Appendix D - MMSE Estimation

Appendix D

Minimum Mean-Square Estimation, Normed Vector Spaces, Spectral Factorization, and Filter Realization

Minimum Mean-Square Error (MMSE) Estimation is fundamental to data transmission on channels with Gaussian noise. For these channels, with proper interpretation, all Shannon results, multi-user optima and capacity regions, and many suboptimal approximations all have basis in MMSE Theory. This appendix provides some fundamentals of MMSE that are pertinent to this text's developments. Chapter 2 uses often MMSE estimates and biases in its multi-user developments, as well as ties MMSE to the more traditional information-theoretic quantities of entropy and mutual information in the Gaussian case. Chapter 3 heavily uses MMSE theory for equalization developments and analysis, while Chapter 4's parallel channel developments fundamentally observe that MMSE estimates of linear weighted sums are linear weighted sums of the individual estimates, which allows MMSE theory also to govern independent parallel channels and multi-dimensional transmission. Chapter 5's unifying GDFE theory depends heavily on MMSE theory. Chapter 7's adaptive systems largely attempt to realize MMSE estimation.

Section D.1 begins with the general MMSE estimate, before focusing rapidly on jointly Gaussian processes for which this estimate is always a linear function of observed random variables or processes. The autocorrelation and cross-correlation matrices that have fundamental roles in transmission and also completely characterize zero-mean Gaussian processes also have formal definition. It also notes that the so-called "EVM" (error-vector magnitude) that some wireless and cable engineers use (apparently unknown to them) the mean-square error estimate long earlier used to characterize transmission performance. Section D.2 develops the **Orthogonality Principal** that this text uses throughout to minimize mean square errors, both for scalar and vector processes. Vector spaces and associated norms for scalars, vectors, and matrices all have formal definition, as does the matrix norm. This section provides the basic partial-derivative formulation that assures the simplifying orthogonality principle indeed solves the MMSE criteria. A useful generalization of the Pythagorean Theorem appears that can often assist understanding of various transmission SNR results. Section D.2 also details the so-called bias estimate and the result that pervades transmission theory that namely the "biased" MMSE estimate of a process from another has a signal to noise ratio that always adds 1 to the unbiased estimate's SNR, the latter is what transmission engineers always use as a figure of merit. This fundamentally ties MMSE to transmission. Section D.3 explores the deep relationship between signal processing transforms and factorizations and MMSE estimation. This section's topics include Fourier, Laplace, and D (Z) transforms and in particular the relationship between scalar spectrum factorization and filter realization that simplifies solutions to many MSE problems, especially for canonical data transmission and noise whitening. Section D also explains and derives the famed (scalar) bf Paley-Weiner Criterion. This section also develops Cholesky factorization as transform's spectral-factorization finite-length equivalent. All these topics develop from MMSE theory in this section, reinforcing intuition throughout the text. Section D.4 address a vector/matrix generalization of Paley Wiener Theory that this text calls MIMO Paley-Weiner, and which

provides elegant solution generalizations to many MIMO situations of increasing interest. In particular, this section develops in detail the relationship between MIMO finite space-time vector/matrix infinite-time/frequency series. Many problems not well understood for transmission in the literature appear here with solution and examples.

D.1 General MMSE estimation

Generally, the MMSE estimate of one random variable or process x based on other random variables or processes, organized in a vector $\boldsymbol{y}$, is some function $\hat{x} = f(\boldsymbol{y})$ that minimizes

$$\arg\min_{f(\boldsymbol{y})} \mathbb{E}\left[(x - f(\boldsymbol{y}))^2\right] \quad . \tag{D.1}$$

In this text, the quantities x and $\boldsymbol{y}$ have zero mean[1]. The vector $\boldsymbol{y}$'s dimensional elements generally have any organization, which can include time samples, frequency samples, spatial samples or other dimensional generalizations. The estimate is a function of a given specific random-variable sample $\boldsymbol{y} = \boldsymbol{v}$ and thus provides a specific estimate $f(\boldsymbol{v})$ that if averaged over all values for joint distribution of x and $\boldsymbol{y}$ minimizes the squared difference. Thus, the quantity

$$e(\boldsymbol{y}) \stackrel{\Delta}{=} x - f(\boldsymbol{y}) \tag{D.2}$$

is the error, and the quantity

$$MSE = \mathbb{E}\left[|e|^2\right] = \mathbb{E}\left[|x - f(\boldsymbol{y})|^2\right] \tag{D.3}$$

is the **mean-square error**, which averages over all possible joint random possibilities of x and $\boldsymbol{y}$. The MMSE estimate has a general form in the following theorem.

Theorem D.1.1 **[Minimum-Mean-Square-Error Estimate]** *The minimum mean-square-error estimate is generally* $\hat{x} = f(\boldsymbol{y}) = \mathbb{E}(x/\boldsymbol{y})$.
Proof: *For any given value of* $\boldsymbol{y} = \boldsymbol{v}$, *the conditional mean-square error is*

$$\begin{aligned} MSE/\boldsymbol{v} &= \mathbb{E}\left[(x - f(\boldsymbol{v}))^2/\boldsymbol{y} = \boldsymbol{v}\right] && \text{(D.4)} \\ \frac{dMSE}{df} &= \mathbb{E}\left[2 \cdot \Re\left\{x - f(\boldsymbol{v})\right\}\right] = 0 && \text{(D.5)} \\ f(\boldsymbol{v}) &= \mathbb{E}\left[x/\boldsymbol{y} = \boldsymbol{v}\right] \ . && \text{(D.6)} \end{aligned}$$

Since (D.6) holds for any $\boldsymbol{y} = \boldsymbol{v}$, *then*

$$\mathbb{E}\left[|e|^2\right] = E_{\boldsymbol{y}}\left[MSE/\boldsymbol{y} = \boldsymbol{v}\right] \ , \tag{D.7}$$

which since all $\boldsymbol{y}$ *probability-distribution values are non-negative has minimum when* $\hat{x}_{MMSE} = f(\boldsymbol{y}) = \mathbb{E}\left[x/\boldsymbol{y}\right]$ *for any specific random* $\boldsymbol{y}$ *observation. The MMSE is thus*

$$\sigma^2_{mmse} = \mathbb{E}\left[|x - \mathbb{E}(x/\boldsymbol{y})|^2\right] \quad \textbf{QED.} \tag{D.8}$$

This entire proof can now repeat with $x \to \boldsymbol{x}$ *and* $\mathbb{E}\left[|e|^2\right] \to \mathbb{E}\left[\|\boldsymbol{e}\|^2\right]$ *and partial derivatives taken for each term in (D.5) and this theorem now generalizes to*

$$\hat{\boldsymbol{x}}_{MMSE}(\boldsymbol{y}) = \mathbb{E}\left[\boldsymbol{x}/\boldsymbol{y}\right] \ . \tag{D.9}$$

The MMSE estimate is $\mathbb{E}(\boldsymbol{x}/\boldsymbol{y})$ in general for any x and $\boldsymbol{y}$ distribution(s). Because the expectation operation E is linear itself, MMSE estimates of linear combinations are the same combinations of the individual estimates so if

$$\boldsymbol{x} = \alpha \cdot \boldsymbol{x}_1 + \beta \cdot \boldsymbol{x}_2 \ , \tag{D.10}$$

then

$$\hat{\boldsymbol{x}}_{mmse} = \alpha \cdot \hat{\boldsymbol{x}}_1 + \beta \cdot \hat{\boldsymbol{x}}_2 \ . \tag{D.11}$$

[1] Any nonzero mean is non-information bearing and can be subtracted from the random process so for instance $x \to x - \mathbb{E}[x]$, without altering theoretical analysis.

D.1.1 Gaussian MMSE, autocorrelation, and cross-correlation

Gaussian and also linear MMSE estimates have two constituent correlation quantities:

Definition D.1.1 [autocorrelation and cross-correlation matrices]

The $N \times N$ **autocorrelation matrix** for any N-dimensional random vector $\boldsymbol{y}$ is

$$R_{\boldsymbol{yy}} \triangleq E\left[\boldsymbol{y} \cdot \boldsymbol{y}^*\right] \quad . \tag{D.12}$$

When $N = 1$, the autocorrelation matrix becomes simply the random vector's energy σ_y^2. When stationary random vector process $\boldsymbol{y}_k$ has discrete (continuous) time index k (t), then the autocorrelation matrix becomes itself a time sequence (function)

$$R_{\boldsymbol{yy},k} \triangleq \mathbb{E}\left[\boldsymbol{y}_l \cdot \boldsymbol{y}_{l-k}^*\right] \tag{D.13}$$

$$R_{\boldsymbol{yy}}(t) \triangleq \mathbb{E}\left[\boldsymbol{y}(u) \cdot \boldsymbol{y}^*(u-t)\right] \quad . \tag{D.14}$$

In this case when $N = 1$, the autocorrelation-matrix sequence becomes the random process' autocorrelation function. Similarly, the **cross-correlation matrix** for any two N_x- and N_y-dimensional random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is the $N_x \times N_y$ matrix

$$R_{\boldsymbol{xy}} \triangleq \mathbb{E}\left[\boldsymbol{x} \cdot \boldsymbol{y}^*\right] \quad . \tag{D.15}$$

Also, clearly $R_{\boldsymbol{yx}} = R_{\boldsymbol{xy}}^*$. When $N_x = N_y = 1$, the cross-correlation matrix becomes simply the two random vector's correlation $R_{x,y}$. The cross correlation between a scalar x and a vector $\boldsymbol{y}$ is thus a $1 \times N_y$ row vector $R_x\boldsymbol{y}$. When the two jointly stationary random vector processes $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ have discrete (continuous) time index k (or t), then the cross-correlation matrix becomes itself a time sequence (function)

$$R_{\boldsymbol{xy},k} \triangleq \mathbb{E}\left[\boldsymbol{x}_l \cdot \boldsymbol{y}_{l-k}^*\right] \tag{D.16}$$

$$R_{\boldsymbol{xy}}(t) \triangleq \mathbb{E}\left[\boldsymbol{x}(u) \cdot \boldsymbol{y}^*(u-t)\right] \quad . \tag{D.17}$$

When $N = 1$, the autocorrelation-matrix sequence becomes the random processes' cross-correlation function $R_{xy}(t)$.

Often AWGN channel noise is Gaussian, which (see Chapter 2) leads to best transmitted signals also being Gaussian, and then all signals being Gaussian. Thus, the Gaussian distribution's form has particular interest in digital transmission. Multivariate Gaussian variables, organized in terms of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ (each with possibly many elements), then depend only on the autocorrelation and cross-correlation matrices, organized into a single matrix R where

$$R \triangleq \mathbb{E}\left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} [\boldsymbol{x}^* \boldsymbol{y}^*] \right\} = \begin{bmatrix} R_{\boldsymbol{xx}} & R_{\boldsymbol{xy}} \\ R_{\boldsymbol{yx}} & R_{\boldsymbol{yy}} \end{bmatrix} \tag{D.18}$$

Definition D.1.2 [Multivariate Gaussian Distribution] It is helpful to provide both the real and complex forms. The real-valued and complex-valued multivariate zero-mean-vector Gaussian distributions are, using autocorrelation matrix R in (D.18),

$$\text{real: } p(\boldsymbol{x}, \boldsymbol{y}) = (2\pi)^{-\frac{N_x+N_y}{2}} \cdot |R|^{-1/2} \cdot e^{-\frac{1}{2}\left\{ \left[\boldsymbol{x}^* \boldsymbol{y}^*\right] \cdot R^{-1} \cdot \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \right\}} \tag{D.19}$$

$$\text{complex: } p(\boldsymbol{x},\boldsymbol{y}) \;=\; (\pi)^{-[N_x+N_y]}\cdot|R|^{-1}\cdot e^{-\left\{\left[\boldsymbol{x}^*\boldsymbol{y}^*\right]\cdot R^{-1}\cdot\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix}\right\}} \quad . \tag{D.20}$$

The complex case has N_x and N_y as complex dimensions (so twice as many real dimensions). The two distributions are indeed equivalent when viewed in terms of this text's per-dimensional normalizations. With nonzero mean $\boldsymbol{\mu} = \mathbb{E}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\mu_x}\\ \boldsymbol{\mu_y}\end{bmatrix}$, then $\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} \rightarrow \begin{bmatrix}\boldsymbol{x}-\boldsymbol{\mu_x}\\ \boldsymbol{y}-\boldsymbol{\mu_y}\end{bmatrix}$ in (D.19) and (D.20).

Singularity: When R is singular, then $R = \sum_{n=1}^{N_r<N}\lambda_n\cdot\boldsymbol{v}_n\boldsymbol{v}_n^*$ where λ_n are the nonzero eigenvalues and $\boldsymbol{v}_n$ are the corresponding eigenvectors; correspondingly $\lambda_{n>N_r}=0$. Singularity means the Gaussian process's projections on the eigenvvectors corresponding to zero eigenvalues are deterministic and the process is equal to the mean vector's projection on these corresponding eigenvectors with probability 1. With Gaussian channels $R_{\boldsymbol{yy}}$ is never singular when the noise is nonsingular $|R_{\boldsymbol{nn}}|>0$. However, $R_{\boldsymbol{xx}}$ can be singular in some cases that Chapter 5 studies in detail. Equations (D.19) and (D.20) for all Gaussian processes remain useful, but with $R^{-1}\rightarrow R^{+}$ (the pseudoinverse - see Appendix C) and with $|R|\rightarrow\prod_{n=1}^{N_r}\lambda_n$.

For jointly Gaussian random variables and processes, the MMSE estimate is linear in $\boldsymbol{y}$, which can be found through somewhat tedious algebra by taking the ratio $p(\boldsymbol{x},\boldsymbol{y})/p(\boldsymbol{y})$ and simplifying to a Guassian form and reading the expected value:

$$\hat{\boldsymbol{x}}_{mmse} = \mathbb{E}[\boldsymbol{x}/\boldsymbol{y}] = R_{\boldsymbol{xy}}\cdot R_{\boldsymbol{yy}}^{-1}\cdot\boldsymbol{y} \quad . \tag{D.21}$$

The MMSE estimate's autocorrelation is

$$R_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}} = R_{\boldsymbol{x}/\boldsymbol{y}} = R_{x\boldsymbol{y}}\cdot R_{\boldsymbol{yy}}^{-1}\cdot R_{\boldsymbol{y}x} \quad . \tag{D.22}$$

The error $\boldsymbol{x}-\mathbb{E}(\boldsymbol{x}/\boldsymbol{y})$ has autocorrelation

$$R_{\boldsymbol{ee}} = R_{\boldsymbol{xx}} - R_{\boldsymbol{xy}}\cdot R_{\boldsymbol{yy}}^{-1}\cdot R_{\boldsymbol{yx}} \triangleq R_{\boldsymbol{x}/\boldsymbol{y}}^{\perp} \quad . \tag{D.23}$$

In the case of scalar $x=\boldsymbol{x}$, then the result is readily seen to be the MMSE[2]

$$\sigma_{mmse}^2 \triangleq \sigma_{x/\boldsymbol{y}}^2 = \mathcal{E}_x - R_{x\boldsymbol{y}}\cdot R_{\boldsymbol{yy}}^{-1}\cdot R_{\boldsymbol{y}x} \quad . \tag{D.24}$$

When $R_{\boldsymbol{yy}}$ is singular, then replace inverse with pseudoinverse in this text; however with practical channels there is always some noise, no matter how small so this singularity is not likely of practical data-transmission interest. When $\boldsymbol{x}$ corresponds to a two-dimensional QAM constellation with corresponding error vector $\boldsymbol{e}$ measured and averaged with respect to constellation points, engineers sometimes call the average value of $\|\boldsymbol{e}\|^2$ the error vector and then the MSE is the **error-vector magnitude**. This is the MSE and nothing more and is superfluous unnecessary nomenclature as MSE was long in use for data transmission prior to introducing another name for the same thing.

[2]This appendix will soon show the trace of the matrix $R_{\boldsymbol{x}/\boldsymbol{y}}$ is corresponding vector MMSE.

D.2 The Orthogonality Principle and Linear MMSE estimation

This section shows that a linear MMSE estimator with any jointly stationary distributions leads to the same MMSE-estimate form as in (D.21) - (D.24). This section's **orthogonality principle** provides a simple way to derive such linear MMSE estimates, but also leads to rich intuition, as later used throughout this text.

The random variable x's linear MMSE estimate depends on a linear combination of observations of the random variables $\{y_{n=1,...,N}\}$. That linear combination uses parameters w_n, with the index n having one distinct value for each observation of y_n that is used. This linear MMSE minimizes

$$\mathbb{E}\left[|e|^2\right] \tag{D.25}$$

where this error is

$$e \triangleq x - \sum_{n=0}^{N-1} w_n \cdot y_n = x - \boldsymbol{w} \cdot \boldsymbol{y} \quad . \tag{D.26}$$

$N \to \infty$ without difficulty. The row vector $\boldsymbol{w}$ is not random. The column vector $\boldsymbol{y}$ can be random, as can be the scalar x. The orthogonality principle determines the values of w_n and also generalizes to the potential of several errors being combined into separately optimized error-vector elements:

Theorem D.2.1 (Orthogonality Principle) *The minimum MSE must meet the following scalar condition*

$$\mathbb{E}\left[e \cdot y_n^*\right] = 0 \quad \forall\, n = 1, ..., N \ , \tag{D.27}$$

or more generally in vector form where the error has L_x components, and $\boldsymbol{y}_n$ has L_y components, as the matrix condition

$$\mathbb{E}\left\{ \begin{bmatrix} e_{L_x} \\ \vdots \\ e_1 \end{bmatrix} \cdot [y_N^* \ ... \ y_1] \right\} = \begin{bmatrix} 0 & ... & 0 \\ 0 & \ddots & 0 \\ 0 & ... & 0 \end{bmatrix} = \mathbb{E}\left[\boldsymbol{e} \cdot \boldsymbol{y}_n^*\right] = 0 \ , \tag{D.28}$$

where the zeros matrix is $L_x \times N$. Indeed further expanding $\boldsymbol{y}^* = \left[\boldsymbol{y}_{L_y} ... \boldsymbol{y}_1\right]^*$ simply creates a larger $Ly \cdot L_x \times 1$ vector for the principle also applies.

Proof: Writing first the scalar $|e|^2 = [\Re(e)]^2 + [\Im(e)]^2$ allows partial MSE differentiation with respect to both x's real and imaginary parts for each n. The real and imaginary derivatives pertinent parts are (realizing that all other w_i, $i \neq n$, will drop from the corresponding partial derivatives)

$$e_r = x_r - w_{r,n} \cdot y_{r,n} + w_{i,n} \cdot y_{i,n} \tag{D.29}$$

$$e_i = x_i - w_{i,n} \cdot y_{r,n} - w_{r,n} \cdot y_{i,n} \quad , \tag{D.30}$$

where subscripts of r and i denote real and imaginary part in the obvious manner. Then, optimization over $w_{r,n}$ and $w_{i,n}$ yields,

$$\frac{\partial |e|^2}{\partial w_{r,n}} = 2e_r \cdot \frac{\partial e_r}{\partial w_{r,n}} + 2e_i \cdot \frac{\partial e_i}{\partial w_{r,n}} = -2 \cdot (e_r \cdot y_{r,n} + e_i \cdot y_{i,n}) = 0 \tag{D.31}$$

$$\frac{\partial |e|^2}{\partial w_{i,n}} = 2e_r \cdot \frac{\partial e_r}{\partial w_{i,n}} + 2e_i \cdot \frac{\partial e_i}{\partial w_{i,n}} = 2 \cdot (e_r \cdot y_{i,n} - e_i \cdot y_{r,n}) = 0 \quad . \tag{D.32}$$

The desired result is found by taking expectations and rewriting the series of results above in (D.27)'s form. This can be repeated for each e_ℓ, $\ell = 1, ..., L_x$ because if the MSE is generalized to the $L_x \times L_x$ matrix of components $MSE = trace\{\mathbb{E}\left[\boldsymbol{e} \cdot \boldsymbol{e}^*\right]\} = \mathbb{E}\left[\boldsymbol{e}^* \cdot \boldsymbol{e}\right]$. Since the MSE is quadratic in the parameters $w_{r,n}$ and $w_{i,n}$ for each of these L_x terms and separable in expanding $w_{r,n} \to w_{r,n,\ell_x}$ and $w_{i,n} \to w_{i,n,\ell_x}$ $\ell_x = 1, ..L_x$, this setting is a global minimum. A final step repeats the same for each index $\ell_y = 1, ..., L_y$. The simple aggregating expression is $\mathbb{E}[\boldsymbol{e} \cdot \boldsymbol{y}^*] = 0$. **QED**.

