – or equivalently the time-averaged autocorrelation function $r_x(\tau)$ is uncorrelated at any particular instant in time. Substituting back into Equation D.20, \[ r_x(\tau) = r_I(\tau) \cos(\omega_c \tau) - r_{QI}(\tau) \sin(\omega_c \tau) \] (D.21)

Equation D.21 expresses the autocorrelation $r_x(\tau)$ in a quadrature decomposition and thus \[ \dot{r}_x(\tau) = r_I(\tau) \sin(\omega_c \tau) + r_{QI}(\tau) \cos(\omega_c \tau) \] (D.22)

Further algebra leads to \[ r_{bb}(\tau) = E \{ x_{bb}(t)x_{bb}^*(t - \tau) \} \] (D.23)
\[ = 2 (r_I(\tau) + j r_{QI}(\tau)) \] (D.24)
\[ r_A(\tau) = r_{bb}(\tau) e^{j \omega_c \tau} \] (D.25)

The power spectral density is \[ S_A(\omega) = S_{bb}(\omega - \omega_c) \] . (D.26)

If $S_x(\omega)$ is symmetric about $\omega_c$, then $S_{bb}(\omega)$ is symmetric about $\omega = 0$ (recall that the spectrum $S_A(\omega)$ is a scaled version of the positive frequencies of $S_X(\omega)$ and $S_{bb}(\omega)$ is $S_A(\omega)$ shifted down by $\omega_c$). In this case $r_{bb}(\tau)$ is real, and using Equation D.24, $r_{QI}(\tau) = 0$. Equivalently, the inphase and quadrature components of a random process are uncorrelated at any lag $\tau$ (not just $\tau = 0$) if the power spectral density is symmetric about the carrier frequency. Finally, \[ r_x(\tau) = \frac{1}{2} \Re \{ r_{bb}(\tau)e^{j \omega_c \tau} \} = \frac{1}{2} \Re \{ r_A(\tau) \} \] . (D.27)

If $x(t)$ is a random modulated waveform, by construction it is usually true that $r_I(\tau) = r_Q(\tau)$ and $r_{IQ}(\tau) = -r_{QI}(\tau) = 0$, so that the constructed $x(t)$ is WSS. For AWGN, $n(t)$ is usually WSS so that $r_I(\tau) = r_Q(\tau)$ and $r_{IQ}(\tau) = -r_{QI}(\tau) = 0$. When a QAM waveform is such that $r_{IQ}(\tau) \neq -r_{QI}(\tau)$ or $r_I(\tau) \neq r_Q(\tau)$, then $x(t)$ is WS cyclostationary with period $\pi/\omega_c$. 

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Appendix E

Random Processes (By Dr. James Aslanis)

This appendix briefly outlines the fundamental definitions of random processes used in this textbook. Random variables and processes are specified by upper-case letters in this appendix, and the corresponding lower case letters are used as variables of integration. Other sections and appendices relax notation and use the lower-case notation for both.

The natural mathematical construct to describe a noisy communication signal is a random process. Random processes are simply a generalization of random variables. Consider a single sample of a random process, which is described by a random variable $X$ with probability density $P_X(x)$. (Often the random process is assumed to be Gaussian but not always). This random variable only characterizes the random process for a single instant of time. For a finite set of time instants, a random vector $X = [X_{t_1}, \ldots, X_{t_n}]$ with a joint probability density function $P_X(x)$ describes the noise. Extension to a countably infinite set of random variables, indexed by $i$, defines a discrete-time random sequence.

**Definition E.0.1 (Discrete-Time Random Sequence)** A Discrete-Time Random Sequence $\{X_i\}$ is a countably infinite, indexed set of random variables described by a joint density function $P_X(x)$ where $X = [X_{t_i}, \ldots, X_{t_i+n}]$, where $i$ is an integer and $n$ is a positive integer.

The random variables in a random process need not be independent nor identically distributed, although the random variables are all defined on the same sample space. The probability density functions that define a random process do depend on the indices. If the index $i$ indicates time, then the probability densities are time-varying. Similarly, statistical averages, i.e. $E[f(X_i)]$, become functions of the index as well. These statistical averages, also known as ensemble averages, should not be confused with averaging over the time index (sometimes referred to as time averaging). For example the mean value of a random variable associated with the tossing of a die is approximated by averaging the values over many independent tosses. The time average is said to converge to the ensemble average (a property also referred to as mean ergodic). In a random process, each random variable in the collection of random variables may have a different probability density function. Time averaging over successive samples may not yield any information about ensemble averages.