If the autocorrelation matrix $R_{\boldsymbol{yy}}$ is strictly positive-definite, then the linear-MMSE estimator is unique.

D.2.1 Some expansion of linear MMSE to vector estimates

While the scalar estimate follows simply as in Theorem D.2.1, estimates of multi-dimensional error signals require explanation. This subsection formalizes vector/Hilbert spaces to that objective.

Definition D.2.1 (Vector Space) *A vector space $\mathscr{V}$ for a scalar field $\mathscr{F}$ and for all vectors $\boldsymbol{v} \in \mathscr{V}$ has two operations vector addition "+" and scalar multiplication "$\cdot$". The vector addition operation maps $\mathscr{V} \otimes \mathscr{V} \to \mathscr{V}$ while scalar multiplication maps $\mathscr{F} \otimes \mathscr{V} \to \mathscr{V}$.* The vector space and operations must satisfy the following:

1. **closure of vector addition**, so that $\forall\, \boldsymbol{u} \in \mathscr{V}$ and $\boldsymbol{v} \in \mathscr{V}$, $\boldsymbol{u} + \boldsymbol{v} \in \mathscr{V}$.
2. **closure of scalar-vector multiplication**, so that $\forall\, \alpha \in \mathscr{F}$ and $\boldsymbol{v} \in \mathscr{V}$, $\alpha \cdot \boldsymbol{v} \in \mathscr{V}$.
3. **commutativity** $(\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u})$ and **associativity** $(\ [\boldsymbol{u} + \boldsymbol{v}] + \boldsymbol{w} = \boldsymbol{u} + [\boldsymbol{v} + \boldsymbol{w}])$ of vector addition.
4. existence of vector addition's **zero element** $\mathbf{0} \in \mathscr{V} \ni \mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$ and **inverse element** $-\boldsymbol{v} \in \mathscr{V} \ni \boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$.
5. **closure of scalar-scalar multiplication**, so that $\forall\, \alpha, \beta \in \mathscr{F}$ and $\boldsymbol{v} \in \mathscr{V}$, $\alpha \cdot \beta \cdot \boldsymbol{v} \in \mathscr{V}$.
6. **commutativity and associativity of scalar-scalar** multiplication so that $\forall\, \alpha\ \&\ \beta \in \mathscr{F}$ and $\boldsymbol{v} \in \mathscr{V}$, $\alpha \cdot (\beta \cdot \boldsymbol{v}) = (\alpha \cdot \beta) \cdot \boldsymbol{v} = \beta \cdot \alpha \cdot \boldsymbol{v}$.
   item multiplicative identify $1 \ni 1 \cdot \boldsymbol{v} = \boldsymbol{v}$
7. **distributivity** of scalar multiplication over vector and field addition, $\alpha \cdot (\boldsymbol{u} + \boldsymbol{v}) = \alpha \cdot \boldsymbol{u} + \alpha \cdot \boldsymbol{v}$ and $(\alpha + \beta) \cdot \boldsymbol{v} = \alpha \cdot \boldsymbol{v} + \beta \cdot \boldsymbol{v}$.

A random vector space will select vectors $\boldsymbol{v} \in \mathscr{V}$ according to a probability density/distribution $p(\boldsymbol{v})$, and then will have autocorrelation $R_{\boldsymbol{uu}}$ and cross-correlation $R_{\boldsymbol{vv}}$ matrices respectively. A vector space may have an infinite number of vector members. Simple examples are the N-dimensional vectors of real (or complex) numbers, $\mathscr{V} = \mathscr{R}^N(\mathscr{C}^N)$, with scalar field of real $\mathscr{R}$ and complex $\mathscr{C}$ scalars respectively. Chapter 2 also had vector codewords selected from the vector space of N-dimensional integers (or Gaussian integers) with finite-field arithmetic and scalar multiplication.

A normed vector space assigns a non-negative real scalar $\|\boldsymbol{v}\| \in \mathscr{R}^+$ to all $\boldsymbol{v} \in \mathscr{V}$ that measures length or size.

Definition D.2.2 (Normed Vector Space) *A* **normed vector space** *has an additional functional mapping, or* **norm** $\|\boldsymbol{v}\| \geq 0$*, to non-negative real numbers, $\mathscr{V} \to \mathscr{R}^+$, as well as positive-real scalar-multiplication norm simply written as the absolute value $|\alpha| \geq 0$.* The **vector norm** is $\|\boldsymbol{v}\|\ \forall\, \boldsymbol{v} \in \mathscr{V}$ and has the following properties:

1. satisfaction of the **triangle inequality** $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.
2. **uniform scaling** $\|\alpha \cdot \boldsymbol{v}\| = |\alpha| \cdot \|\boldsymbol{v}\|$.
3. **unique zero** norm (length) $\|\boldsymbol{v}\| = 0 \Leftrightarrow \boldsymbol{v} = 0$.

A normed vector space typically has an inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ between vector elements, and then becomes a Hilbert space. The inner product in this text will be $\sum_{n=1}^{N} u_n \cdot v_n^*$ when the vectors are in $\mathbb{C}^N$ (or $\mathbb{R}^N$, and will be a limiting form of this sum (an integral) in the same way used in Chapter 1. Normed vector spaces allow specification also of a **distance** between two vectors $d(\boldsymbol{v}, \boldsymbol{u}) = \|\boldsymbol{v} - \boldsymbol{u}\|$, which is zero only if

the two vectors are equal. A normed vector space is a **Hilbert Space** if it is complete, which basically means that any sum of the vector space's norms is finite; equivalently, it has an inner product that is less than infinite if both vectors have finite norms.

This text's far most common norm is the **euclidean norm** for real or complex vectors:

$$\|\boldsymbol{v}\| = \sqrt{\sum_{n=1}^{N} |v_n|^2} \quad . \tag{D.33}$$

However, there are other norms and generally the p-norm is $\|v_n\|_p = \left\{\sum_{n=1}^{N} |v_n|^p\right\}^{1/p}$, which is the eculidean norm for $p = 2$, the sum of the absolute values for $p = 1$, and the maximum element magnitude for $p = \infty$. The euclidean norm aligns particularly well with Gaussian and linear MMSE developments.

A matrix is a collection of column (or row) vectors from a vector space, so vector space properties and norms apply to each of the columns and therefore to the entire matrix, except the norm would translate to a set of real numbers. Matrix multiplication also expands the types of operations, but is of course not commutative. This text however focuses for matrix norm definition **only on positive-semi-definite square matrices:**

Definition D.2.3 *Matrix norm A matrix norm for an $N \times N$ square positive-semi-definite matrix R is*

$$\|R\| = \sqrt{trace\{R\}} \quad . \tag{D.34}$$

This matrix norm satisfies all the properties in Definition D.2.2.

Linear MMSE's expansion to vectors involves semantic bookkeeping that is notationally convenient. The error becomes a vector $\boldsymbol{e}$, which might immediately raise the question of which element to minimize. However the error vector definition expands to

$$\boldsymbol{e} = \boldsymbol{x} - W \cdot \boldsymbol{y} \quad , \tag{D.35}$$

where W becomes an $N_x \times N_y$ matrix. Each row of W is a row vector $\boldsymbol{w}_n$ that estimates x_n from $\boldsymbol{y}$. The mean norm is

$$\mathbb{E}\left[\|\boldsymbol{e}\|^2\right] = \sum_{n=0}^{N_x} |e_n|^2 \quad . \tag{D.36}$$

The (mean) squared error components are variables separable in that each depends only on its corresponding $\boldsymbol{w}_n$. Minimization of $\|\boldsymbol{e}\|^2$ is therefore the same as separate minimization of each component. The orthogonality principle then applies to each, which can be written simply as

$$W = R_{\boldsymbol{xy}} \cdot R_{\boldsymbol{yy}}^{-1} \quad . \tag{D.37}$$

Perhaps more interestingly, the minimized error components are also the diagonal elements of

$$R_{\boldsymbol{ee}} = R_{\boldsymbol{x}/\boldsymbol{y}}^{\perp} = R_{\boldsymbol{xx}} - R_{\boldsymbol{xy}} \cdot R_{\boldsymbol{yy}}^{-1} \cdot R_{\boldsymbol{yx}} \quad , \tag{D.38}$$

which then clearly has a minimimum $\sqrt{trace\{R_{\boldsymbol{ee}}\}} = \|R_{\boldsymbol{ee}}\|$. Linear MMSE is thus the same as minimizing the error-autocorrelation matrix's (squared) norm.

Lemma D.2.1 (Generalized Pythagorean Theorem (GPT)) *The following holds for vector MMSE estimation:*

$$R_{\boldsymbol{xx}} = R_{\boldsymbol{ee}} + R_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}} \quad . \tag{D.39}$$

Proof: The proof follows directly from $\hat{\boldsymbol{x}} = R_{\boldsymbol{xy}} R_{\boldsymbol{yy}}^{-1} \cdot \boldsymbol{y}$ and Equation D.38. **QED.**

The GPT conveniently allows direct analysis of many Chapter 3 receiver structures' performance. The error vector $\boldsymbol{e}$'s dimensionality equals the dimensionality of $\boldsymbol{x}$. In this text, transmitted dimensions may be in time, usually indexed by k for discrete time and infinite or semi-infinite[3]. N often corresponds in Chapter 4 and beyond to a frequency index where the basic functions within a symbol are indexed to frequency. The error vector may also correspond to spatial dimensions $l = 1, ..., L_x$. Whatever the index bookkeeping, the Orthogonality Principle and the GPT hold.

The MMSE minimizes the sum of the eigenvalues of $R_{\boldsymbol{ee}} = Q \cdot \Lambda_e \cdot Q^*$ where Λ_e is a diagonal matrix of non-negative real eigenvalues (see Matlab eig command) and $QQ^* = Q^*Q = I$, because the trace is the sum of the eigenvalues. This is immediately evident because multiplication by Q does not affect the error vector's norm $\|\boldsymbol{e}'\| = \|Q \cdot \boldsymbol{e}\| = \|\boldsymbol{e}\|$. So the MMSE estimator matrix then would be

$$\tilde{\boldsymbol{W}} = Q \cdot \boldsymbol{W} \quad , \tag{D.40}$$

which is still now separable in the $\tilde{\boldsymbol{e}}$ uncorrelated components. This means the determinant $|R_{\boldsymbol{ee}}|$ has also been minimized because the product of the eigenvalues is the determinant (as should be obvious from observing $|Q| = 1$), and indeed is also separable multiplicatively. Thus, often MMSE vector problems are written in terms of minimizing the determinant of the error autocorrelation matrix, but this is the same as minimizing the sum of component MSE's.

> **Theorem D.2.2 (MMSE Equivalence of Trace and Determinants)** *MMSE matrix-trace minimization is equivalent to minimization of the MMSE matrix determinant.* **Singularity:**. *When a matrix is singular, the determinant in the MMSE sense used here means the product of the non-zero eigenvalues. The zero eigenvalues do not contribute to MMSE as is clear they do not contribute to the trace of the matrix.* **Proof:** *The preceding development is the proof.* **QED.**

D.2.1.1 The Matrix AWGN

For the matrix AWGN **forward** channel,

$$\boldsymbol{y} = H \cdot \boldsymbol{x} + \boldsymbol{n} \quad , \tag{D.41}$$

and the MMSE estimation creates a **backward** channel

$$\boldsymbol{x} = W \cdot \boldsymbol{y} + \boldsymbol{e} \quad . \tag{D.42}$$

W and H relate through

$$W = R_{\boldsymbol{xx}} \cdot H^* \left[R_{\boldsymbol{nn}} + H \cdot R_{\boldsymbol{xx}} \cdot H^* \right]^{-1} \quad . \tag{D.43}$$

Similarly

$$H = R_{\boldsymbol{yy}} \cdot W^* \left[R_{\boldsymbol{ee}} + W \cdot R_{\boldsymbol{yy}} \cdot W^* \right]^{-1} \quad . \tag{D.44}$$

With given $R_{\boldsymbol{xx}} = R_{\boldsymbol{xx}}^{1/2} \cdot R_{\boldsymbol{xx}}^{*/2}$ and $R_{\boldsymbol{nn}} R_{\boldsymbol{nn}}^{1/2} \cdot R_{\boldsymbol{nn}}^{*/2}$, Appendix C's singular value decomposition (SVD) and pseudoinverse when $R_{\boldsymbol{nn}}^{1/2}$ is singular in unusual cases related to worst-case noise) applies to

$$\widetilde{H} \triangleq R_{\boldsymbol{nn}}^{-1/2} \cdot H \cdot R_{\boldsymbol{xx}}^{1/2} = F \cdot \Lambda \cdot M^* \quad , \tag{D.45}$$

This admits normalized noise/iinput random vectors $\boldsymbol{y}' = R_{\boldsymbol{nn}}^{-1/2} \cdot \boldsymbol{y}$ (with $R_{\boldsymbol{n}'\boldsymbol{n}'} = I$) and $\boldsymbol{x} = R_{\boldsymbol{xx}}^{1/2} \cdot \boldsymbol{u}$ (with $R_{\boldsymbol{uu}} = I$), with equivalent forward channel:

$$\boldsymbol{y}' = \widetilde{H} \cdot \boldsymbol{u} + \boldsymbol{n}' \tag{D.46}$$

[3] sometimes a maximum number of time dimensions may be provided as N corresponding to a single transmission stream within a symbol, but sometimes the index k corresponds to a symbol and so there may be up to KN dimensions in a concatenation of K successive N-dimensional symbols.

and corresponding backward channel, with $\widetilde{W} \triangleq R_{\boldsymbol{xx}}^{-1/2} \cdot W \cdot R_{\boldsymbol{nn}}^{1/2}$,

$$\boldsymbol{u} = \widetilde{W} \cdot \boldsymbol{y}' + \boldsymbol{e}' \quad . \tag{D.47}$$

$\widetilde{W} = R_{\boldsymbol{xx}}^{-1/2} \cdot W \cdot R_{\boldsymbol{nn}}^{-1/2}$ is the MMSE estimator for $\boldsymbol{u}$ in terms of $\boldsymbol{y}'$ because MMSE estimates are linear.

In terms of the SVD quantities, simple algebra leads to

$$\widetilde{W} = M \cdot \Lambda^* \cdot [I + \Lambda\Lambda^*]^{-1} \cdot F^* \quad , \tag{D.48}$$

which associates a $SNR/(1+SNR)$ like character to the backward channel with the same SVD, where the diagonal elements of Λ effectively define this ratio for each dimension. This is an example of bias (the F and M matrices are 1-to-1 and do not change energy). The backward channel has an SVD that reverses the role of the same F and M from $\widetilde{H}$ with a set of SNR-like diagonal elements. Thus estimating the channels root eigenmodes fully completes the symmetry of forward and backward channels. Equivalently, noise-whitening of $\boldsymbol{y}$ also completes the symmetry.

D.2.1.2 MMSE Matrix SNR

A matrix $\boldsymbol{SNR}$ at the forward MMSE channel output is

$$\boldsymbol{SNR}_{out} = R_{\boldsymbol{yy}} \cdot R_{\boldsymbol{nn}}^{-1} \quad . \tag{D.49}$$

An interpretation of matrix SNR's in MMSE are the two expression lists for $\boldsymbol{SNR}_{mmse}$ and $\boldsymbol{SNR}_{out}$:

$$\begin{aligned}
R_{\boldsymbol{ee}} &= R_{\boldsymbol{xx}} - R_{\boldsymbol{xy}} \cdot R_{\boldsymbol{yy}}^{-1} \cdot R_{\boldsymbol{yx}} && \text{(D.50)}\\
&= R_{\boldsymbol{xx}} - W \cdot H \cdot R_{\boldsymbol{xx}} && \text{(D.51)}\\
&= [I - W \cdot H] \cdot R_{\boldsymbol{xx}} && \text{(D.52)}\\
&= R_{\boldsymbol{xx}} \cdot [I - H^* \cdot W^*] && \text{(D.53)}\\
R_{\boldsymbol{xx}} \cdot R_{\boldsymbol{ee}}^{-1} &= [I - W \cdot H]^{-1} && \text{(D.54)}\\
R_{\boldsymbol{ee}}^{-1} \cdot R_{\boldsymbol{xx}} &= [I - H^* \cdot W^*]^{-1} && \text{(D.55)}\\
&\triangleq \boldsymbol{SNR}_{mmse} && \text{(D.56)}
\end{aligned}$$

The two expressions in (D.54) and (D.55) always have nonsingular matrices in the inversions, and further either is a valid SNR (even if not equal). The SNR of interest is

$$SNR_{mmse} = |\boldsymbol{SNR}_{mmse}| = \frac{|R_{\boldsymbol{xx}}|}{|R_{\boldsymbol{ee}}|} = \frac{1}{|I - W \cdot H|} \quad . \tag{D.57}$$

For the forward channel (with pseudoinverse used on $R_{\boldsymbol{xx}}$ if singular)

$$\begin{aligned}
R_{\boldsymbol{nn}} &= R_{\boldsymbol{yy}} - R_{\boldsymbol{yx}} \cdot R_{\boldsymbol{xx}}^{-1} \cdot R_{\boldsymbol{xy}} && \text{(D.58)}\\
&= R_{\boldsymbol{yy}} - H \cdot W \cdot R_{\boldsymbol{yy}} && \text{(D.59)}\\
&= [I - H \cdot W] \cdot R_{\boldsymbol{yy}} && \text{(D.60)}\\
&= R_{\boldsymbol{yy}} \cdot [I - W^* \cdot H^*] && \text{(D.61)}\\
R_{\boldsymbol{yy}} \cdot R_{\boldsymbol{nn}}^{-1} &= [I - H \cdot W]^{-1} && \text{(D.62)}
\end{aligned}$$

while for the backward channel equally

$$\begin{aligned}
R_{\boldsymbol{ee}}^{-1} \cdot R_{\boldsymbol{xx}} &= [I - W^* \cdot H^*]^{-1} && \text{(D.63)}\\
&\triangleq \boldsymbol{SNR}_{out} && \text{(D.64)}
\end{aligned}$$

Again, the determinant is the same

$$|\boldsymbol{SNR}_{out}| = \frac{|R_{\boldsymbol{yy}}|}{|R_{\boldsymbol{nn}}|} = \frac{1}{|I - H \cdot W|} \quad . \tag{D.65}$$

Appendix C lists **Sylvesters Determinant Identity**, which also becomes clear from Chapter 2's mutual information symmetry, which is

$$|I + A \cdot B| = |I + B \cdot A| \quad , \tag{D.66}$$

so therefore

$$SNR_{out} = SNR_{mmse} \quad . \tag{D.67}$$

This simple relationship basically means the best data rate achievable is the same for the forward and backward channel.

D.2.2 MMSE estimate bias

A **biased estimate** of $\boldsymbol{x}$ has

$$\mathbb{E}\left[\hat{\boldsymbol{x}}/\boldsymbol{x}\right] \neq \boldsymbol{x} \quad . \tag{D.68}$$

If the two are equal, the estimate is **unbiased**. MMSE estimates always have bias if they correspond to any Gaussian (non-zero) noise channel. This follows direct from

$$\begin{aligned} \mathbb{E}\left[\hat{\boldsymbol{x}}_{mmse}/\boldsymbol{x}\right] &= R_{\boldsymbol{xy}} \cdot R_{\boldsymbol{yy}}^{-1} \cdot \mathbb{E}\left[\boldsymbol{y}/\boldsymbol{x}\right] && \text{(D.69)} \\ &= R_{\boldsymbol{xy}} \cdot R_{\boldsymbol{yy}}^{-1} \cdot H \cdot \boldsymbol{x} && \text{(D.70)} \\ &= R_{\boldsymbol{xy}} \cdot R_{\boldsymbol{yy}}^{-1} \cdot R_{\boldsymbol{yx}} \cdot R_{\boldsymbol{xx}}^{-1} \cdot \boldsymbol{x} && \text{(D.71)} \\ &= R_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}} \cdot R_{\boldsymbol{xx}}^{-1} \cdot \boldsymbol{x} \quad . && \text{(D.72)} \end{aligned}$$

Since the estimate autocorrelation matrix is not equal (unless the error vector is identically zero, which only happens with no noise), MMSE estimates have bias. (Any singular dimensions in $R_{\boldsymbol{xx}}$ are discarded and not of interest in estimation, equivalently use the pseudoinverse.) The Generalized Pythagorean Theorem D.2.1 then provides with $R_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}} = R_{\boldsymbol{xx}} - R_{\boldsymbol{ee}}$

$$\mathbb{E}\left[\hat{\boldsymbol{x}}_{mmse}/\boldsymbol{x}\right] = \left(I - R_{\boldsymbol{ee}} \cdot R_{\boldsymbol{xx}}^{-1}\right) \cdot \boldsymbol{x} \quad . \tag{D.73}$$

For the scalar case. $\boldsymbol{x} = x$, Equation (D.73) relates, with $SNR_{mmse} \triangleq \frac{\bar{\mathcal{E}}_x}{\bar{\sigma}^2_{mmse}}$

$$\begin{aligned} \mathbb{E}\left[\hat{x}_{mmse}/x\right] &= \left(1 - \frac{\sigma^2_{mmse}}{\mathcal{E}_x}\right) \cdot x && \text{(D.74)} \\ &= (1 - \frac{1}{SNR_{mmse}}) \cdot x && \text{(D.75)} \\ &= \frac{SNR_{mmse} - 1}{SNR_{mmse}} \cdot x < x \quad , && \text{(D.76)} \end{aligned}$$

confirming the bias. MMSE estimates reduce the noise slightly, at the expense of some bias, so that the trade-off between bias and noise reduction is optimum in the MMSE sense. Receivers can remove bias in anctipcation of ML detection by

$$\hat{x}_{ML} = \frac{SNR_{mmse}}{SNR_{mmse} - 1} \cdot \hat{x}_{mmse} = x \quad . \tag{D.77}$$

The vector/MIMO case requires some caution, but the same bias exists. The vector $\boldsymbol{x}$ decomposes into its eigenvalue decomposition

$$\boldsymbol{x} = Q\boldsymbol{u} \quad , \tag{D.78}$$

with unitary Q that satisfies $QQ^* = Q^*Q = I$, such that

$$R_{\boldsymbol{xx}} = Q \cdot R_{\boldsymbol{uu}} \cdot Q^* \tag{D.79}$$

is the autocorrelation matrix' eigenvalue decomposition, with eigenvalues as the diagonal entries $\mathcal{E}_n$ on the diagonal matrix $R_{\boldsymbol{uu}}$. Since MMSE estimates are linear

$$\hat{\boldsymbol{x}}_{mmse} = Q \cdot \hat{\boldsymbol{u}}_{mmse} \quad . \tag{D.80}$$

and

$$\boldsymbol{e}' = Q^* \cdot \boldsymbol{e} \quad . \tag{D.81}$$

The matrix $R_{\boldsymbol{e}'\boldsymbol{e}'}$ is also diagonal in this case since $R_{\boldsymbol{e}\boldsymbol{e}} = Q \cdot R_{\boldsymbol{e}'\boldsymbol{e}'} \cdot Q^*$ is also a valid eigen-decomposition for $R_{\boldsymbol{e}'\boldsymbol{e}'}$ with diagonal elements $\lambda_{e,n} = \sigma^2_{mmse,n}$. Thus the MMSE estimation SNR's for the vector problem are then:

$$SNR_{mmse,n} = \frac{\mathcal{E}_n}{\sigma^2_{mmse,n}} \tag{D.82}$$

with corresponding indices n matched for $n = 1, ..., N$.

The corresponding bias matrix is then

$$\mathbb{E}\left[\boldsymbol{u}_{mmse}/\boldsymbol{u}\right] = \text{Diag}\left\{\frac{SNR_{mmse,n} - 1}{SNR_{mmse,n}}\right\} \cdot \boldsymbol{u} \quad . \tag{D.83}$$

Thus, the bias removal independently takes each dimension of the MMSE estimate and multiplies by

$$x_{unbiased,n} = \frac{SNR_{mmse,n}}{SNR_{mmse,n} - 1} \cdot \hat{x}_{mmse,n} \quad . \tag{D.84}$$

This is bias exists in all MMSE estimates and its removal in general as above when Q is unitary. However, $R_{\boldsymbol{x}\boldsymbol{x}}$ can be maintained with other invertible matrices A such that $R_{\boldsymbol{x}\boldsymbol{x}} = A \cdot R_{\boldsymbol{u}\boldsymbol{u}} \cdot A^*$, where A may not be unitary. In this case, the MMSE estimate of $\boldsymbol{x}$ remains $A\hat{\boldsymbol{u}}$, but the error vector may no longer have diagonal $R_{\boldsymbol{e}'\boldsymbol{e}'}$. Chapter 5's GDFE will address this situation, where a similar $SNR_{bias} = SNR_{no-bias} + 1$ exists, but specifically only after additional receiver processing removes the remaining interference between dimensions. Simple per-dimensional bias removal in the MIMO case necessitates the MMSE error-vector autocorrelation matrix is diagonal.

D.3 Spectral Factorization and Scalar Filter Realization

This section describes scalar filter-based autocorrelation realization via spectral factorization. The realized filters will be causal, causally invertible (minimum phase), and monic. Such realization invokes the so-called Paley-Wiener Criterion (PWC) that is constructively developed and proven as part of the realization process, basing the proof on discrete sequences but covering also continuous signals[4]. Section D.4 generalizes these filters and/or process and Paley Wiener criterion to MIMO (matrix filters and/or processes), which will require understanding Cholesky Factorization of simple matrices first from Subsection D.3.6 that ends this wection and precedes Section D.4.

D.3.1 Some Transform Basics: D-Transform and Laplace Transform

This subsection addresses use of D-transform notation for sequences[5]:

Definition D.3.1 *[D- Transforms] A sequence* x_k $\forall$ *integer* $k \in (-\infty, \infty)$ *has D-Transform* $X(D) = \sum_{k=-\infty}^{\infty} x_k \cdot D^k$ *for all* $D \in \mathscr{D}_x$*, where* $\mathscr{D}_x$ *is the* **region of convergence** *of complex* D *values for which the sum* $X(D)$ *converges, and* $X(D)$ *is analytic*[6] *in* $\mathscr{D}_x$*. The inverse transform is a clockwise line/contour integral around any closed circle in the region of convergence* $x_k = \frac{1}{2\pi\jmath} \oint_{D \in \mathscr{D}_x} X(D) \cdot D^{1-k} \cdot dD$.[7]

The sequence x_k can be complex. The symbol rate will be presumed $T = 1$ in this appendix[8]. The sequence x^*_{-k} has a D-Transform $X^*(D^{-*}) = \sum_{k=\infty}^{\infty} x^*_k D^{-k}$. When the region of convergence includes the unit circle, the discrete-time sequence's Fourier transform exists as $X(e^{-\jmath\omega}) = X(D)|_{D=e^{-\jmath\omega}}$, and such sequences are considered to be "stable" or "realizable" (non-causal sequencies become realizable with sufficient delay, or infinite delay in some limiting situations).

A sufficient condition for the discrete-time sequence's Fourier transform to exist (the sum $X(D)$ converges) is that the sequence itself be absolutely summable, meaning

$$\sum_{k=-\infty}^{\infty} |x_k| < \infty \quad , \tag{D.85}$$

or equivalently the sequence belongs to the (Hilbert infinite-dimensional vector) space of sequences L^1. Another sufficient condition is that the sequence belongs to L^2 or has finite energy according to

$$\sum_{k=-\infty}^{\infty} |x_k|^2 < \infty \quad . \tag{D.86}$$

The similarity of the form of transform and inverse then allows equivalently that the inverse Fourier Transform ($\frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{\jmath\omega}\right) e^{\jmath\omega k} \cdot d\omega$) exists if:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{-\jmath\omega})) \right| \cdot d\omega < \infty \quad , \tag{D.87}$$

or if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{-\jmath\omega})) \right|^2 \cdot d\omega < \infty \quad . \tag{D.88}$$

[4] In both cases, these proofs are applicable to deterministic magnitude-squared functions or to stationary random processes with consequently non-negative real power spectra.

[5] The reader may insert Z^{-1} for D to relate to other developments where Z transforms are used.

[8] The generalization for $T \neq 1$ is addressed in the specific text sections of this chapter and later chapters when necessary. For instance, see Table **??** for the generalization of transforms under sampling for all T. Sampling should not be confused with the bi-linear transform: the former corresponds to conversion of a continuous waveform to discrete samples, while the latter maps filters or functions from/to continuous to/from discrete time and thus allows use of continuous- (discrete-) time filter realizations in the other discrete- (continuous-) time domain.

While $x_k \in L_1$ or $x_k \in L_2$ are sufficient conditions, this text uses non-L_1-nor-L_2 functions that can have Fourier transforms. These"generalized" functions include the Dirac Delta function $\delta(t)$ or $\delta(\omega)$, so that for instance $\cos(\omega_0 k)$ has Fourier Transform $\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$, or the series $x_k = 1$ has transform $2\pi\delta(\omega)$, even though neither of the sums above in Equations (D.85) and (D.86) converge for these functions. These types of generalized-function-assisted Fourier Transforms are "on the stability margin" where values of D in a convergence-region series that approaches closely the unit circle (outside, but not on) so that existence criteria have limiting values in (D.87) or (D.88) in a "generalized-function" sense.

A continuous-time sequence $x(t)$ has a **Laplace Transform** $X(s)$ defined over a convergence region $\mathcal{S}_x$ as

Definition D.3.2 *[Laplace Transform] A function $x(t)$ has Lapace-Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$, which converges/exists for all $s \in \mathcal{S}_x$, where $\mathcal{S}_x$ is the* **convergence region** *of complex $s \in \mathscr{C}$ values. The inverse transform is $\frac{1}{2\pi\jmath}\oint X(s)\cdot e^{st}\cdot ds$ when the closed contour of integration is in $\mathcal{S}_x$.*

The time function $x(t)$ can be complex. The function $x^*(-t)$ has a LapaceTransform $\int_{-\infty}^{\infty} x^*(-t)e^{-st}dt = X^*(-s^*)$. When the region of convergence includes the $\jmath\omega$ axis, the Fourier transform exists as $X(\omega) = X(s)|_{s=\jmath\omega}$, and such functions are considered to be "stable" or "realizable" (non-causal functions become realizable with sufficient delay, or infinite delay in some limiting situations).

A sufficient condition for the continuous Fourier transform to exist (the integral converges) is that the function be absolutely integrable, meaning

$$\int_{-\infty}^{\infty} |x(t)|\cdot dt \quad < \quad \infty \quad \text{, or equivalently} \tag{D.89}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\cdot d\omega \quad < \quad \infty \quad , \tag{D.90}$$

or equivalently that the function $x(t)$ belongs to the space of continuous functions L^1. Another sufficient condition is that function $x(t)$ belongs to L^2 or has finite energy according to

$$\int_{-\infty}^{\infty} |x(t)|^2\cdot dt \quad < \quad \infty \quad \text{, or equivalently} \tag{D.91}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\cdot d\omega \quad < \quad \infty \tag{D.92}$$

Similar to the discrete-time D-Transform, "generalized" functions complete the capability to handle continuous Fourier Transforms that are "on the stability margin" where values of s in a convergence-region sequence arbitrarily close to the $\jmath\omega$ axis (left of, but not on this axis) will converge so the criteria satisfy (D.91) or (D.92) in a limiting or generalized sense.

D.3.2 Autocorrelation and Non-Negative Spectra Magnitudes

Of particular interest in this text, and generally in digital communication, are the autocorrelation functions and associated power spectra for stationary and wide-sense stationary processes. These concepts are revisited briefly here for discrete processes before returning to filter realization of a given specified non-negative Fourier Transform magnitude.

Definition D.3.3 *[Autocorrelation and Power Spectrum for Sequences] If x_k is any stationary complex sequence, its* **autocorrelation function** *is $r_{xx,j} = \mathbb{E}[x_k x^*_{k-j}]$ with D-Transform $R_{xx}(D)$; symbolically*[9]

$$R_{xx}(D) \stackrel{\Delta}{=} \mathbb{E}\left[X(D) \cdot X^*(D^{-*})\right] \quad . \tag{D.93}$$

By stationarity, $r_{xx,j} = r^*_{xx,-j}$ and $R_{xx}(D) = R^*_{xx}(D^{-*})$. The **power spectrum** of a stationary sequence is the Fourier transform of its autocorrelation function, which is written as

$$R_{xx}(e^{-\jmath\omega}) = R_{xx}(D)|_{D=e^{-\jmath\omega}} \quad , \quad -\pi < \omega \leq \pi \quad , \tag{D.94}$$

which is real and nonnegative for all ω. Conversely, any function $R(e^{-\jmath\omega})$ that is real and nonnegative over the interval $\{-\pi < \omega \leq \pi\}$ is a power spectrum, and has an autocorrelation function satisfying $R(D) = R^*(D^{-*})$.

Generally, conjugate symmetric sequences with $a_k = a^*_{-k}$ have real Fourier transforms $A(e^{-\jmath\omega}) \in \mathscr{R}$ that however can be negative. Thus, a necessary and sufficient condition to be an autocorrelation sequence is that $A(e^{-\jmath\omega}) \geq 0$, or a **positive real** sequence. The term "positive real" used by mathematicians should not be confused to mean that each time-domain sequence value is positive and real[10].

The quantity $E\left[|x_k|^2\right]$ is $\mathcal{E}_{\boldsymbol{x}}$, or $\bar{\mathcal{E}}_{\boldsymbol{x}}$ per dimension, and can be determined from either the autocorrelation function or the power spectrum as follows:

$$\begin{aligned} \mathcal{E}_{\boldsymbol{x}} &= \mathbb{E}\left[|x_k|^2\right] && \text{(D.95)} \\ &= r_{xx,0} && \text{(D.96)} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(e^{-\jmath\omega})d\omega \quad . && \text{(D.97)} \end{aligned}$$

If the matrix sequence R_k, for instance perhaps formed as the inverse transform of a fixed filter $\mathscr{F}^{-1}(H(e^{-\jmath\omega}) \cdot H^*(e^{\jmath\omega}))$, is deterministic, then the averages are not necessary. The power spectra is then essentially the magnitude squared of the Fourier Transform $R(e^{-\jmath\omega}) \stackrel{\Delta}{=} |H(e^{-\jmath\omega})|^2 \geq 0$ for discrete time. These Fourier Transforms' magnitudes can be thought of also as power spectra, and the corresponding inverse transforms as autocorrelation functions in this text.

Definition D.3.4 *[Autocorrelation and Power Spectrum for Continuous-time Functions] If $x(t)$ is any stationary (or WSS, see Chapter 1's Appendix A) complex function, its* **autocorrelation function** *is $r_{x_c x_c}(t) = \mathbb{E}[x_c(u)x^*_c(u-t)]$ with Laplace Transform $R_{x_c x_c}(s)$; symbolically*[11]

$$R_{x_c x_c}(s) \stackrel{\Delta}{=} \mathbb{E}\left[X(s) \cdot X^*(-s^*)\right] \quad . \tag{D.98}$$

By stationarity, $r_{x_c x_c}(t) = r^*_{x_c x_c}(-t)$ and $R_{x_c x_c}(s) = R^*_{x_c x_c}(-s^*)$. The **power spectrum** *of a stationary continuous-time process is the Fourier transform of its autocorrelation function, which is written as*

$$R_{x_c x_c}(\omega_c) = R_{x_c x_c}(s)|_{s=\jmath\omega_c} \quad , \quad -\infty < \omega_c \leq \infty \quad , \tag{D.99}$$

which is real and nonnegative for all ω_c. Conversely, any function $R(\omega)$ that is real and nonnegative over the interval $\{-\infty < \omega_c \leq \infty\}$ is a power spectrum and has autocorrelation function $r(t) = r^*(-t)$.

[10] Positive values for all discrete time instants means that the corresponding Fourier transform is an "autocorrelation" function of frequency.

Generally, conjugate symmetric functions with $a(t) = a^*(-t)$ have real Fourier transforms $A(\omega) \in \mathscr{R}$ that however can be negative. Thus, a necessary and sufficient condition to be an autocorrelation sequence is that $A(\omega) \geq 0$, or a **positive real** sequence. The term "positive real" used by mathematicians should not be confused to mean that each time-domain function value is positive and real[12], but instead refers to the Fourier Transform being "positive real."

The quantity $E\left[|x_c(t)|^2\right]$ is P_x, or the power of the random continuous-time process, and can be determined from either the autocorrelation function or the power spectrum as follows:

$$\begin{aligned} P_x &= \mathbb{E}\left[|x_c(t)|^2\right] && \text{(D.100)} \\ &= r_{x_c x_c}(0) && \text{(D.101)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{x_c x_c}(\omega_c) \cdot d\omega_c \quad . && \text{(D.102)} \end{aligned}$$

If the sequence in question (a fixed filter for instance) is deterministic, then the averages are not necessary above. The power spectra is then essentially the magnitude squared of the Fourier Transform $R(\omega_c) \triangleq |X_c(\omega_c)|^2 \geq 0$ for continuous time. These Fourier Transforms' magnitudes can viewed as power spectra, and the corresponding inverse transforms as autocorrelation functions in this text.

D.3.3 The Bilinear Transform and Spectral Factorization

This section denotes a continuous-time function's Fourier Transform radian frequency by ω_c while the discrete-time sequence's Fourier Transform variable will be ω (with no subscript of c). Similarly all continuous-time quantities will use a subscript of c to avoid confusion with discrete time. For the transforms, if a transform $X(D)$ or $X_c(s)$ exists in their respective regions of convergence, then the transforms $e^{X(D)}$ and $e^{X_c(s)}$ also exist in that same region of convergence[13]. Similarly then, $\ln(X(D))$ and $\ln(X_c(s))$ also have the same regions of convergence.

A filter-design technique for discrete-time filters uses what is known as the "bi-linear" transform to map a filter designed in continuous time into a discrete-time filter (or vice versa):

Definition D.3.5 *[Bilinear Transforms] The* **bilinear transform** *maps between discrete-time D Transforms and continuous-time Laplace Transforms according to*

$$s = \frac{1-D}{1+D} \tag{D.103}$$

and conversely

$$D = \frac{1-s}{1+s} \quad . \tag{D.104}$$

The bilinear transform can also relate discrete-time and continuous-time Fourier Transforms by inserting $D = e^{-\jmath\omega}$ and $s = \jmath\omega_c$. The $\jmath\omega_c$ (complex) axis from 0 to $\pm\jmath\infty$ corresponds to mapping $D = e^{-\jmath\omega}$ along the unit circle (centered at origin of D plane) from the (real, imaginary) D-plane point $[1, 0]$ of 0 radians to the point of π radians (or $[-1, 0]$) clockwise for positive ω_c (counter clockwise for negative ω_c). This blinear transform scales or compresses the infinite range of frequencies $\omega_c \in (-\infty, \infty)$ for the continuous-time Fourier Transform to the finite range of frequencies $\omega \in [-\pi, \pi]$ for the discrete-time Fourier Transform. The bilinear transform does not correspond to sampling (and why $T = 1$ here to avoid confusion) to go from continuous time to discrete time. The filter designer needs to convert the

[12] Positive values for all time means that the corresponding Fourier transform is an "autocorrelation" function of frequency.

[13] This follows from the derivative of $e^{f(x)}$ for any function $f(x)$ is $e^{f(x)} \cdot f'(x)$ so if the function existed at x, then $e^{f(x)}$ also exists at all argument values, and then since $f'(x)$ also exits, then so does the derivative. This argument can be recursively applied to all successive derivatives that of course exist for $f(x)$ in its domain of convergence.

filter cut-off frequencies with this compression/scaling in mind. A stable design in continuous time corresponds to all poles on/in the left-half plane of s (or the region of convergence includes, perhaps in limit, the $\jmath\omega_c$ axis). These poles will map (with the Bilinear Transform's frequency-scale conversion) into corresponding points through the bilinear transform in the region outside (or in limiting sense on) the unit circle $|D| = 1$, and vice-versa. Similarly a minimum-phase design (all poles and zeros in LHP) will map to all poles/zeros outside the unit circle, and vice-versa.