Communication systems often observe a realization (function of time) of the random process, also called a sample function. For repetition of the experiment, a different sample function is observed. Thus, the random process $X_t$ also can be considered an ensemble of sample functions. Statistical averages are computed over the ensemble of sample functions. For certain processes the time averages (i.e. averaging over a single sample function the sequential values in time) may approximate well the ensemble averages.

Random processes are classified by statistical properties that their density functions obey, the most important of which is stationarity.
### E.1 Stationarity

**Definition E.1.1 (Strict Sense Stationarity)** A random process \( X_t \) is called **strict sense stationary (SSS)** if the joint probability density function

\[
P_{X_{t_1}, \ldots, X_{t_n}}(x_{t_1}, \ldots, x_{t_n}) = P_{X_{t_1+t}, \ldots, X_{t_n+t}}(x_{t_1+t}, \ldots, x_{t_n+t}) \quad \forall \ n, t, \{t_1, \ldots, t_n\} . \tag{E.1}
\]

Roughly speaking the statistics of \( X_t \) are invariant to a time shift; i.e. the placement of the origin \( t = 0 \) is irrelevant.

This text next considers commonly calculated functions of a random process: These functions encapsulate properties of the random processes, and a linear-systems analysis sometimes calculates these functions for processes without knowing the exact statistics of the process. Certain random processes, such as stationary Gaussian processes, are completely described by a collection of these functions.

**Definition E.1.2 (Mean)** The mean of a random process \( X_t \) is

\[
E(X_t) = \int_{-\infty}^{\infty} x_t \cdot P_X(x_t) \cdot dx_t \overset{\Delta}{=} \mu_X(t) . \tag{E.2}
\]

In general, the mean is a function of the index \( t \).

**Definition E.1.3 (Autocorrelation)** The autocorrelation of a random process \( X_t \) is

\[
E(X_{t_1}X_{t_2}^*) = \int_{-\infty}^{\infty} x_{t_1} \cdot x_{t_2}^* \cdot P_{X_{t_1}X_{t_2}}(x_{t_1}, x_{t_2}) \cdot dx_{t_1} \cdot dx_{t_2} \overset{\Delta}{=} r_X(t_1, t_2) \tag{E.3}
\]

In general, the autocorrelation is a two-dimensional function of the pair \( \{t_1, t_2\} \). For a stationary process, the autocorrelation is a one-dimensional function of the time difference \( t_1 - t_2 \) only:

\[
E(X_{t_1}X_{t_2}^*) = r_X(t_1 - t_2) = r_X(\tau) \tag{E.4}
\]

The stationary process’ autocorrelation also satisfies a Hermitian property \( r_X(\tau) = r_X^*(-\tau) \).

Using the mean and autocorrelation functions, also known as the first- and second-order statistics, engineers often define a weaker form of stationarity.

**Definition E.1.4 (Wide Sense Stationarity)** A random process \( X_t \) is called **wide sense stationary (WSS)** if

a. \( E(X_t) = \text{constant} \),

b. \( E(X_{t_1}X_{t_2}^*) = r_X(t_1 - t_2) \), i.e. a function of the time difference only.

One can show that SSS \( \Rightarrow \) WSS, but WSS \( \not\Rightarrow \) SSS. Often, random processes’ analysis only considers their first- and second-order statistics. Such results do not reveal anything about the random process’ higher-order statistics; however, in one special case the stationary random process is completely defined by the lower order statistics. In particular,

**Definition E.1.5 (Gaussian Random Process)** The joint probability density function of a **stationary real Gaussian random process** for any set of \( n \) indices \( \{t_1, \ldots, t_n\} \) is

\[
P_X(x) = \frac{1}{(2\pi)^{n/2}(\det \Lambda)^{1/2}} \exp \left[ -\frac{1}{2}(x - \mu)\Lambda^{-1}(x - \mu)' \right] , \tag{E.5}
\]

where the mean vector is given by \( \mu = E[X] \), and the covariance matrix is defined as \( \Lambda = E[(X - \mu)(X - \mu)'] \).