For the Fourier Transform, the blinear transformation maps frequency according to

$$\begin{aligned}
\jmath\omega_c &= \frac{1 - e^{\jmath\omega}}{1 + e^{\jmath\omega}} && \text{(D.105)} \\
&= \jmath \tan\left(\frac{\omega}{2}\right) && \text{(D.106)} \\
\omega &= 2\arctan(\omega_c) && \text{(D.107)} \\
d\omega &= \frac{d\omega_c}{1 + \omega_c^2} \quad . && \text{(D.108)}
\end{aligned}$$

The **spectral factorization** of a discrete-time autocorrelation function's D-Transform is:

Definition D.3.6 *[Factorizability for Sequences] An autocorrelation function $R_{xx}(D)$, or equivalently any non-negative real $R_{xx}(e^{\jmath\omega})$ so that $r_k = r_{-k}^*$, will be called* **factorizable** *if it can be written in the form*

$$R_{xx}(D) = S_{x,0} \cdot G_x(D) \cdot G_x^*(D^{-*}), \tag{D.109}$$

where $S_{x,0}$ is a finite positive real number and $G_x(D)$ is a canonical filter response. A filter response $G_x(D)$ is called **canonical** if it is **causal** ($g_{x,k} = 0$ for $k < 0$), **monic** ($G_x(s = 0) = 1$), and **minimum-phase** (all of its poles and zeros are outside or on the unit circle). If $G_x(D)$ is canonical, then $G_x^*(D^{-*})$ is **anticanonical**; i.e., anticausal, monic, and maximum-phase (all poles and zeros inside or on the unit circle).

The region of convergence for factorizable $R_{xx}(D)$ clearly includes the unit circle, as do the regions for both $G_x(D)$ and $G_x^*(D^{-*})$. If $G_x(D)$ is a canonical response, then $\|g_x\|^2 \triangleq \sum_j |g_{x,k}|^2 \geq 1$, with equality if and only if $G_x(D) = 1$, since $G_x(D)$ is monic. Further, the inverse also factorizes similarly into

$$R_{xx}^{-1}(D) = (1/S_{x,0}) \cdot G_x^{-1}(D) \cdot G_x^{-*}(D^{-*}) \quad . \tag{D.110}$$

Clearly if $R_{xx}(D)$ is a ratio of finite-degree polynomials in D, then it is factorizable (simply group poles/zeros together for inside and outside of the circle - any on unit circle will also appear in conjugate pairs so easily separated). For the situation in which $R_{xx}(D)$ is not already such a polynomial, the next section generalizes through the Paley-Wiener Criterion. Also, if $R_{xx}(D)$ is factorizable, then the corresponding $R_{x_cx_c}(s) = R_{xx}\left(\frac{1-s}{1+s}\right)$ is also factorizable into

$$R_{x_cx_c}(s) = S_{x_c,0} \cdot G_{x_c}(s) \cdot G_{x_c}^*(-s^*) \quad . \tag{D.111}$$

Definition D.3.7 *[Factorizability for Continuous Functions] An autocorrelation function $R_{x_cx_c}(s)$, or equivalently any non-negative real power spectrum $R_{x_cx_c}(\omega)$ so that $r(t) = r^*(-t)$, will be called* **factorizable** *if it can be written in the form*

$$R_{x_cx_c}(s) = S_{x_c,0} \cdot G_{x_c}(s) \cdot G_{x_c}^*(-s^*), \tag{D.112}$$

where $S_{x_c,0}$ is a finite positive real number and $G_{x_c}(s)$ is a canonical filter response. A filter response $G_{x_c}(s)$ is called **canonical** *if it is* **causal** *($g_{x_c}(t) = 0$ for $t < 0$),* **monic** *($g_{x_c}(0) = 1$), and* **minimum-phase** *(all of its poles and zeros are in the left half plane). If $G_{x_c}(s)$ is canonical, then $G^*_{x_c}(-s^*)$ is* **anticanonical***; i.e., anticausal, monic, and maximum-phase (all poles and zeros inside in the right half plane).*

The region of convergence for factorizable $R_{x_cx_c}(s)$ clearly includes the $\jmath\omega_c$ axis, as do the regions for both $G_{x_c}(s)$ and $G^*_{x_c}(-s^*)$.

If $G_{x_c}(s)$ is a canonical response, then $\|g_{x_c}\|^2 \triangleq \int_{-\infty}^{\infty} |g_{x_c}(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{x_c}(\omega)|^2 d\omega$. Further, the inverse also factorizes similarly into

$$R^{-1}_{x_cx_c}(s) = (1/S_{x_c,0}) \cdot G^{-1}_{x_c}(s) \cdot G^{-*}_{x_c}(-s^*) \quad . \tag{D.113}$$

Clearly if $R_{xx}(s)$ is a ratio of finite-degree polynomials in s, then it is factorizable (simply group poles/zeros together for left and right half planes - any on imaginary axis will also appear in conjugate pairs, and so are easily separated). When not already such a polynomial, the next section generalizes this situation through the Paley-Wiener Criterion.

D.3.4 The Paley-Wiener Criterion

Minimum-phase signals or filters are of interest in data transmission not only because they are causal and admit causal invertible inverses (one of the reasons for their study more broadly in digital signal processing) but because they allow best results with Decision Feedback as in Section **??**. These minimum-phase filters are also useful in noise whitening.

Calculation of $S_{x,0}$ for a factorizable D-Transform follows Equations (**??**) to (**??**) in Section **??** as

$$S_{x,0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln[R_{xx}(e^{\jmath\omega})]\cdot d\omega} \text{ or} \tag{D.114}$$

$$\ln(S_{x,0}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln[R_{xx}(e^{\jmath\omega})] \cdot d\omega \quad , \tag{D.115}$$

which has components corresponding to the last two terms in (D.109) integrating to zero because they are periodic with no constant components and being integrated over one period of the fundamental frequency. The integral in (D.115) must be finite for the exponent in (D.114) to be finite (also, any exponent of a real number is a real positive number, so $S_{x,0} > 0$ and real, consistent with the power spectral density). The integral of a power spectral density's natural log is fundamental in filter realization and in the Paley Wiener Criterion to follow. Again, $\mathscr{D}_x$ is the same for $R_{xx}(D)$ as for $\ln[R_{xx}(D)]$ and includes the unit circle; this also means the region of convergence for $\ln[G_x(D)]$ also is the same as for $G_x(D)$ and includes the unit circle. Further the region of convergence for $G_x^{-1}(D)$ also includes the unit circle and is the same as for $\ln\left[G_x^{-1}(D)\right]$.

The calculation of $S_{x,0}^{-1}$ has a very similar form to that of $S_{x,0}$:

$$S^{-1}_{x,0} = e^{-\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln[R_{xx}(e^{\jmath\omega})]\cdot d\omega} \text{ or} \tag{D.116}$$

$$\ln\left(S^{-1}_{x,0}\right) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln[R_{xx}(e^{\jmath\omega})] \cdot d\omega \quad , \tag{D.117}$$

Because (D.116) and (D.117) are similar, just differing in sign, and because any functions of G_x (including in particular ln or $|\bullet|$) are all periodic, factorizability also implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\ln[R_{xx}(e^{\jmath\omega})]| \cdot d\omega < \infty \tag{D.118}$$

(or the function $\ln[R_{xx}(D)]$ exists because the autocorrelation in L1 is absolutely integrable). Essentially this integral's finite value corresponds to a factorizable $R_{xx}(D)$ and means that the sequence's Fourier

Transform has no frequencies (except point frequencies of non-zero measure) at which it can be either zero or infinite. The non-infinite is consistent with the basic criterion (D.85) to be absolutely integrable, but the the non-zero portion corresponds intuitively to saying any filter that has some kind of "dead band" is singular. Any energy in these singular bands would be linear combinations of energy at other frequencies. Singular bands thus carry no new information, and can be viewed as useless: A signal with such a dead band is wasting energy on components transmitted already at other frequencies that exactly cancel. A filter with such a dead band would block any information transmitted in that band, making reliable data-detection/communication impossible (kind of an inverse to the reversibility concept and theorem in Chapter 1). Such a filter would not be reversible (causally or otherwise). Both situations should be avoided. Section **??** deals with such singularity far more precisely.

The following Paley-Wiener Theorem from discrete-time spectral factorization theory formalizes when an autocorrelation function is "factorizable." The ensuing development will essentially prove the theorem while developing a way to produce the previous subsection's factors $G_x(D)$ and thus $G_x^*(D^{-*})$. This development also finds a useful way to handle the continuous-time case, which can be useful in noise-whitening. ***The reader is again reminded that any non-negative (real) spectrum and corresponding inverse transform is a candidate for spectral factorization.***

Theorem D.3.1 *[Paley Wiener Criterion] If $R_{xx}(e^{-\jmath\omega})$ is any power spectrum such that both $R_{xx}(e^{-\jmath\omega})$ and $\ln R_{xx}(e^{-\jmath\omega})$ are absolutely integrable over $-\pi < \omega \leq \pi$, and $R_{xx}(D)$ is the corresponding autocorrelation function, then there is a canonical discrete-time response $G_x(D)$ that satisfies the equation*

$$R_{xx}(D) = S_{x,0} \cdot G_x(D) \cdot G_x^*(D^{-*}), \tag{D.119}$$

where the finite constant $S_{x,0}$ is given by

$$\ln S_{x,0} = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \ln R_{xx}(e^{-\jmath\omega}) d\omega \; . \tag{D.120}$$

For $S_{x,0}$ to be finite, $R_{xx}(e^{-\jmath\omega})$ must satisfy the **discrete-time Paley-Wiener Criterion (PWC)**

$$\frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} |\ln R_{xx}(e^{-\jmath\omega})| d\omega < \infty \quad . \tag{D.121}$$

The PWC's continuous-time equivalent of this PWC is that the Fourier Transform of the continuous-time autocorrelation function is factorizable

$$R_{x_c x_c}(s) = S_{x_c,0} \cdot G_{x_c}(s) \cdot G_{x_c}^*(-s^*) \quad , \tag{D.122}$$

where $G_{x_c}(s)$ is minimum phase (all poles and zeros in the left half plane or on axis in limiting sense) and "monic" $g_{x_c}(t)|_{t=0} = 1$, whenever

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\ln R_{x_c x_c}(\omega_c)|}{1+\omega_c^2} d\omega_c < \infty \quad . \tag{D.123}$$

Constructive Proof: The equivalence of the two PW criteria in (D.121) and (D.123) (discrete- and continuous-time) follows directly from Equations (D.105) to (D.108). However, it remains to show that the condition is necessary and sufficient for the factorization to exist. The necessity of the criterion followed previously when it was shown that factorizability lead to the PWC being satisfied. The sufficiency proof will be constructive from the criterion itself.

The desired non-negative real in (D.114) and (D.115) frequency has a (positive or zero) real square root $R_{xx}^{1/2}(e^{\jmath\omega})$ at each frequency, and this function in turn has a natural log

$$A(e^{-\jmath\omega}) \triangleq \ln\left[R_{xx}^{1/2}(e^{-\jmath\omega})\right] \quad . \tag{D.124}$$

$A(e^{-\jmath\omega})$ itself is also periodic and real, and by the PWC integral equation, is absolutely integrable and so has a corresponding Fourier representation

$$A(e^{-\jmath\omega}) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{-\jmath\omega k} \tag{D.125}$$

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(e^{-\jmath\omega}) \cdot e^{\jmath\omega} \cdot d\omega \quad . \tag{D.126}$$

Because this (Fourier Transform) $A(e^{-\jmath\omega})$ is purely real, then $a_k = a_{-k}^*$, and the D-Transform simplifies to

$$A(D) = a_0 + \sum_{k=1}^{\infty} a_k \cdot D^k + \sum_{l=-1}^{-\infty} a_l \cdot D^l \quad , \tag{D.127}$$

and then by letting $k = -l$ in the second sum,

$$A(D) = a_0 + \sum_{k=1}^{\infty} a_k \cdot D^k + \sum_{k=1}^{\infty} a_{-k} \cdot D^k \tag{D.128}$$

$$= a_0 + \sum_{k=1}^{\infty} [a_k + a_{-k}] \cdot D^k \tag{D.129}$$

$$= a_0 + 2 \cdot \sum_{k=1}^{\infty} \Re[a_k] \cdot D^k \quad , \tag{D.130}$$

which defines a causal sequence a_k that corresponds to $\ln\left[R_{xx}^{1/2}(D)\right]$. The sequence is causally invertible because $\ln\left[R_{\boldsymbol{xx}}^{-1/2}(D)\right]$ can be handled in the same way following (D.121). So,

$$R_{xx}(D) = e^{A(D)} \cdot e^{A^*(D^{-*})} \quad . \tag{D.131}$$

Then, the desired canonical factorization has the factors

$$S_{x,0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left[R_{xx}(e^{-\jmath\omega})\right]\cdot d\omega} \tag{D.132}$$

$$G_x(D) = \frac{e^{A(D)}}{\sqrt{S_{x,0}}} \quad . \tag{D.133}$$

The corresponding continuous-time spectrum factorization then would be found with $R_{x_c x_c}(s) = R_{xx}\left(\frac{1-s}{1+s}\right)$ and thus $A_c(s) = A\left(\frac{1-s}{1+s}\right)$. Then, with $s \to \jmath\omega_c$

$$S_{x_c,0} = e^{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\ln\left[R_{x_c x_c}(\omega_c)\right]}{1+\omega_c^2}\cdot d\omega_c} \tag{D.134}$$

$$G_{x_c}(s) = \frac{e^{A_c(s)}}{\sqrt{S_{x_c,0}}} \quad . \tag{D.135}$$

If the original desired spectra were defined in continuous time, then it could be mapped into discrete time through $\omega_c \to \tan(\frac{\omega}{2})$ and then proceeding with that discrete-time mapped equivalent through the process above, ultimately leading to Equations (D.134) and (D.135). Sufficiency has thus been established in both discrete- and continuous-time. **QED.**

Minimum-phase functions and sequences have several interesting properties that can help understand their utility. If a "phasor" diagram were drawn from each pole and zero to a point on the unit circle (or imaginary axis for continuous time), the magnitude is the ratio of the zero-phasor-length products to the pole-phasor-length products while the phase is the sum of the zero phases minus the sum of the pole phases. For the minimum-phase D-Transform the phasor angle is measured from a horizontal line to the left (while for the minimum-phase Laplace Transform it is measured from a horizontal line to the right). These phase contributions are always the smallest with respect to the "other choice" of the zero/pole from the maximum-phase factor. Whence the name "minimum phase." Perhaps more importantly, one can see as frequency increases the rate of change of the angle (the magnitude of delay) is smallest for this same choice. Equivalently, each frequency for the particular magnitude spectrum of interest is delayed the smallest possible amount. With respect to all other pole/zero choices, the energy is maximally concentrated towards zero for any time period (among all waveforms with the same spectrum). In Section **??**, the DFE feedback section thus has largest ratio of first tap magnitude to remaining taps' summed magnitude, so thus smallest loss with respect to Chapter 3's matched filter bound. Such minimal-energy delay allows inversion of the function with the also-minimum-phase/delay that is actually the negative of the first delay. In effect, this only occurs when the function is causal and causally invertible.

D.3.4.1 Illustrative Concepts

A first illustration investigates the general realization's simplification when the function to be factored is already a ratio of finite polynomials. In this case, the following log power spectrum has a term like:

$$\sum_k \left[(1 + z_k D) \cdot (1 + z_k^* D^{-1})\right] \tag{D.136}$$

for the numerator where z_k are the zeros, divided into min-phase set for first term and max-phase set for second term. There is a similar pole-term for the denominator. The example focuses on the log-of-square-root term

$$\ln(1 + z_k D) = z_k D - \frac{(z_k D)^2}{2} + \frac{(z_k D)^3}{3} - \frac{(z_k D)^4}{4} + ... \ , \tag{D.137}$$

expanded via Taylor Series. The positive real $A(e^{-\jmath\omega T}) = \ln|(1 + z_k e^{-\jmath\omega})|$ is periodic in ω and therefore itself has a "Fourier Series" representation in terms of ω:

$$A(e^{-\jmath\omega T}) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{-\jmath\omega k} \ . \tag{D.138}$$

Since $A(e^{-\jmath\omega T})$ is real, then $a_k = a_{-k}^*$ and thus

$$A(e^{-\jmath\omega T}) = a_0 + 2 \cdot \sum_{k=1}^{\infty} \Re\left\{a_k) \cdot e^{-\jmath\omega k}\right\} \ . \tag{D.139}$$

Then $A(D)$ where $e^{-\jmath\omega} \to D$ is also minimum phase and clearly causal. Indeed then

$$R(D) = e^{A(D)} \cdot e^{A^*(D^{-*})} = (1 + z_k D) \cdot (1 + z_k^* D^{-1}). \ . \tag{D.140}$$

Because (D.136)'s sum, this process can be continued for each term, resulting $A(D)$ be sum of such terms and thus $e^{A(D)}$ being the product of corresponding terms. Pole terms can be handled similarly with a minus sign in front. Thus, the construction process leaves an

$$R_{xx}(D) = \prod_{k=-\mathcal{L}}^{\mathcal{L}} \frac{(1 + z_k D)}{(1 + p_k D)} \cdot \frac{(1 + z_k^* D^{-1})}{(1 + p_k^* D^{-1})} \ . \tag{D.141}$$

While the result could be produced by simple factoring, it illustrates the more general construction's thought processes. The more general procedure essentially corresponds to $\mathscr{L} \to \infty$ in the example.

This prevents simple factorization in some cases, even with poles. Essentially, the extended process corresponds to polynomials that cannot be factored, as the following illustrates.

The periodic function

$$R(e^{-\jmath\omega}) \triangleq 1 - \frac{|\omega|}{\pi} \quad \forall |\omega| \leq \pi \tag{D.142}$$

is clearly positive real. It also clearly satisfies PWC. It's log-square-root function is also period and equal to

$$A(e^{-\jmath\omega}) = \ln\left\{\sqrt{1 - \frac{|\omega|}{\pi}}\right\} = \frac{1}{2}\ln\left(1 - \frac{|\omega|}{\pi}\right) \quad . \tag{D.143}$$

This function has a Fourier Series representation

$$A(\omega) = \sum_k f_k \cdot e^{-\jmath\omega k} \tag{D.144}$$

with $f_{-k} = f_k^*$, with

$$f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(1 - \frac{|\omega|}{\pi}\right) \cdot e^{-\jmath\omega k} \cdot d\omega \quad , \tag{D.145}$$

which simplifies to

$$f_{k\geq 0} = \frac{1}{2\pi}\int_{0}^{\pi} \left(1 - \frac{|\omega|}{\pi}\right) \cdot \cos(\omega k) \cdot d\omega \quad . \tag{D.146}$$

Matlab can be used to integrate numerically and find the coefficients (for $k < 0$, take conjugate).

```
>>  F=zeros(1,11);
>> F(2) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(x)) ,0,pi);
>> F(3) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(2*x)) ,0,pi);
>> F(4) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(3*x)) ,0,pi);
>> F(5) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(4*x)) ,0,pi);
>> F(6) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(5*x)) ,0,pi);
>> F(7) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(6*x)) ,0,pi);
>> F(8) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(7*x)) ,0,pi);
>> F(9) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(8*x)) ,0,pi);
>> F(10) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(9*x)) ,0,pi);
>> F(11) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi).*cos(10*x)) ,0,pi);
>> F(1) = (1/(2*pi))*integral(@(x) (log(1-abs(x)/pi)) ,-pi,pi)  =
   -1.0000    0.2947   -0.1129    0.0888   -0.0594    0.0520   -0.0403    0.0367
   -0.0305    0.0284   -0.0245
```

So,

$$\begin{aligned} A(D) \quad \approx \quad & -1 + .2947D - .1129D^2 + .0888D^3 - .0594D^4 + .052D^5 - \\ & .0403D^6 + .0367D^7 - .0305D^8 + .0284D^9 - .0245D^{10} \quad . \end{aligned}$$

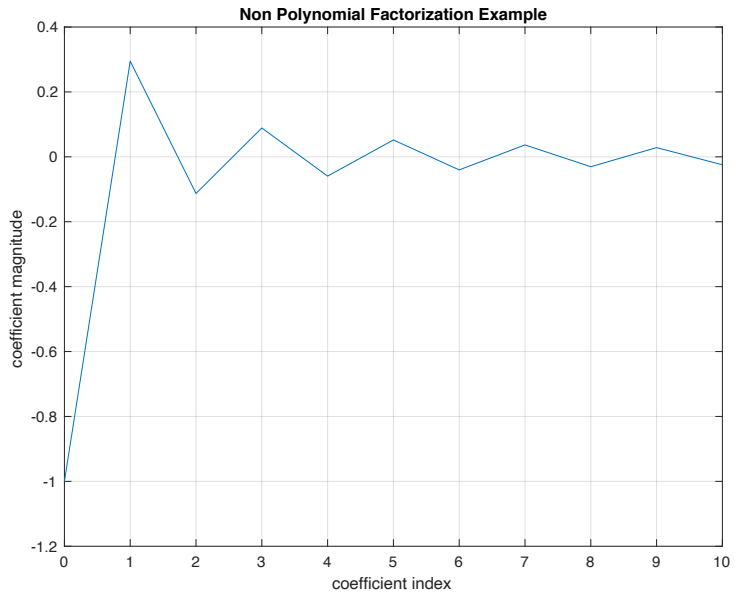

Figure D.1: Illustration of non pole-zero spectral factorization.

Figure ?? illustrates the positive-index coefficients for the first 10 values, which are clearly decreasing in magnitude. It is not hard to show by induction that the magnitude of the coefficients is monotonically decreasing, and of course since this function already satisfies PWC, it is absolutely convergent. The canonical factor is therefore:

$$G(D) = \frac{1}{\sqrt{S_0}} \cdot e^{-1+.2947D-.1129D^2+.0888D^3-.0594D^4+.052D^5-.0403D^6+.0367D^7-.0305D^8+.0284D^9-.0245D^{10}} . \tag{D.147}$$

Using $e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}$ in (D.147) will lead to all positive powers in D and a multi-term expression for $\cdot\sqrt{S_0} \cdot G(D)$ that could be truncated in practice as the division by $n!$ decreases the value of higher-order terms. $\sqrt{S_0}$ is this expansion's constant term.

D.3.5 Linear Prediction

The inverse of $R_{xx}(D)$ is also an autocorrelation function and can be factored when $R_{xx}(D)$ also satisfies the PW criterion with finite $\S_{x,0}$. In this case, as with the MMSE-DFE in Section **??**, the inverse autocorrelation factors as

$$R_{xx}^{-1}(D) = S_{x,0}^{-1} \cdot \bar{G}(D) \cdot \bar{G}^*(D^{-*}) \quad , \tag{D.148}$$

where $\bar{G}(D) = G_x^{-1}(D)$.

If $A(D)$ is any causal and monic sequence, then $1 - A(D)$ is a strictly causal sequence that may be used as a prediction filter, and the prediction error sequence $E(D)$ is given by

$$E(D) = X(D) - X(D) \cdot [1 - A(D)] = X(D) \cdot A(D) \quad . \tag{D.149}$$

The autocorrelation function of the prediction error sequence is

$$R_{ee}(D) = R_{xx}(D) \cdot A(D) \cdot A^*(D^{-*}) = \frac{A(D) \cdot A^*(D^{-*})}{S_{x,0}^{-1} \cdot \bar{G}(D) \cdot \bar{G}^*(D^{-*})} \quad , \tag{D.150}$$

so its average energy satisfies $\mathcal{E}_e = S_0 \cdot \|(1/g) * a\|^2 \geq S_0$ (since $\frac{A(D)}{\bar{G}(D)}$ is monic), with equality if and only if $A(D)$ is chosen as the **whitening filter** $A(D) = \bar{G}(D)$. The process $X(D) \cdot \bar{G}(D) = \frac{X(D)}{G_x(D)}$ is often called the **innovations** of the process $X(D)$, which has mean square value $S_{x,0} = 1/S_0$. Thus, $S_{x,0}$ of the direct spectral factorization is the mean-square value of the innovations process or equivalent of the MMSE in linear prediction. $X(D)$ can be viewed as being generated by inputting a white innovations process $V(D) = \bar{G}(D) \cdot X(D)$ with mean square value $S_{x,0}$ into a filter $G_x(D)$ so that $X(D) = G_x(D) \cdot V(D)$.

The inverse's factorization and $\bar{G}(D)$'s resultant linear-prediction-fiilter interpretation helps develop Section **??**'s interesting MMSE-DFE interpretation where the MS-WMF output sequence D-Transform replaces $X(D)$.

D.3.6 Cholesky Factorization - Finite-Length Spectral Factorization

Cholesky Factorization is spectral factorization's finite-length equivalent for an N-dmensional symbol/packet. There are really two equivalent Cholesky Factorizations, both of which converge to the infinite-length spectral factorization when the process is stationary with successive time-indexed dimensions as $N \to \infty$.

D.3.6.1 Cholesky Form 1 - Forward Prediction

Cholesky factorization of a positive-definite (nonsingular)[14] $N \times N$ matrix $R_{xx}(N)$ produces a unique upper triangular monic (ones along the diagonal) matrix $G_x(N)$ and a unique diagonal positive-definite diagonal matrix $S_x(N)$ of Cholesky factors such that[15]

$$R_{xx}(N) = G_x(N) \cdot S_x(N) \cdot G_x^*(N) \quad . \tag{D.151}$$

The matrix $R_{xx}(N)$ is often an autocorrelation matrix for N samples of some random vector process x_k with ordering

$$\boldsymbol{X}_N = \begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix} \quad . \tag{D.152}$$

A corresponding order of $G_x(N)$'s and $S_x(N)$'s elements is then

$$G_x(N) = \begin{bmatrix} \boldsymbol{g}_{N-1} \\ \boldsymbol{g}_{N-2} \\ \vdots \\ \boldsymbol{g}_0 \end{bmatrix} \text{ and } S_x(N) = \begin{bmatrix} s_{N-1} & 0 & ... & 0 \\ 0 & s_{N-2} & ... & 0 \\ & \vdots & \ddots & 0 \\ 0 & 0 & ... & s_0 \end{bmatrix} \quad . \tag{D.153}$$

Since $G_x(N)$ is monic, it is convenient to write

$$\boldsymbol{g}_i = \begin{bmatrix} \boldsymbol{0}_{N-1-i}^* & 1 & \tilde{\boldsymbol{g}}_i \end{bmatrix} \quad , \tag{D.154}$$

where $\boldsymbol{0}_j$ in general is a column vector with j zeros in it, and $\tilde{\boldsymbol{g}}_0 = \emptyset$ or $\boldsymbol{g}_0 = 1$. The determinant of $\boldsymbol{R}_N$ is

$$S_{x,0} = |R_{xx}(N)| = \prod_{n=0}^{N-1} s_n \quad . \tag{D.155}$$

(or $\ln S_{x,0} = \ln |R_{xx}(N)|$ in the limit as $N \to \infty$). A convenient recursive description of $R_{xx}(N)$'s components is, with $r_n = \mathbb{E}\left[|x_N|^2\right]$ and $\boldsymbol{r}_{N-1} = \mathbb{E}\left[\boldsymbol{X}_n x_N^*\right]$:

$$R_{xx}(N) = \begin{bmatrix} r_N & \boldsymbol{r}_{N-1}^* \\ \boldsymbol{r}_{N-1} & R_{xx}(N-1) \end{bmatrix} \quad . \tag{D.156}$$

[14] This is the equivalent of $|\ln |R_{xx}(N)|| < \infty$ where $|R_{xx}(N)|$ is the determinant of $R_{xx}(N)$. Thus the determinant cannot be infinite nor can it be zero, eliminating singularity. The determinant is the product of the eigenvalues of $|R_{xx}|$ so the ln transforms that product into a sum of the log eigenvalues - analogous to the integral in the PWC summing the log transform values. The Fourier transform values are the eigenvalues at each frequency for infinite time extent.

[15] Dots are used in this subsection to help notation, even though they here may correspond to matrix multiplication - it just makes it easier to read here.

The submatrix $R_{xx}(N-1)$ also has a Cholesky Factorization

$$R_{xx}(N-1) = G_x(N-1) \cdot S_x(N-1) \cdot G_x^*(N-1) \quad , \tag{D.157}$$

which because of the 0 entries in the triangular and diagonal matrices shows the recursion inherent in Cholesky decomposition; the $G_x(N-1)$ matrix is the lower right $(N-1)\times(N-1)$ submatrix of $G_x(N)$, which is also upper triangular. Thus, the corresponding recursion description for G_x is

$$G_x(N) = \begin{bmatrix} 1 & \tilde{\boldsymbol{g}}_{N-1} \\ \mathbf{0}_{N-1} & G_x(N-1) \end{bmatrix} \quad , \tag{D.158}$$

so then

$$\begin{bmatrix} r_N & \boldsymbol{r}_{N-1}^* \\ \boldsymbol{r}_{N-1} & R_{xx}(N-1) \end{bmatrix} = \begin{bmatrix} 1 & \tilde{\boldsymbol{g}}_{N-1} \\ \mathbf{0}_{N-1} & G_x(N-1) \end{bmatrix} \begin{bmatrix} s_{N-1} & 0 \\ 0 & S_x(N-1) \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}_{N-1}^* \\ \tilde{\boldsymbol{g}}_{N-1}^* & G_x^*(N-1) \end{bmatrix} \tag{D.159}$$

Equation (D.159) then admits by observation these recursions:

$$\boldsymbol{r}_{N-1}^* = \tilde{\boldsymbol{g}}_{N-1} \cdot S_x(N-1) \cdot G_x^*(N-1) \tag{D.160}$$

or equivalently to compute $\tilde{\boldsymbol{g}}_{N-1}$ in terms of previously known quantities

$$\tilde{\boldsymbol{g}}_{N-1} = \boldsymbol{r}_{N-1}^* \cdot G_x^{-*}(N-1) \cdot S_x^{-1}(N-1) \quad , \tag{D.161}$$

and

$$s_{N-1} = r_{N-1} - \tilde{\boldsymbol{g}}_{N-1} \cdot S_x(N-1) \cdot \tilde{\boldsymbol{g}}_{N-1}^* \quad . \tag{D.162}$$

The inverse of $R_{xx}(N)$ has a Cholesky factorization

$$R_{xx}^{-1}(N) = G_x^{-*}(N) \cdot S_x^{-1}(N) \cdot G_x^{-1}(N) \quad , \tag{D.163}$$

where $G_x^{-1}(N)$ is also upper triangular and monic with ordering

$$G_x^{-1}(N) = \begin{bmatrix} \bar{\boldsymbol{g}}_{N-1} \\ \bar{\boldsymbol{g}}_{N-2} \\ \vdots \\ \bar{\boldsymbol{g}}_0 \end{bmatrix} = \begin{bmatrix} 1 & -\tilde{\boldsymbol{g}}_{N-1} \cdot G_x^{-1}(N-1) \\ 0 & G_x^{-1}(N-1) \end{bmatrix} \quad , \tag{D.164}$$

where the use of $\tilde{\boldsymbol{g}}$ from $G_x(N)$'s Cholesky Factorization follows easily through multiplication of $G_x(N) \cdot G_x^{-1}(N) = I$. Also, because it is monic,

$$\bar{\boldsymbol{g}}_i = \begin{bmatrix} \mathbf{0}_{N-1-i}^* & 1 & \tilde{\boldsymbol{g}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N-1-i}^* & 1 & -\tilde{\boldsymbol{g}}_{N-1-i} \cdot G_x^{-1}(N-1-i) \end{bmatrix} \quad . \tag{D.165}$$

D.3.6.1.1 Cholesky and finite-length linear prediction :

Cholesky factorization also derives from a linear-prediction interpretation. The **innovations**, $\boldsymbol{V}_N$, of the N samples of X_N are defined by

$$\boldsymbol{X}_N = G_x(N) \cdot \boldsymbol{V}_N \tag{D.166}$$

where $\mathbb{E}\left[\boldsymbol{V}_N \boldsymbol{V}_N^*\right] = \boldsymbol{S}_N$, and the individual innovations are independent (or uncorrelated if not Gaussian), $\mathbb{E}\left[v_i \cdot v_j^*\right] = S_{\boldsymbol{x}}(i) \cdot \delta_{ij}$. Then

$$\boldsymbol{V}_N = \begin{bmatrix} v_{N-1} \\ \vdots \\ v_0 \end{bmatrix} \quad . \tag{D.167}$$

Also,

$$\boldsymbol{V}_N = G_x^{-1}(N) \cdot \boldsymbol{X}_N \quad . \tag{D.168}$$

The cross-correlation between $\boldsymbol{X}_N$ and $\boldsymbol{V}_N$ is

$$R_{\boldsymbol{vx}} = S_x(N) \cdot G_x^*(N) \quad , \tag{D.169}$$

which is lower triangular. Thus,

$$\mathbb{E}\left[v_k \cdot x_{k-i}^*\right] = 0 \;\; \forall \, i \geq 1 \quad . \tag{D.170}$$

Since $\boldsymbol{X}_{N-1} = G_x(N-1) \cdot \boldsymbol{V}_{N-1}$ shows a reversible mapping from $\boldsymbol{V}_{N-1}$ to $\boldsymbol{X}_{N-1}$, then (D.170) relates that the sequence v_k is a set of growing-order MMSE prediction errors for x_k in terms of x_{k-1} ... x_0 (i.e., (D.170) is the orthogonality principle for linear prediction). Thus,

$$v_N = x_N - \boldsymbol{r}_{N-1}^* \cdot R_{xx}^{-1}(N-1) \cdot \boldsymbol{X}_{N-1} \tag{D.171}$$

since $\boldsymbol{r}_{N-1}$ is the cross-correlation between x_N and $\boldsymbol{X}_{N-1}^*$ in (D.156). The top row of Equation ((D.166) also says

$$\begin{aligned}
x_N &= v_N + \boldsymbol{r}_{N-1}^* \cdot R_{xx}^{-1}(N-1) \cdot \boldsymbol{X}_{N-1} && \text{(D.172)} \\
&\quad \text{using the top row of Equation (D.171) with argument } N-1 \text{ and also (D.153)} && \text{(D.173)} \\
&= v_N + \underbrace{\boldsymbol{r}_{N-1}^* \cdot G_x^{-*}(N-1) \cdot S_x^{-1}(N-1)}_{\tilde{\boldsymbol{g}}_{N-1}} \cdot \underbrace{G_x^{-1}(N-1) \cdot \cdot \boldsymbol{X}_{N-1}}_{\boldsymbol{V}_{N-1}} && \text{(D.174)} \\
&= \boldsymbol{g}_{N-1} \cdot \boldsymbol{V}_N \quad , && \text{(D.175)}
\end{aligned}$$

confirming that

$$\boldsymbol{g}_{N-1} = \begin{bmatrix} 1 & \boldsymbol{r}_{N-1}^* \cdot \tilde{\boldsymbol{g}}_{N-1}^* \cdot S_x^{-1}(N-1) \end{bmatrix} \quad . \tag{D.176}$$

Then, from (D.168), (D.171), and (D.174),

$$\bar{\boldsymbol{g}}_{N-1} = \begin{bmatrix} 1 & -\tilde{\boldsymbol{g}}_{N-1} \cdot G_x(N-1) \end{bmatrix} \quad . \tag{D.177}$$

Finally, the mean-square error recursion is

$$\begin{aligned}
s_N &= \mathbb{E}\left[v_{N-1} \cdot v_{N-1}^*\right] && \text{(D.178)} \\
&= \mathbb{E}\left[x_{N-1} \cdot x_{N-1}^*\right] - \boldsymbol{r}_{N-1}^* \cdot R_{xx}^{-1}(N-1) \cdot \boldsymbol{r}_{N-1} && \text{(D.179)} \\
&= r_N - \tilde{\boldsymbol{g}}_{N-1} \cdot S_x(N-1) \cdot \tilde{\boldsymbol{g}}_{N-1}^* \quad . && \text{(D.180)}
\end{aligned}$$

D.3.6.1.2 Forward [Upper Triangular] Cholesky Algorithm:

For nonsingular R_N:
Set $\boldsymbol{g}_0 = G_x(1) = G_x^{-1}(1) = \bar{\boldsymbol{g}}_0 = 1$, $S_x(1) = s_0 = E|x_0|^2$, and $\boldsymbol{r}_i^*$ the last i upper row entries in $R_{xx}(i+1)$ as per (D.156).
For $n = 2$ ... N:

1. $\tilde{\boldsymbol{g}}_n = \boldsymbol{r}_{n-1}^* \cdot G_x^{-*}(n-1) \cdot S_x^{-1}(n-1)$.
2. $G_x(n) = \begin{bmatrix} 1 & \tilde{\boldsymbol{g}}_{n-1} \\ 0 & G_x(n-1) \end{bmatrix}$.
3. $G_x^{-1}(n) = \begin{bmatrix} 1 & -\tilde{\boldsymbol{g}}_{n-1} \cdot G_x^{-1}(n-1) \\ 0 & G_x^{-1}(n-1) \end{bmatrix}$.
4. $S_x(n) = r_n - \tilde{\boldsymbol{g}}_{n-1} \cdot S_x(n-1) \cdot \tilde{\boldsymbol{g}}_{n-1}^*$.

A singular R_N means that $s_n = 0$ for at least one index $n = i$, which is equivalent to $v_i = 0$. This means that x_i can be exactly predicted from the samples x_{i-1} ... x_0 or equivalently can be exactly constructed from v_{i-1} ... v_0. Such a singular process has $\ln|R_N| = 0$ and would not as $N \to \infty$ satisfy the PWC. In the singular case, Cholesky factorization is not unique. Chapter 5 introduces a generalized Cholesky factorization for singular situations that essentially corresponds to doing Cholesky factorization for the nonsingular process, and then generating the deterministic parts that are singular and depend entirely on the nonsingular parts from those nonsingular parts. This will be found equivalent there to independent sampling of each remaining nonsingular processes.

D.3.6.2 Backward [Lower Triangular] Cholesky Algorithm

Backward Cholesky essentially corresponds to time-order reversal for the finite group of N samples (for infinite-length sequences, this corresponds to $G_x(D^{-*})$). Time reversal sets $\boldsymbol{x}_N \leftarrow J_N \boldsymbol{X}_N$ where J_N is the $N \times N$ matrix with ones on the anti-diagonal and zeros everywhere else. Backward prediction is of x_0 from $x_{k=1,\ldots,N-1}$. Further, $J_N^* = J_N$, and $J_N^2 = I$. For this time reversal, the autocorrelation matrices follow

$$\bar{R}_{xx}(N) \leftarrow J_N \cdot R_{xx}(N) \cdot J_N \tag{D.181}$$

So the operation in Equation (D.181) is the autocorrelation matrix corresponding to the time reversal of x_k's components. This operation reversal basically "flips" the matrix about it's antidiagonal[16]. For a REAL Toeplitz matrix (stationary sequence), this flipping does not change the matrix; however for a complex Toeplitz matrix, the new matrix is the conjugate of the original matrix. Further, the operation $J_N \cdot G_x(N) \cdot J_N$ converts $G_x(N)$ from upper triangular to lower triangular, with the ones down the diagonal (monic) retained. The operation

$$J_N \cdot R_{xx}(N) \cdot J_N = \underbrace{[J_N \cdot G_x(N) \cdot J_N]}_{\tilde{G}_x^*(N)} \cdot \underbrace{[J_N \cdot S_x(N) \cdot J_N]}_{\tilde{S}_x(N)} \cdot \underbrace{[J_N \cdot G_x^*(N) \cdot J_N]}_{\tilde{G}_x(N)} \quad , \tag{D.182}$$

which is the desired lower-diagonal-upper or "Backward-Cholesky" factorization. Thus, the backward algorithm can start with the forward algorithm, and then just use the "tilded" quantities defined in (D.182) as the backward Cholesky factorization (including $\tilde{G}_x^{-1}(N) \to J_N \cdot G_x^{-1}(N) \cdot J_N$).

D.3.6.3 Infinite-length convergence

Extension to infinite-length stationary sequences takes the limit as $N \to \infty$ in either forward or backward Cholesky factorization. In this case, the matrix sequence $R_{xx}(N)$ (and therefore $G_x(N)$ and $S_x(N)$) must be nonsingular to satisify the Paley-Weiner Criterion. The equivalence to spectral factorization is evident from both the finite-length and infinite length linear prediction discussions.