A complex Gaussian random variable has independent Gaussian random variables in both the real and imaginary parts, both with the same variance, which is half the variance of the complex random variable. Then, the distribution is

\[
P_X(x) = \frac{1}{(\pi)^{n}(\det \Lambda)} \exp \left[ -\frac{1}{2}(x - \mu)\Lambda^{-1}(x - \mu)' \right] . \tag{E.6}
\]

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For a Gaussian random process, the set of random variables \( \{X_{t_1}, \ldots, X_{t_n}\} \) are jointly Gaussian. A Gaussian random process also satisfies the following two important properties:

a. The output response of a linear time-invariant system to a Gaussian input is also a Gaussian random process.

b. A WSS, real-valued, Gaussian random process is SSS.

Much of the analysis in this textbook will consider Gaussian random processes passed through linear time-invariant systems. As a result of the properties listed above, the designer only requires the mean and autocorrelation functions of these processes to characterize them completely. Fortunately, one can calculate the effect of linear time-invariant systems on these functions without explicitly using the probability densities of the random process.

In particular, for a linear time-invariant system defined by an impulse response \( h(t) \), the mean of the output random process \( Y_t \) is

\[
\mu_Y(t) = \mu_X(t) \star h(t) .
\]  
(E.7)

The autocorrelation of the output can also be found as

\[
r_Y(t + \tau, t) = r_X(t + \tau, t) \star h(t + \tau) \star h^*(-t) .
\]  
(E.8)

In addition, many analyses use the correlation between the input and output random processes:

**Definition E.1.6 (Cross-correlation)** The cross-correlation between the random processes \( X_t \) and \( Y_t \) is given by

\[
E(X_{t_1}Y_{t_2}^*) \Delta r_{XY}(t_1, t_2) .
\]  
(E.9)

For a jointly WSS random processes, the cross-correlation only depends on the time difference

\[
r_{XY}(t_1, t_2) = r_{XY}(t_1 - t_2) = r_{XY}(\tau) .
\]  
(E.10)

The cross-correlation \( r_{XY}(\tau) \) does not satisfy the Hermitian property that the autocorrelation obeys, but

\[
r_{XY}(\tau) = r_{YX}^*(\tau) .
\]  
(E.11)

Further

\[
r_{XY}(\tau) = r_{X}(\tau) \star h^*(-\tau) \]  
(E.12)

\[
r_{YX}(\tau) = r_{X}(\tau) \star h(\tau) \]  
(E.13)

A more general form of stationarity is **cyclostationarity**, wherein the statistics of the random process are invariant only to specific shifts in the indices.

**Definition E.1.7 (Strict Sense Cyclostationarity)** A random process is strict sense cyclostationary if the joint probability density function satisfies

\[
P_{X_{t_1}, \ldots, X_{t_n}}(x_{t_1}, \ldots, x_{t_n}) = P_{X_{t_1+T}, \ldots, X_{t_n+T}}(x_{t_1+T}, \ldots, x_{t_n+T}) \forall n, \{t_1, \ldots, t_n\},
\]  
(E.14)

where \( T \) is called the period of the process.

That is, \( X_{t+kT} \) is statistically equivalent to \( X_t \) \( \forall t, k \). Cyclostationarity accounts for the regularity in communication transmissions that repeat a particular operation at specific time intervals; however, within a particular time interval, the statistics are allowed to vary arbitrarily. As with stationarity, a weaker form for cyclostationarity that depends only on the first and second order statistics of the random process is.

**Definition E.1.8 (Wide Sense Cyclostationarity)** A random process is wide sense cyclostationary if
a. \( E(X_t) = E(X_{t+kT}) \forall t, k. \)

b. \( r_X(t + \tau, t) = r_X(t + \tau + kT, t + kT) \forall t, \tau, k. \)

Thus, the mean and autocorrelation functions of a WS cyclostationary process are periodic functions with period \( T \). Many random signals in communications, such as an ensemble of modulated waveforms, satisfy the WS cyclostationarity properties.

The periodicity of a WS cyclostationary random process would complicate the study of modulated signals without use of the following convenient property. Given a WS cyclostationary random process \( X_t \) with period \( T \), the random process \( X_{t+\theta} \) is WSS if \( \theta \) is a uniform random variable over the interval \([0, T] \). Thus, analysis often shall include (or assume) a random phase \( \theta \) to yield a WSS random process.

Alternatively for a WS cyclostationary random process, there is a time-averaged autocorrelation function.