From a stationary perspective, forward and backward prediction are the same except that the backward predictor reverses the time index (and conjugates when complex) of the forward predictor's coefficients (and vice versa). This is the equivalent (with re-index of time 0) of $G_x^*(D^{-*})$ being the reverse of $G_x(D)$ with conjugate coefficients.

Thus, the inverse autocorrelation function factors as

$$R_{xx}^{-1}(D) = S_{x,0}^{-1} \cdot G_x^{-1}(D) \cdot G_x^{-*}(D^{-*}) \quad , \tag{D.183}$$

where $G_x^{-1}(D)$ is the forward prediction polynomial (and its time reverse specified by $G_x^*(D^{-*})$ is the backward prediction polynomial). The series $\{R_{xx,n}\}_{n=1:\infty}$ formed from the coefficients of $R_{xx}(D)$ creates a series of linear predictors $\{G_x(N)\}_{N=1:\infty}$ with D-transforms $G_{x,N}(D)$. In the limit as $N \to \infty$ for a stationary nonsingular series,

$$\lim_{N\to\infty} G_{x,N}(D) = G_x(D) \quad . \tag{D.184}$$

Similarly,

$$\lim_{N\to\infty} G_{x,N}^*(D) = G_x^*(D^{-*}) \quad . \tag{D.185}$$

[16] "Flip" is like transpose but around the anti-diagonal.

As $N \to \infty$, the prediction-error variances S_{N-1}, will tend to a constant, namely $S_{x,0}$. Finally, defining the geometric-average determinants as $S_{x,0}(N) \stackrel{\Delta}{=} |R_{xx}|^{1/N}$ and $S_{x,0}^{-1}(N) = |R_{xx}^{-1}|^{1/N}$

$$\lim_{N\to\infty} S_{x,0}(N) = S_{x,0} = e^{\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(R_{xx}(e^{-\jmath\omega}))d\omega\right\}} \tag{D.186}$$

$$\lim_{N\to\infty} \S_{x,0}^{-1}(N) = S_{x,0}^{-1} = e^{-\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(R_{xx}(e^{-\jmath\omega}))d\omega\right\}} \quad . \tag{D.187}$$

The convergence to these limits implies that the series of filters converges or that the bottom row (last column) of the Cholesky factors tends to a constant repeated row/column. Chapter 5 has examples of this effect.

Interestingly, Chapter 5's Generalized Cholesky Factorization of a singular process exists only for finite lengths. Using the modifications to this appendix section's Cholesky Factorization with "resampling" in each disjoint PWC-satisfying frequency band, it becomes obvious why such the original combined random process cannot converge to a constant limit. So only nonsingular processes have (infinite-length) spectral factorization. A singular process' infinite-length factorization first separates that process into a sum of subprocesses, each of which is resampled at a new sampling rate that satisfies the PWC over each of the frequency bands associated with these processes. This is equivalent to the multiple Cholesky's for each of the process' nonsingular components at infinite length. For more, see Chapter 5.

D.4 MIMO Spectral Factorization

D.4.0.1 Vector Transforms

The D-Transform's extension to an L-dimensional vector sequence $\boldsymbol{x}_k$ is

$$\boldsymbol{X}(D) = \sum_{k=\infty}^{\infty} \boldsymbol{x}_k \cdot D^k \quad , \tag{D.188}$$

for all scalar complex D in the convergence region, $D \in \mathscr{D}_{\boldsymbol{x}}$. The inverse D-Transform is given by $\boldsymbol{x}_k = \frac{1}{2\pi\jmath} \oint_{D \in \mathscr{D}_x} \boldsymbol{X}(D) \cdot D^{1-k} \cdot dD$.[17] When the unit circle is in the convergence region $|D| = 1 \in \mathscr{D}_{\boldsymbol{x}}$, then a vector Fourier Transform exists and is

$$\boldsymbol{X}(e^{-\jmath\omega}) = \sum_{k=\infty}^{\infty} \boldsymbol{x}_k \cdot e^{-\jmath\omega k} \quad . \tag{D.189}$$

Sufficient conditions for convergence of the Fourier Transform generalize to:

$$\|\boldsymbol{x}\|_1 = \sum_{k=\infty}^{\infty} |\boldsymbol{x}_k|^1 < \infty \quad , \tag{D.190}$$

or

$$\|\boldsymbol{x}\|_2 = \sum_{k=\infty}^{\infty} \|\boldsymbol{x}_k\|^2 < \infty \quad . \tag{D.191}$$

Vector D-Transforms with poles on the unit circle are handled in the same limiting sense of approaching the unit circle arbitrarily closely from outside, which essentially allows generalized functions to be used in the frequency domain. The vector sequence's Fourier Transform then also exists. The transform's similarity to its inverse then allows equivalently that the inverse Fourier Transform:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \boldsymbol{X}(e^{-\jmath\omega})) \right| \cdot d\omega < \infty \quad , \tag{D.192}$$

or if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\boldsymbol{X}(e^{-\jmath\omega}))\|^2 \cdot d\omega < \infty \quad . \tag{D.193}$$

Rather, than repeat all for a continuous-time vector function's Laplace Transform, the Vector Laplace Transform for continuous-time vector process $\boldsymbol{x}(t)$ is

$$\boldsymbol{X}(s) = \int_{-\infty}^{\infty} \boldsymbol{x}(t) \cdot e^{-st} dt \, , \tag{D.194}$$

with convergence region $s \in \mathcal{S}_{\boldsymbol{x}}$. When the imaginary axis is in the convergence region $\mathcal{S}_{\boldsymbol{x}}$, the Fourier Transform is

$$\boldsymbol{X}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{x}(t) \cdot e^{-\jmath\omega t} dt \, . \tag{D.195}$$

Convergence conditions on the vector continuous-time process are then

$$\int_{-\infty}^{\infty} |\boldsymbol{x}(t)| \cdot dt \quad < \quad \infty \quad , \text{ or equivalently} \tag{D.196}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\boldsymbol{X}(\omega)| \cdot d\omega \quad < \quad \infty \quad , \tag{D.197}$$

[17] There is not a separate D value for each dimension of the vector $\boldsymbol{x}_k$ - just one scalar D value for all vector elements.

This means that the vector function $\boldsymbol{x}(t)$ belongs to the space of continuous functions L^1. Another sufficient condition is that vector function $\boldsymbol{x}(t)$ belongs to L^2 or has finite energy according to

$$\int_{-\infty}^{\infty} \|\boldsymbol{x}(t)\|^2 \cdot dt \quad < \quad \infty \ , \text{ or equivalently} \tag{D.198}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\boldsymbol{X}(\omega)\|^2 \cdot d\omega \quad < \quad \infty \quad . \tag{D.199}$$

Similarly "generalized" functions complete the capability to handle continuous Fourier Transforms that are "on the stability margin" where values of s in a region of convergence arbitrarily close to the $\jmath\omega$ axis (left of, but not on this axis). These will converge so the criteria are considered satisfied in a limiting or generalized sense.

Similarly a matrix time series can also have a matrix D-Transform (Laplace Transform) that contains the D-Transform (Laplace Transform) of each and every time-series element. There is only one common D variable for all these matrix elements. The convergence region is the intersection of the elements' convergence regions. When that overall convergence region includes the unit circle (complex axis), the Fourier Transform also exists and the inversion formula are all the obvious extensions. For square matrices, Fourier-Transform sufficiency criteria simplifies to

$$\sum_{k=-\infty}^{\infty} |R_k| < \infty \tag{D.200}$$

where the summed entities are the matrix-time-series coefficients' determinants, so equivalently Definition D.2.3's summed individual matrix norms. Basically the sum of the matrix-time-series coefficients' norms for transforms (or integral for Laplace) has to be finite as a sufficient condition for any valid norm's existence. So in this case the chosen norm is the trace, but that is equivalent to summing determinants in terms of existence. As Definition D.2.3, the square norm has essentially two equivalent forms for MMSE estimation.

The bilinear transform is unchanged with respect to scalar processes.

D.4.1 Autocorrelation and Power Spectra for vector sequences

This subsection generalizes scalar D-Transforms factorization to autocorrelation functions and associated power spectra for stationary and wide-sense stationary vector processes.

Definition D.4.1 *[Autocorrelation and Power Spectrum for Vector Sequences] If* $\boldsymbol{x}_k$ *is any stationary complex vector sequence, its* **autocorrelation matrix** *is* $r_{\boldsymbol{x}\boldsymbol{x},j} = \mathbb{E}[\boldsymbol{x}_k \boldsymbol{x}_{k-j}^*]$ *with D-Transform* $R_{\boldsymbol{x}\boldsymbol{x}}(D)$*; symbolically*[18]

$$R_{\boldsymbol{x}\boldsymbol{x}}(D) \triangleq \mathbb{E}\left[\boldsymbol{X}(D) \cdot \boldsymbol{X}^*(D^{-*})\right] \quad . \tag{D.201}$$

By stationarity, $r_{\boldsymbol{x}\boldsymbol{x},j} = r^*_{\boldsymbol{x}\boldsymbol{x},-j}$ *and* $R_{\boldsymbol{x}\boldsymbol{x}}(D) = R^*_{\boldsymbol{x}\boldsymbol{x}}(D^{-*})$*. The* **power spectrum matrix** *of a stationary vector sequence is the Fourier transform of its autocorrelation matrix function, which is*

$$R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega}) = R_{\boldsymbol{x}\boldsymbol{x}}(D)|_{D=e^{-\jmath\omega}} \ , \quad -\pi < \omega \leq \pi \ , \tag{D.202}$$

which is positive semi-definite[19] *for all* ω*, namely*

$$\boldsymbol{x}^* \cdot R(e^{-\jmath\omega}) \cdot \boldsymbol{x} \geq 0 \ , \ \forall \ \boldsymbol{x} \in \mathscr{C}^{L_x} \text{ and } \forall \ \omega \in (-\pi, \pi) \quad . \tag{D.203}$$

Consequently the determinant and the norm are both non-negative:

$$\left|R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})\right| \quad \geq \quad 0 \ , \ \forall \ \omega \in (-\pi, \pi) \tag{D.204}$$

$$trace\{R_{\boldsymbol{x}\boldsymbol{x}}\} \quad \geq \quad 0 \ , \ \forall \ \omega \in (-\pi, \pi) \quad . \tag{D.205}$$

Conversely, any positive semi-definite function $R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})$ that is real and nonnegative definite over the interval $\{-\pi < \omega \leq \pi\}$ is a power-spectrum matrix and has corresponding matrix autocorrelation function $R_{\boldsymbol{x}\boldsymbol{x}}(D) = R^*_{\boldsymbol{x}\boldsymbol{x}}(D^{-*})$, or equivalently $r_{\boldsymbol{x}\boldsymbol{x},k} = r^*_{\boldsymbol{x}\boldsymbol{x},-k}$.

The quantity $E\left[|\boldsymbol{x}_k|^2\right]$ is $\mathcal{E}_{\boldsymbol{x}}$, and can be determined from either the autocorrelation matrix or the power spectrum matrix as follows:

$$\begin{aligned} \mathcal{E}_{\boldsymbol{x}} &= \mathbb{E}\left[|\boldsymbol{x}_k|^2\right] && \text{(D.206)} \\ &= \text{trace}\left\{r_{\boldsymbol{x}\boldsymbol{x},0}\right\} && \text{(D.207)} \\ &= \frac{1}{2\pi}\text{trace}\left\{\int_{-\pi}^{\pi} R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})d\omega\right\} \quad . && \text{(D.208)} \end{aligned}$$

If a matrix sequence R_k in question is deterministic such as might be formed by inverse Fourier Transform of the filter $R(e^{-\jmath\omega}) = \boldsymbol{H}(e^{-\jmath\omega}) \cdot \boldsymbol{H}^*(e^{\jmath\omega})$, then the averages are not necessary above. The power spectra matrix is then the positive semi-definite matrix $R(e^{-\jmath\omega})$ for discrete time. These Fourier Transforms' magnitudes can be thought of also as power spectra matrices, and the corresponding inverse transforms as autocorrelation functions in this text.

Definition D.4.2 *[Continuous-Time Vector Autocorrelation & Power Spectra] If $\boldsymbol{x}(t)$ is any stationary complex vector function, its* **autocorrelation matrix** *is $r_{\boldsymbol{x}_c\boldsymbol{x}_c}(t) = \mathbb{E}[\boldsymbol{x}_c(u) \cdot \boldsymbol{x}_c^*(u-t)]$ with Laplace Transform $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s)$; symbolically*[20]

$$R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) \triangleq \mathbb{E}\left[\boldsymbol{X}(s) \cdot \boldsymbol{X}^*(-s^*)\right] \quad . \tag{D.209}$$

By stationarity, $r_{\boldsymbol{x}_c\boldsymbol{x}_c}(t) = r^*_{\boldsymbol{x}_c\boldsymbol{x}_c}(-t)$ and $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) = R^*_{\boldsymbol{x}_c\boldsymbol{x}_c}(-s^*)$. The **power spectrum matrix** of a stationary continuous-time vector process is the Fourier transform of its autocorrelation matrix function, which is

$$R_{\boldsymbol{x}_c\boldsymbol{x}_c}(\omega_c) = R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s)|_{s=\jmath\omega_c} \quad , \quad -\infty < \omega_c \leq \infty \quad , \tag{D.210}$$

and which is real and nonnegative definite for all ω_c, namely

$$|R_{\boldsymbol{x}_c\boldsymbol{x}_c}(\omega_c)| \geq 0 \;\; -\infty < \omega_c < \infty \quad . \tag{D.211}$$

Conversely, any function $R(\omega_c)$ that is real and nonnegative definite over the interval $\{-\infty < \omega_c \leq \infty\}$ is a power-spectrum matrix and has autocorrelation matrix function satisfying $R(s) = R^*(-s^*)$.

The quantity $E\left[|\boldsymbol{x}_c(t)|^2\right]$ is the power P_x and can be determined from either the autocorrelation function or the power spectrum as follows:

$$P_x = \mathbb{E}\left[|\boldsymbol{x}_c(t)|^2\right] = \text{trace}\left\{r_{\boldsymbol{x}_c\boldsymbol{x}_c}(0)\right\} = \frac{1}{2\pi}\text{trace}\left\{\int_{-\infty}^{\infty} R_{\boldsymbol{x}_c\boldsymbol{x}_c}(\omega_c) \cdot d\omega_c\right\} \quad . \tag{D.212}$$

If a matrix function $r(t)$ in question is deterministic such as might be formed by inverse Fourier Transform of the filter $R(\jmath\omega) = \boldsymbol{H}(\jmath\omega) \cdot \boldsymbol{H}^*(\jmath\omega)$, then the averages are not necessary above. The power spectra matrix is then the positive semi-definite matrix $R(\jmath\omega)$ for continuous time. These Fourier Transforms' magnitudes can be thought of also as power spectra matrices, and the corresponding inverse transforms as autocorrelation functions in this text.

D.4.2 Factorizability for MIMO Processes

Vector sequences have two types of dimensions, discrete (or continuous) **time** with infinite index span $k \in Z$ (or $t \in \mathscr{R}$) and **space** with finite index $l = 1, ..., L$. The infinite time index corresponds to a continuous transform variable D for discrete time and s for continuous time, while the finite space index leads to Subsection D.2's vector and matrix generalizations. Sometimes this text will group time-domain samples into finite symbols with $k \in [0, ..., N-1]$ and the corresponding infinite transforms (D and s) sampled (on the unit circle or imaginary axis respectively) also to a discrete frequency index n. In this case, the matrix indexing, k or n, corresponds to a SISO system, and $L = 1$. Chapter 3 uses discretization in Section **??**'s FIR Equalizers to compute all equalizers (linear or decision-feedback). Chapters 4 and 5 show how each sampled frequency may be viewed in MIMO as a separate MIMO system, so there will be two levels of matrix application: (1) time-frequency and (2) space. Those designs are finite-length implementable subsets of the infinite-length (time-frequency) MIMO processes appearing in Section **??**.

Matrix "factorization" has some definition flexibility; this text will combine Cholesky Factorization and infinite-length factorization to create a MIMO factorization theory that this text uses, particularly Section **??**. One contributor to non-unique factorization is the order of the spatial dimensions. Each order produces a different factorization, but they all clearly from a high-level intuitive perspective must be the same. The spatial indexing is arbitrary from that view point. However, this order leads to different MMSE's on each of the spatial dimensions, which favors certain dimensions. Such favored dimensions will have application in Chapter 5's multi-user MMSE theory.

At time of writing, the author is aware of no treatment anywhere that approaches this appendix, including the classic text by [1], which calls this area intractable. However, the short paper by [2] was helpful in developing the methods of this subsection, which the author hopes readers find useful for the general MIMO theory of equalization and MMSE in general.

The **spectral factorization** of a discrete-time vector autocorrelation function's D-Transform is:

Definition D.4.3 *[Factorizability for Vector Sequences] An autocorrelation function $R_{\boldsymbol{xx}}(D)$, or equivalently any positive semi-definite real function $R_{\boldsymbol{xx}}(e^{-\jmath\omega})$ at all $\omega \in (-\pi, \pi)$ and corresponding inverse transform $r_{\boldsymbol{xx},k} = r^*_{\boldsymbol{xx},-k}$, will be called* **factorizable** *if it can be written in the form*

$$R_{\boldsymbol{xx}}(D) = G_{\boldsymbol{x}}(D) \cdot S_{\boldsymbol{x},0} \cdot G^*_{\boldsymbol{x}}(D^{-*}), \tag{D.213}$$

where $S_{\boldsymbol{x},0}$ is a constant postive-real diagonal matrix and where $G_{\boldsymbol{x}}(D)$ is a canonical matrix filter response. The **canonical** matrix-filter factor $G_{\boldsymbol{x}}(D)$ for a factorizable $R_{\boldsymbol{xx}}(D)$ must also be causal $G_{\boldsymbol{x}}(D) = \sum_{k=0}^{\infty}$ or $G_{\boldsymbol{x},k} = 0 \ \forall \ k < 0$, have **upper triangular** $G_{\boldsymbol{x}}(0)$ that is also monic Diag $\{G_{\boldsymbol{x}}(0)\} = I$, and **minimum phase** (all of its poles and zeros are outside or on the unit circle). If $G_{\boldsymbol{x}}(D)$ is canonical, then $G^*_{\boldsymbol{x}}(D^{-*})$ is **anticanonical**; i.e., anticausal, monic lower triangular $G_{xvec}(\infty)$, and maximum-phase (all poles and zeros inside or on the unit circle).

The convergence region for a *factorizable* $R_{\boldsymbol{xx}}(D)$ clearly includes the unit circle, as do the regions for both $G_{\boldsymbol{x}}(D)$ and $G^*_{\boldsymbol{x}}(D^{-*})$. If canonical $G_{\boldsymbol{x}}(D) = G_0$, a constant, then $|G_{\boldsymbol{x}}(D)| = 1$ and $trace\{G_{\boldsymbol{x}}(D)\} = L_x$, which are clearly their minimum norm values for the more general canonical $G_{\boldsymbol{x}}(D)$ because positive semi-definite terms $G_{\boldsymbol{x},k>0} \cdot G^*_{\boldsymbol{x},-k<0}$ would only contribute non-negative value to their norm. The inverse $G^{-1}_{\boldsymbol{x}}(D)$ is also clearly canonical and thus can be realized. This inverse also factorizes similarly into

$$R^{-1}_{\boldsymbol{xx}}(D) = G^{-1}_{\boldsymbol{x}}(D) \cdot S^{-1}_{\boldsymbol{x},0} \cdot G^{-*}_{\boldsymbol{x}}(D^{-*}) \quad . \tag{D.214}$$

The determinant $|R_{\boldsymbol{xx}}(D)|$ will capture any and all the factorization's poles and zeros[21]. If $|R_{xx}(D)|$ is a ratio of finite-degree polynomials in D, then $R_{\boldsymbol{xx}}(D)$ is factorizable, which becomes evident as this

[21] Hidden cancellations will not be important in practice; In the non-polynomial-ratio case, the number of zeros may be infinite.

section progresses with a procedure to find the canonical factors. This procedure generalizes "group poles/zeros together for inside and outside of the circle - any poles/zeros on the unit circle will also appear in conjugate pairs and so are easily separated," as in Subsection D.4. Handling of autocorrelation matrices whose determinant is not already such a finite-degree polynomial ratio appears in the Subsection **??** through the MIMO Paley-Wiener Criterion.

Also, if $R_{\boldsymbol{x}\boldsymbol{x}}(D)$ is factorizable, then the corresponding $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) = R_{\boldsymbol{x}\boldsymbol{x}}\left(\frac{1-s}{1+s}\right)$ is also factorizable into

$$R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) = G_{\boldsymbol{x}_c}(s) \cdot S_{\boldsymbol{x}_c} \cdot G^*_{\boldsymbol{x}_c}(-s^*) \quad . \tag{D.215}$$

Definition D.4.4 *[Factorizability for Continuous Vector Functions] An autocorrelation matrix $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s)$, or equivalently any positive semi-definite $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(\omega)$ at all $\omega \in (-\infty, \infty)$ and corresponding inverse transform $r_{\boldsymbol{x}_c\boldsymbol{x}_c}(t) = r^*_{\boldsymbol{x}_c\boldsymbol{x}_c}(-t)$, will be called* **factorizable** *if it can be written in the form*

$$R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) = G_{\boldsymbol{x}_c}(s) \cdot S_{\boldsymbol{x}_c,0} \cdot G^*_{\boldsymbol{x}_c}(-s^*), \tag{D.216}$$

where $S_{\boldsymbol{x}_c,0}$ is a finite positive real diagonal matrix and $G_{\boldsymbol{x}_c}(s)$ is a canonical filter response. The **canonical** matrix-filter response $G_{\boldsymbol{x}_c}(s)$ for a factorizable $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s)$ must also be **causal** ($g_{\boldsymbol{x}_c}(t) = 0$ for $t < 0$), have **monic** upper triangular $G_{\boldsymbol{x}_c}(s = 0)$ so that (Diag $\{G_{\boldsymbol{x}_c}(0)\} = I$), and **minimum-phase** (all of its poles and zeros are in the left half plane). If $G_{\boldsymbol{x}_c}(s)$ is canonical, then $G^*_{\boldsymbol{x}_c}(-s^*)$ is **anticanonical**; i.e., anticausal, monic, **lower triangular** $G_{\boldsymbol{x}_c}(0)$, and maximum-phase (all poles and zeros inside in the right half plane). Further, (note use of bold and plain fonts on "x")

$$S_{x,0} \stackrel{\Delta}{=} |S_{\boldsymbol{x},0}| \quad . \tag{D.217}$$

The region of convergence for *factorizable* $R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s)$ clearly includes the $\jmath\omega_c$ axis, as do the regions for both $G_{\boldsymbol{x}_c}(s)$ and $G^*_{\boldsymbol{x}_c}(-s^*)$.

If $G_{\boldsymbol{x}_c}(s)$ is a canonical response, then $\|g_{\boldsymbol{x}_c}\|^2 \stackrel{\Delta}{=} \text{trace}\left\{\int_{-\infty}^{\infty} |g_{\boldsymbol{x}_c}(t)|^2\right\} \geq L_x$, with equality if $G_{\boldsymbol{x}_c}(s) = I$, since $G_{\boldsymbol{x}_c}(s)$ is monic. Further, the inverse also factorizes similarly into

$$R^{-1}_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) = G^{-1}_{\boldsymbol{x}_c}(s) \cdot S^{-1}_{\boldsymbol{x}_c,0} \cdot G^{-*}_{\boldsymbol{x}_c}(-s^*) \quad . \tag{D.218}$$

The determinant of $|R_{\boldsymbol{x}\boldsymbol{x}}(s)|$ will capture all poles and zeros in any and all terms of the factorization (and hidden cancellations will not be important in practice).

D.4.3 Finite-Degree MIMO polynomial factorization

Any matrix autocorrelation function that is a ratio of finite-degree polynomials in all entries is factorizable. This subsection provides a direct calculation of the factors by expanding on a method suggested by [2]. The square autocorrelation matrix of polynomials is

$$R_{\boldsymbol{x}\boldsymbol{x}}(D) = \begin{bmatrix} R_{\boldsymbol{x}\boldsymbol{x},L_x,L_x}(D) & \dots & R_{\boldsymbol{x}\boldsymbol{x},L_x,1}(D) \\ \vdots & \ddots & \vdots \\ R_{\boldsymbol{x}\boldsymbol{x},1,L_x}(D) & \dots & R_{\boldsymbol{x}\boldsymbol{x},1,1}(D) \end{bmatrix} \tag{D.219}$$

Each element of $R_{\boldsymbol{x}\boldsymbol{x}}(D)$, here simplified to $R(D)$ has the form

$$R_{i,j}(D) = \frac{K_{i,j} \cdot \left(\prod_{q=1}^{\mathcal{Q}_z}(1 - z_q D)\right) \cdot \left(\prod_{q=1}^{\mathcal{Q}_z}(1 - z_q D^{-1})\right)}{\mathcal{Z}_{i,j} \cdot \left(\prod_{m=1}^{\mathcal{M}_p}(1 - p_m D)\right) \cdot \left(\prod_{m=1}^{\mathcal{M}_p}(1 - p_m D^{-1})\right)} \cdot \quad , \tag{D.220}$$

with $z_q, q = 1, ..., \mathcal{Q}_z$ with $|z_q| < 1$ are the max-phase zeros inside the unit circle and $1/z_q$ are the minimum-phase zeros outside the unit circle. Complex zeros occur in conjugate pairs. Zeros on the unit circle will also occur in pairs, where one in each unit-magnitude pair is minimum phase and the other is maximum phase. The poles $p_m, m = 1, ..., \mathcal{M}_p$ with $|p_m| < 1$ are the max-phase poles inside the unit circle, and $1/p_m$ are the minimum-phase poles outside the unit circle. Complex poles occur in conjugate pairs. Poles on the unit circle will also occur in pairs, where one in each unit-magnitude pair is minimum phase and the other is maximum phase. $\mathcal{Q}_z < \infty$ is the maximum degree of the min- (or max-) phase numerator component, while $\mathcal{M}_p < \infty$ is the maximum degree of the min- (or max-) phase denominator component. These maximums can be taken over all elements in the matrix and simple $z_i = 0$ or $p_i = 0$ terms used for those terms of lower degree where the pole zero terms drop. The constant $K_{i,j}$ will be the ratio of the highest-degree non-zero coefficient in $R_{i,j}(D)$'s numerator to the corresponding highest-degree non-zero coefficient in its denominator.

Prescripts will be used to denote generations of quantities within the algorithm that follows, as will become evident.

D.4.3.1 STEP ZERO - INITIALIZATION

A initial step computes the least common multiple of all $R_{i,j}(D)$ elements' pole factors $\Delta(D) \cdot \Delta^*(D^{-*})$, and saves them, replacing $R(D)$ then with this new polynomial of finite maximum degree $2\mathcal{Q}_z$. The pole factors return in a final step because they easily separate into minimum and maximum phase and then will multiply the algorithm's corresponding $G(D)$ output factors that arise. The factorization then proceeds of the remaining all-zero polynomial $R(D)$.

$$ {}_0R(D) = \Delta(D) \cdot R(D) \cdot \Delta(D) \quad . \tag{D.221} $$

D.4.3.2 STEP ONE - Polynomial Cholesky Factorization

Cholesky Factorization applies also to transform polynomials even though presented earlier for constant matrices. The Cholesky factorization of a symmetric positive definite (semi-definite in case of polynomials at any zeros) matrix $R(D)$ for the 2×2 case is:

$$ R(D) = \begin{bmatrix} a(D) & b^*(D^{-*}) \\ 0 & g(D) \end{bmatrix} \cdot \begin{bmatrix} a^*(D^{-*}) & 0 \\ b(D) & g^*(D^{-*}) \end{bmatrix} \tag{D.222} $$

$$ = \begin{bmatrix} a(D) \cdot a^*(D^{-*}) + b^*(D^{-*}) \cdot b(D) & b^*(D^{-*}) \cdot g^*(D^{-*}) \\ g(D)c \cdot b(D) & g(D) \cdot g^*(D^{-*}) \end{bmatrix} \quad , \tag{D.223} $$

In the 2×2 case for ${}_0R(D)$, Cholesky factors are easily determined by factoring the lower right-side term (half the 11 term) in ${}_0R(D)$

$$ g(D) = \sqrt{K_{1,1}} \cdot \prod_{q=1}^{\mathcal{Q}_z} (1 - z_{q11} \cdot D). \quad , \tag{D.224} $$

and then

$$ b(D) = \frac{{}_0R_{1,2}(D)}{g(D)} \tag{D.225} $$

and finally

$$ a(D) = \sqrt{{}_0R_{2,2}(D) - b^*(D^{-*}) \cdot b(D)} \quad . \tag{D.226} $$

The square root in (D.226) is chosen through this all-zero polynomial's factorization terms $(1 - z \cdot D)$ with $|z| \leq 1$ and their complimentary terms $(1 - z \cdot D^{-1})$. The algorithm retains the former, along with the square root of a highest-degree-D coefficient. Equation (D.226)'s $a(D)$ then is the minimum-phase square root.

Once the above 2×2 recursion's steps are understood, progress to the $L \times L$ general version for $R(D)$ follows as the STEP ONE recursion that takes advantage of the $G(D)$ "upper triangular matrix"

growing in size at each iteration:

$$R_l(D) = \begin{bmatrix} r_{l,l}(D) & \boldsymbol{r}_l^*(D^{-*}) \\ \boldsymbol{r}_l(D) & R_{l-1}(D) \end{bmatrix} \tag{D.227}$$

$$= \begin{bmatrix} a_l(D) & \boldsymbol{b}_l^*(D^{-*}) \\ 0 & G_{l-1}(D) \end{bmatrix} \cdot \begin{bmatrix} a_l^*(D^{-*}) & 0 \\ \boldsymbol{b}_l(D) & G_{l-1}^*(D^{-*}) \end{bmatrix} \tag{D.228}$$

$$= \begin{bmatrix} a_l(D) \cdot a_l^*(D^{-*}) + \boldsymbol{b}_l^*(D^{-*}) \cdot \boldsymbol{b}_l(D) & \boldsymbol{b}_l^*(D^{-*}) \cdot G_{l-1}^*(D^{-*}) \\ G_{l-1}(D) \cdot \boldsymbol{b}_l(D) & G_{l-1}(D) \cdot G_{l-1}^*(D^{-*}) \end{bmatrix} \quad , \tag{D.229}$$

with iteration l solution for $l = 2, ..., L$ as[22]

$$\boldsymbol{b}(D) = G_{l-1}^{-1} \cdot \boldsymbol{r}_l(D) \; , \tag{D.230}$$

then minimum-phase square root for $a(D)$ as

$$a(D) = \sqrt{r_{l,l}(D) - \boldsymbol{b}_l^*(D^{-*}) \cdot \boldsymbol{b}_l(D)} \quad , \tag{D.231}$$

and finally $G(D)$ is then given in (D.229). When $l = L$, a valid square-root, but not necessarily minimum phase, has been found for the original matrix $R(D)$. The remaining steps then drive to a canonical factor from this square root. Thus,

$${}_1G(D) \cdot {}_1G^*(D^{-*}) = {}_0R(D) \quad . \tag{D.232}$$

D.4.3.3 STEP TWO: Left Extraction of Poles

Cholesky factorization will leave the upper left entry of $G(D)$ as the ratio of the determinants $|R_L(D)|$ to $|R_{L-1}(D)|$, and similarly for $l = L-1, ..., 2$. Because of the minimum-phase square root choices made in STEP ONE, the denominators of these determinants will have poles outside the unit circle. The factor ${}_1G(D)$ thus may now have poles that will be stable (outside or on the unit circle), which STEP TWO extracts to the left through a diagonal-matrix multiple removal so that

$${}_2G(D) = \begin{bmatrix} \frac{1}{\prod_{m=1}^{\mathcal{M}_p}(1-p_{m,l,l}D)} & 0 & ... & 0 \\ 0 & \frac{1}{\prod_{m=1}^{\mathcal{M}_p}(1-p_{m,l-1,l-1}D)} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & 1 \end{bmatrix} \cdot {}_1G_l(D) \tag{D.233}$$

Poles at infinity, $p_{m,j,j} = 0$ are simply the factor $1/D$ that can be ignored without altering the square root (because an D factor on the right side for $G^*(D^{-*})$ eventually cancels them or equivalently and advance of a stationary sequence does not change its autocorrelation). Pole extraction is not necessary but simplifies the remaining steps. These poles will necessarily cancel with zeros in the final $G(D)$ because the determinant is ${}_0R(D)$ an all-zero polynomial. That determinant is maintained in the product of the square root with its paraunitary factor ${}_0R^{1/2} \cdot {}_0R^{*/2}(D^{-*}) = {}_0R(D)$.

D.4.3.4 STEP THREE: Compute and factor the determinant of G

The current square root ${}_2G(D)$ may have zeros inside the unit circle and is thus not yet minimum phase. These zeros are found by computing the determinant $|{}_2G(D)|$ and factoring this scalar polynomial:

$$|{}_2G(D)| = k_{g2} \cdot \left(\prod_{q=1}^{\mathcal{Q}_z} (1 - z_{q2} \cdot D) \right) \quad , \tag{D.234}$$

where z_{q2} are the zeros and k_{g2} is a constant equal to the coefficient of lowest power of D (typically D^0 at this point) in the original determinant. The zeros inside the unit circle where $|z_{q2}| > 1$ will need to be removed in the next step, and STEP THREE identifies them.

[22] The inverse $G_{l-1}^{-1}(D)$ is determined easily by adding a new row at the top of the value for $G_{l-1}^{-1}(D)$, and there is essentially only 1 polynomial divide operation per iteration step.

D.4.3.5 STEP FOUR: Remove the maximum-phase zeros from $G(D)$

Any determinant zero causes ${}_2G(D) = 0$ and thus means that its columns are linearly dependent at this value of $D = z_{q2}$. There consequently exists a unitary constant transformation U_3 (with $U_3 U_3^* = I = U_3^* U_3$) for each such zero that rotates ${}_3G(D) = {}_2G(D) \cdot U_3$ to zero any selected column. Such a zeroed column has a common zero factor at z_{q2} for all elements. Such a transform preserves ${}_3G(D)$ as a square root. Usually finding U_3 is trivial (as a later example shows), but generally it is found from the constant matrix ${}_2G(z_{q2}$'s null space (which is of dimension at least one because this matrix is singular) and placing a normalized null-space basis vector in the last column of U_3 so that the last column of the product ${}_2G(z_{q2}) \cdot U_3$ is zero, which also means that column has a common zero at z_{q2} in all its elements.

STEP FOUR extracts this common zero to the right through the multiplication

$$ {}_4G(D) = \underbrace{{}_2G(D) \cdot U_3}_{{}_3G(D)} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1-1/z_{q2} \cdot D}{D \cdot (1-1/z_{q2} D^{-1})} \end{bmatrix}}_{\widetilde{U}_3(D)} \tag{D.235} $$

The matrix $\widetilde{U}_3(D)$ is paraunitary and will cancel with is opposite phase equivalent in the parallel formation of ${}_4G^*(D^{-*})$. The zero will be cancelled and replaced by a minimum phase zero at $1/z_{q2}$. This fourth step repeats until there are no maximum-phase zeros remaining in ${}_4G(D)$. ${}_4G(D)$ is clearly a valid minimum-phase square root.

D.4.3.6 STEP FIVE - remove constants to the center

The term ${}_4G(0)$ is not necessarily upper triangular, nor monic. STEP FIVE continues with matrix scaling so that ${}_5G(0) = I$ and thus

$$ {}_5G(D) = {}_4G(D) \cdot {}_4G^{-1}(0) \quad . \tag{D.236} $$

Then

$$ {}_1R(D) = {}_5G(D) \cdot \underbrace{{}_5R}_{{}_4G(0) \cdot {}_4G^*(0)} \quad {}_5G^*(D^{-*}) \tag{D.237} $$

The middle matrix is not yet diagonal, but ${}_5G(0)$ is now monic.

D.4.3.7 STEP SIX - Constant-Matrix Cholesky Factorization and adjustment

STEP SIX factors the constant middle matrix through normal Cholesky Factorization as

$$ {}_5R = {}_6G(0) \cdot S \cdot {}_6G^*(0) \tag{D.238} $$

and so now

$$ {}_6G(D) \triangleq {}_5G(D) \cdot {}_6G(0) \quad . \tag{D.239} $$

${}_6G(D)$ is a valid square root and canonical for ${}_1R(D)$.

D.4.3.8 STEP SEVEN - Restore the original poles

Step 0 removed poles from $R(D)$, and this STEP SEVEN now restores them.

$$ G(D) = \frac{1}{\Delta(D)} \cdot {}_6G(D) \quad , \tag{D.240} $$

and finally the desired canonical factorization is

$$ R(D) = G(D) \cdot S \cdot G^*(D^{-*}) \tag{D.241} $$

D.4.3.9 Swenson's Example

Dr. Norm Swenson, a former student and as a visiting Scholar in 2020, suggested the following $L = 2$ MIMO channel for factorization. He encountered it in an optical mixed-mode situation and challenged the author to factor:

The matrix to be factored is:

$$R(D) = \begin{bmatrix} 8D^{-1} + 23 + 8D & 7D^{-1} + 7 - D \\ -D^{-1} + 7 + D & -6D^{-1} + 18 - 6D \end{bmatrix} \tag{D.242}$$

The following matrix commands for the matrix and find its roots, determinant, and STEP ONE Cholesky Factor (there are no poles):

```
>> R11=[-6 18 -6];
>> R22=[8 23 8];
>> R12=[-1 7 7 ];
>> R21=[7 7 -1];
>> roots(R11)
ans =    2.6180
             0.3820
>> z11 = 0.3820;
>> sr11 = sqrt(R11(1)/-z11) = 3.9634
>> roots(R22)
ans =   -2.4702
            -0.4048
>> z22 = -0.4048
>> sr22=sqrt(R22(1)/-z22) = 4.4454
>> detR=conv(R22,R11)-conv(R21,R12)
detR =  -41   -36   219   -36   -41
>> roots(detR)
   -2.8306
    1.7267
    0.5791
   -0.3533
>> z1=  -0.3533
>> z2 = 0.5791
>> sdetR=sqrt(detR(1)/(-z1*-z2))  = 14.1559
>> GA22=sdetR*conv([1 -z1],[1 -z2]) =  14.1559   -3.1973   -2.8963
>> GA11=sr11*[-z11 1] =   -1.5139    3.9634
>> GA21 = [     7     7    -1 ];
GA12 =[ 0    0  ];
```

So the original matrix can also be written with factorization of its diagonal components as:

$$R(D) = \begin{bmatrix} 4.45^2 \cdot (1 + .405D)(1 + .405D^{-1}) & 7D^{-1} + 7 - D \\ -D^{-1} + 7 + D & 3.96^2 \cdot (1 - .382D)(1 - .382D^{-1}) \end{bmatrix} \tag{D.243}$$

and

$$|R(D)| = -41D^{-2} - 36D^{-1} + 219 - 36D - 41D^2 = 14.16^2 \cdot (1 - .579D) \cdot (1 + .353D)(1 + .353D^{-1}) \cdot (1.579D^{-1}) \ . \tag{D.244}$$

The matlab code uses the label GA, GB, etc. instead of prescripts.