**Definition E.1.9 (Time-Averaged Autocorrelation)** The time-averaged autocorrelation of a WS cyclostationary random process \( X_t \) is

\[
\bar{r}_X(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} r_X(t + \tau, t) dt .
\]

Since the autocorrelation function \( r_X(t + \tau, t) \) is periodic, integration could be over any closed interval of length \( T \) in the preceding equation.

As in the study of deterministic signals and systems, frequency domain descriptions are often useful for analyzing random processes. First, this appendix continues with the definitions for deterministic signals.

**Definition E.1.10 (Energy Spectral Density)** The energy spectral density of a finite energy deterministic signal \( x(t) \) is \( |X(\omega)|^2 \) where

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \triangleq \mathcal{F}\{x(t)\}
\]

is the Fourier transform of \( x(t) \). Thus, the energy is calculable as

\[
\mathcal{E}_x \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega < \infty
\]

If the finite energy signal \( x(t) \) is nonzero for only a finite time interval, say \( T \), then the time average power in the signal equals \( P_x = \mathcal{E}_x/T \).

Communication signals are usually modeled as repeated patterns extending from \((-\infty, \infty)\), in which case the energy is infinite, although the time average power may be finite.

**Definition E.1.11 (Power Spectral Density)** The power spectral density of a finite power signal defined as

\[
S_x(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T}
\]

where \( X_T(\omega) = \mathcal{F}\{x_T(t)\} \) is the Fourier transform of the truncated signal

\[
x_T(t) = \begin{cases} x(t) & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}
\]

Thus, the time average power is calculable as \( P_x \triangleq \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(\omega) d\omega < \infty \)

For deterministic signals, the power \( P_x \) is a time-averaged quantity.

For a random process, we cannot simply take its Fourier transform to specify the power spectral density. Even if the integral of the random process is well-defined (and we do not present the requisite
mathematical tools in this text), the result would be another random process. Ensemble averages are required for frequency-domain analysis.

For a random process $X_t$, the ensemble average power, $P_{X_t} \triangleq E[|X_t|^2]$, may vary instantaneously over time. For a WSS random process, however, $P_{X_t} = P_X$ is a constant.

**Definition E.1.12 (Power Spectral Density)** For a WSS continuous-time random process $X_t$, the power spectral density is

$$S_X(\omega) = \mathcal{F}\{r_x(\tau)\}.$$  

(E.20)

One can show that $\int_{-\infty}^{\infty} S_X(\omega)d\omega = P_X$.

For a WS cyclostationary random process $X_t$ the autocorrelation function $r_X(t+\tau,t)$, for a fixed time lag $\tau$, is periodic in $t$ with period $T$. Consequently the autocorrelation function can be expanded using a Fourier series.

$$r_X(t+\tau,t) = \sum_{n=-\infty}^{\infty} \gamma_n(\tau) \cdot e^{j2\pi nt/T},$$  

(E.21)

where $\gamma_n(\tau)$ are the Fourier coefficients

$$\gamma_n(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} r_X(t+\tau,t) \cdot e^{-j2\pi nt/T} dt.$$  

(E.22)

The time average autocorrelation function is then

$$\bar{r}_X(\tau) = \gamma_0(\tau).$$  

(E.23)

The average power for the WS cyclostationary random process is

$$P_{X_t} = \frac{1}{T} \int_{-T/2}^{T/2} E[|X_t|^2]dt = \bar{r}_X(0) = \gamma_0(0) = \int_{-\infty}^{\infty} G_0(f)df$$  

(E.24)

where $G_0(f)$ is the Fourier transform of the $n=0$ Fourier coefficient

$$\mathcal{F}\{\gamma_0(\tau)\} = G_0(f) = \mathcal{F}\{\bar{r}_X(\tau)\}.$$  

(E.25)

The function $G_0(f)$ is the power spectral density of the WS cyclostationary random process $X_t$ associated with the time average autocorrelation $\bar{r}_X(\tau)$.

For a nonstationary random process, the average power must be calculated by both time and ensemble averaging, i.e. $P_{X_t} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} E[|X_t|^2]dt$.

### E.2 Linear Systems

For the linear system defined by $y(t) = h(t) \ast x(t)$, with a stationary input $x(t)$ and fixed deterministic impulse response $h(t)$, the following relationships are easily proved:

$$r_{YY}(\tau) = h(\tau) \ast h^*(-\tau) \ast r_{XX}(\tau)$$  

(E.26)

and thus

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega).$$  

(E.27)

These same relations hold in discrete time also.
Bibliography