So far then, the factoring has produced (since the Cholesky Factor 22 entry is the ratio of the overall square-root determinant to the corresponding square root of 11 entry), extracting the common top-row's $1/(3.96(1 - .382D))$ divide-by-g pole term to the left.

$$G_A(D) = \begin{bmatrix} \frac{1}{3.96(1-.382D)} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 14.15(1 + .353D)(1 - .579D) & 7 + 7D - D^2 \\ 0 & 3.96(-.382 + D) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D^{-1} \end{bmatrix} \tag{D.245}$$

STEP THREE and STEP FOUR identify the zero at .382 and remove it from last column.

```
>> GA22atz11 = GA22(1)+ GA22(2)*z11 + GA22(3)*z11*z11  = 12.5120
>> GA21atz11 = GA21(1)+ GA21(2)*z11 + GA21(3)*z11*z11 =  9.5279
>> GA12atz11=GA12(1) + GA12(2)*z11 = 0
>> GA11atz11=GA11(1) + GA11(2)*z11 = 0
```

$$G_A(0.382) = \begin{bmatrix} 12.512 & 9.5279 \\ 0 & 0 \end{bmatrix}. \ , \tag{D.246}$$

which makes it obvious that the unitary matrix to zero is a simple two-dimensional rotation with a cosine and sine computed as:

```
>> normuA=sqrt(GA22atz11^2+GA21atz11^2) =   15.7268
>> cA=GA22atz11/normuA =  0.7956
>> sA=GA21atz11/normuA  =   0.6058
>> GA21tilde=GA22*(-sA)+GA21*cA  =   -3.0070    7.5062    0.9591
>> roots(GA21tilde) =
    2.6180
   -0.1218
>> GA22tilde=GA22*cA+GA21*sA  =  15.5031    1.6971   -2.9101
>> GA12tilde=GA12*cA+GA11*sA  =   -0.9172    2.4011
>> GA11tilde=GA12*-sA+GA11*cA  =   -1.2044    3.1532
>> roots(GA11tilde) = >> roots([  -1.2044    3.1532]) = 2.6181
checks both 11 term and 21 term in last column now have the max-phase zero at 1/2.618, which can now
>> detGAtilde =  conv(GA22tilde,GA11tilde)-conv(GA21tilde,GA12tilde) =  -21.4301   60.9450   -8.2873
>> roots(detGAtilde) =
    2.6180 ( corresponds to zero inside unit circle)
    0.5791
   -0.3533
   (checks because 1/2.618 = .382, so the zero at .382 can be extracted by RHS multiply)
>> roots(GA21tilde) =
    2.6180
   -0.1218
>> X1 =    2.6180
>> X2  =   -0.1218
>> GA21tilde(1)*conv([1 -X1],[1 -X2]) =   -3.0070    7.5062    0.9591 (checks!)
```

So, after rotation on the right, the new focal point (middle matrix ignoring poles on left) is

$$\tilde{G}_A(D) = \begin{bmatrix} 15.5031 + 1.6971D - 2.9101D^2 & -3.0070 + 7.5062D + .9591D^2 \\ -0.9172 + 2.4011D & -1.2044 + 3.1532D \end{bmatrix} \tag{D.247}$$

```
>> GB21=GA21tilde(1)*-X1*[1 -X2] =    7.8725    0.9591
>> GB22=GA22tilde =   15.5031    1.6971   -2.9101
>> GB12=GA12tilde =   -0.9172    2.4011
>> GB11=GA11tilde(1)*-X1  =    3.1532
>> roots(GB22) =
   -0.4914
    0.3820
>> zB22 =   -0.4914
>> zB21 =   -0.1218
To do a check, see the 1,1 term
We left the [1 -z11] zero in the RHS factor (but used its pole to cancel zeros
in last column of GAtilde in forming GB)
To check, we need to put it back in.
 >> GB11ch=conv(GB11, [1 -z11]) =   3.1532   -1.2044
```

```
>> rch =conv(GB12(1:2),GB12(2:-1:1))+conv(GB11ch,GB11ch(2:-1:1)) =

   -6.0000   18.0000   -6.0000.  (looks good, at least on this term so far)
    ------------------

>> detGB=conv(GB22,GB11)-conv(GB21,GB12) =   66.2215  -37.9255  -14.7060
>> roots(detGB) =
    0.8378
   -0.2651
>> X1B = 0.8378
>> X2B = -0.2651
we're done already because roots are all min phase.
-----------------------------------------------------
another check
>> rch22 = (1/(-z11*sr110*sr110))*(conv(GB22(1:3),GB22(3:-1:1))+conv(GB21ch,GB21ch(3:-1:1))) =
    8.0000   -1.0000  -53.0000   -1.0000    8.0000
>> roots(rch22) =
    2.6180
   -2.4702
   -0.4048
    0.3820
>> rch22(1)*conv([1 -ans(2)],[1 -ans(3)]) =
    8.0000   23.0000    8.0000
    checks.
 ----------------------------------------
```

The matrix of interest is now

$$G_B(D) = \tilde{G}_A(D) \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-.382D}{D(1-.382D^{-1})} \end{bmatrix} \tag{D.248}$$

$$= \begin{bmatrix} 15.5031 + 1.6971D - 2.9101D^2 & 7.8725 + .9591D \\ -0.9172 + 2.4011D & 3.1532 \end{bmatrix} \tag{D.249}$$

with determinant

$$|G_B(D)| = 66.2215 \cdot (1 + .2651D) \cdot (1 - .8378D) \ , \tag{D.250}$$

which has all roots outside unit circle. Matlab commands above checked the 22 term also. Continuing by reabsorbing the pole on the left:

```
>> GC22=(GB22(1)/sr110)*[1 -zB22] =  3.9116    1.9223
>> GC11=GB11*[1 -z11]  =                     3.1532   -1.2044
>> GC12=GB1 =                                   -0.9172    2.4011
>> GC21=(GB21(1)/sr110)*[1 -zB21] =   1.9863    0.2420

>> conv(GC22,GC22(2:-1:1))+conv(GC21,GC21(2:-1:1)) =.   8.0000   23.0000    8.0000
>> conv(GC12,GC22(2:-1:1))+conv(GC11,GC21(2:-1:1)) =   -1.0000    7.0000    7.0000
>> conv(GC22,GC12(2:-1:1))+conv(GC21,GC11(2:-1:1)) =    7.0000    7.0000   -1.0000
>> conv(GC12,GC12(2:-1:1))+conv(GC11,GC11(2:-1:1)) =   -6.0000   18.0000   -6.0000
Checks.
```

Now the conversion to monic in STEPS FIVE and SIX:

```
>> GC0=[GC22(1) GC21(1)
               GC12(1) GC11(1)] =
    3.9116    1.9863
```

```
   -0.9172    3.1532

>> GC0*GC0' =
   19.2462    2.6757
    2.6757   10.7839
>> J = [     0     1
                1     0 ];
>> R=J*chol(J*GC0*GC0'*J)*J =
    4.3107         0
    0.8148    3.2839
>> R=R'
    4.3107    0.8148
         0         3.2839
>> R*R' = 19.2462    2.6757
                     2.6757   10.7839. checks.
>> S=diag(diag(R))*diag(diag(R)). =
   18.5823         0
         0   10.7839
>> G=R*inv(diag(diag(R))) =
    1.0000    0.1890
         0         1.0000
>> G*S*G' =
   19.2462    2.6757
    2.6757   10.7839.   (checks)
>> GC1=[GC22(2) GC21(2)
GC12(2) GC11(2)] =
    1.9223    0.2420
    2.4011   -1.2044
>> tempG1=GC1*inv(GC0) =
    0.4439   -0.2029
    0.4568   -0.6697
>> tempG0=GC0*inv(GC0) =
    1.0000    0.0000
         0         1.0000              (checks)
>> inv(S) =
    0.0538         0
         0         0.0927
>> S =   18.5823         0
                          0   10.7839
repeated, these are MMSE prediction error variances based on past sequence values
and spatial estimation of upper mode 2 from lower mode 1.

>> G22=[G0(1,1) G1(1,1)]
G21=[G0(1,2) G1(1,2)]
G12=[G0(2,1) G1(2,1)]
G11=[G0(2,2) G1(2,2)]

G22 = 1.0000    0.4439
G21 = 0.2481   -0.0927
G12 =         0    0.4568
G11 = 1.0000   -0.5564
>> S(1,1)*conv(G22,G22(2:-1:1))+S(2,2)*conv(G21,G21(2:-1:1)) = 8.0000   23.0000    8.0000
>> S(1,1)*conv(G22,G12(2:-1:1))+S(2,2)*conv(G21,G11(2:-1:1)) = 7.0000    7.0000   -1.0000
>> S(1,1)*conv(G12,G22(2:-1:1))+S(2,2)*conv(G11,G21(2:-1:1)) =-1.0000    7.0000    7.0000
```

```
>> S(1,1)*conv(G12,G12(2:-1:1))+S(2,2)*conv(G11,G11(2:-1:1)) =-6.0000    18.0000    -6.0000
```

Dr. Swenson's desired factorization is

$$\begin{aligned} & \begin{bmatrix} 8D^{-1}+23+8D & 7D^{-1}+7-D \\ -D^{-1}+7+D & -6D^{-1}+18-6D \end{bmatrix} \\ = & \begin{bmatrix} 1+.4439D & .2481-.0927D \\ .4568D & 1-.5564D \end{bmatrix} \cdot \begin{bmatrix} 18.5823 & 0 \\ 0 & 10.7839 \end{bmatrix} \cdot \begin{bmatrix} 1+.4439D^{-1} & .4568D^{-1} \\ .2481-.0927D^{-1} & 1-.5564D^{-1} \end{bmatrix} \end{aligned} \tag{D.251}$$

D.4.4 MIMO Paley Wiener Criterion and Matrix Filter Realization

Minimum-phase vector signals and matrix filters are of interest in data transmission not only because they are causal and admit causal invertible inverses (one of the reasons for their study more broadly in digital signal processing) but because they allow best results with MIMO Decision Feedback as in Section **??**. These minimum-phase matrix filters are also useful in noise whitening. While Section D.4 provided a direct construction of $G_{\boldsymbol{x}}(D)$ when $R_{\boldsymbol{xx}}(D)$ has fractional fraction polynomials, this subsection provides a more general construction for any more general function that satisfies the MIMO Paley Wiener criterion provided within.

D.4.4.1 Analytic Functions of Matrices

Analytic functions, like $\ln(x)$ and e^x as used here, have convergent power-series representations like

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+... \tag{D.252}$$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m} \tag{D.253}$$

$$\ln(x) = (x-1)-\frac{(x-1)}{2}+\frac{(x-1)^2}{2}-\frac{(x-1)^3}{3}+... \tag{D.254}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\cdot(x-1)^m}{m} \tag{D.255}$$

for all values of x. When the argument of the function is a square matrix R, the value of the corresponding output matrix of the same dimension can be found by insertion of this matrix into the power series, so[23]

$$e^R = \sum_{m=0}^{\infty} \frac{R^m}{m} \tag{D.256}$$

$$\ln(R) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\cdot(R-I)^m}{m} \quad . \tag{D.257}$$

With some care on aversion of commuting matrices, the following will hold:

$$\ln(R_1\cdot R_2) = \ln(R_1)+\ln(R_2) \tag{D.258}$$

$$e^{R_1+R_2} = e^{R_1}\cdot e^{R_2} \quad . \tag{D.259}$$

D.4.4.2 Necessity and the Sum-Rate Equivalent

Calculation of $S_{\boldsymbol{x},0}$ and $S_{x,0}=|S_{\boldsymbol{x},0}|$ for a factorizable D-Transform autocorrelation matrix generalizes Equations (**??**) to (**??**) in Section **??** as the matrix generalizations:

$$S_{\boldsymbol{x},0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln[R_{\boldsymbol{xx}}(e^{j\omega})]\cdot d\omega} \text{ or} \tag{D.260}$$

[23] For non-square matrices, it is usually possible to achieve desired results by forming a square matrix RR^* and applying the power series of the function to that "squared" matrix, and then finding the positive square root through Cholesky Factorization on the result and absorbing the square root of the diagonal matrix of Cholesky factors into each of the triangular matrices.

$$\ln\left(S_{\boldsymbol{x},0}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left[R_{\boldsymbol{xx}}(e^{\jmath\omega})\right]\cdot d\omega \tag{D.261}$$

$$S_{x,0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|R_{\boldsymbol{xx}}(e^{\jmath\omega})|\cdot d\omega} \text{ or} \tag{D.262}$$

$$\ln\left(S_{x,0}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|R_{\boldsymbol{xx}}(e^{\jmath\omega})|\cdot d\omega \quad , \tag{D.263}$$

which have components corresponding to the first and last terms of the factorization in (D.213) integrating to zero because they are periodic (see D.254) for $\ln(x+1)$) and being integrated over one period of the fundamental frequency, except for the constant term that is monic and then also integrates to zero because $\ln(1)=0$. The integral in (D.263) must be finite for the exponent in (D.262) to be finite (also, any exponent of a real function is a real positive number, so $S_{x,0}>0$ and real, consistent with the power spectral density). That integral of the natural log of a power-spectral density is fundamental in filter realization and in the MIMO Paley Wiener Criterion. Again, $\mathscr{D}_{\boldsymbol{x}}$ is the same for $R_{\boldsymbol{xx}}(D)$ as for $\ln\left[R_{\boldsymbol{xx}}(D)\right]$ and includes the unit circle; this also means the convergence region for $\ln\left[G_{\boldsymbol{x}}(D)\right]$ also is the same as for $G_{\boldsymbol{x}}(D)$ and includes the unit circle. Further the convergence region for $G_{\boldsymbol{x}}^{-1}(D)$ also includes the unit circle and is the same as for $\ln\left[G_{\boldsymbol{x}}^{-1}(D)\right]$.

The calculation of $S_{\boldsymbol{x},0}^{-1}$ has a very similar form to that of $S_{x,0}$:

$$S_{\boldsymbol{x},0}^{-1} = e^{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|R_{\boldsymbol{xx}}(e^{\jmath\omega})|\cdot d\omega} \text{ or} \tag{D.264}$$

$$\ln\left(S_{\boldsymbol{x},0}^{-1}\right) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|R_{\boldsymbol{xx}}(e^{\jmath\omega})|\cdot d\omega \quad , \tag{D.265}$$

Because (D.263) and (D.265) are similar, just differing in sign, and because any functions of G_x (including in particular ln or $|\bullet|$) are all periodic in ω, factorizability also implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\ln\left\{\left|R_{xx}(e^{-\jmath\omega})\right|\right\}\right|\cdot d\omega < \infty \tag{D.266}$$

(or the function $\ln\{|R_{xx}(e^{-\jmath\omega})|\}$ exists because this log-autocorrelation's determinant is absolutely integrable). Essentially the finite nature of this integral corresponding to factorizable $R_{\boldsymbol{xx}}(D)$ means that the sequence's Fourier Transform has no frequencies (except point frequencies of non-zero measure) at which it can be either zero or infinite. The non-infinite nature is consistent with the basic criterion for the norm to be absolutely integrable, but the the non-zero portion corresponds intuitively to saying any non-satisfying matrix filter has some singular "null components." Any energy in these null components would be linear combinations of energy of other components. Null components thus carry no new information, and can be viewed as useless: A signal with such a null components is wasting energy on these components that exist elsewhere already. A matrix filter with such a null space would block any information transmitted in that corresponding null space, making reliable data-detection/communication impossible (kind of a MIMO inverse to the reversibility concept and theorem in Chapter 1). Such a filter would not be reversible (causally or otherwise). Both infinite size and singular situations should be avoided. Chapter 5 will deal with such singularity and null spaces far more precisely.

The following MIMO Paley-Wiener Theorem from discrete-time spectral factorization theory formalizes when an autocorrelation function is "factorizable." The ensuing development will essentially prove the theorem while theoretically developing a way to produce generally the factors $G_{\boldsymbol{x}}(D)$ and thus $G_{\boldsymbol{x}}^{*}(D^{-*})$ of the previous subsection. This development also finds a useful way to handle the continuous-time case, which can be useful in noise-whitening. ***The reader is again reminded that any positive-semidefinite matrix and corresponding inverse transform is a candidate for spectral factorization.***

Theorem D.4.1 *[Paley Wiener Criterion] If $R_{\boldsymbol{xx}}(e^{-\jmath\omega})$ is any power spectrum such that both $|R_{\boldsymbol{xx}}(e^{-\jmath\omega})|$ and thus $|\ln\{|R_{xx}(e^{-\jmath\omega})|\}|$ are absolutely integrable over $-\pi <$*

$\omega \leq \pi$, and $R_{\boldsymbol{x}\boldsymbol{x}}(D)$ is the corresponding autocorrelation matrix, then there exists a canonical discrete-time response $G_{\boldsymbol{x}}(D)$ that satisfies the equation

$$R_{\boldsymbol{x}\boldsymbol{x}}(D) = G_{\boldsymbol{x}}(D) \cdot S_{\boldsymbol{x},0} \cdot G^*_{\boldsymbol{x}}(D^{-*}), \tag{D.267}$$

where $G_{\boldsymbol{x}}(D)$ is canonical, and where the diagonal matrix of all positive elements $S_{\boldsymbol{x},0}$ is given by

$$\ln\left[S_{\boldsymbol{x},0}\right] = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \ln R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})d\omega \ . \tag{D.268}$$

$$\ln\left|S_{\boldsymbol{x},0}\right| = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left|\ln R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})\right| d\omega \ . \tag{D.269}$$

For $S_{\boldsymbol{x},0}$ to be finite, $R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})$ must satisfy the **discrete-time MIMO Paley-Wiener Criterion (PWC)**

$$\ln\left|S_{\boldsymbol{x},0}\right| = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} |\ln R_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})|d\omega < \infty \quad . \tag{D.270}$$

The continuous-time equivalent of this MIMO PWC is that the Fourier Transform of the continuous-time autocorrelation function is factorizable

$$R_{\boldsymbol{x}_c\boldsymbol{x}_c}(s) = G_{\boldsymbol{x}_c}(s) \cdot S_{\boldsymbol{x}_c,0} \cdot G^*_{\boldsymbol{x}_c}(-s^*) \quad , \tag{D.271}$$

where $G_{\boldsymbol{x}_c}(s)$ is minimum phase (all poles and zeros in the left half plane or on axis in limiting sense), with upper triangular monic $|G_{\boldsymbol{x}_c}(0)|$, whenever

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\ln R_{\boldsymbol{x}_c\boldsymbol{x}_c}(\omega_c)|}{1+\omega_c^2} d\omega_c < \infty \quad . \tag{D.272}$$

Constructive Proof: The equivalence of the two PW criteria in (D.270) and (D.272) (discrete- and continuous-time) follows directly from Equations (D.105) to (D.108). However, it remains to show that the condition is necessary and sufficient for the factorization to exist. The necessity of the criterion followed previously when it was shown that factorizability lead to the PWC being satisfied. The sufficiency proof will be constructive from the criterion itself.

The desired positive-definite matrix is any square root $R^{1/2}_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})$ that can, for instance, be found by Cholesky Factorization (and absorbing positive diagonal square-root equally into the two upper and lower factors), and this function in turn has a natural-log real matrix

$$\mathcal{A}(e^{-\jmath\omega}) \triangleq \ln\left[R^{1/2}_{\boldsymbol{x}\boldsymbol{x}}(e^{-\jmath\omega})\right] \quad . \tag{D.273}$$

$\mathcal{A}(e^{-\jmath\omega})$ itself is periodic and by the MIMO PWC integral equation is absolutely integrable and so has a corresponding Fourier representation

$$\mathcal{A}(e^{-\jmath\omega}) = \sum_{k=-\infty}^{\infty} A_k \cdot e^{-\jmath\omega k} \tag{D.274}$$

$$A_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{A}(e^{-\jmath\omega}) \cdot e^{\jmath\omega} \cdot d\omega \ . \tag{D.275}$$

Because the Fourier Transform $\mathcal{A}(e^{-\jmath\omega})$ is positive real, then $A_k = A^*_{-k}$, and the D-Transform simplifies to

$$\mathcal{A}(D) = A_0 + \sum_{k=1}^{\infty} A_k \cdot D^k + \sum_{l=-1}^{-\infty} A_l \cdot D^l \quad , \tag{D.276}$$

and then by letting $k = -l$ in the second sum,

$$\begin{aligned} \mathcal{A}(D) &= A_0 + \sum_{k=1}^{\infty} A_k \cdot D^k + \sum_{k=1}^{\infty} A_{-k} \cdot D^k & \text{(D.277)} \\ &= A_0 + \sum_{k=1}^{\infty} [A_k + A_{-k}] \cdot D^k & \text{(D.278)} \\ &= a_0 + 2 \cdot \sum_{k=1}^{\infty} \Re[A_k] \cdot D^k \quad , & \text{(D.279)} \end{aligned}$$

which defines a causal sequence A_k that corresponds to $\ln\left[R^{1/2}_{\boldsymbol{xx}}(D)\right]$. So,

$$R_{\boldsymbol{xx}}(D) = e^{\mathcal{A}(D)} \cdot e^{\mathcal{A}^*(D^{-*})} \quad . \tag{D.280}$$

Then, the canonical factorization's MIMO components are:

$$\begin{aligned} \mathcal{S}_{\boldsymbol{x},0} &= e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left[R_{\boldsymbol{xx}}(e^{-\jmath\omega})\right]\cdot d\omega} & \text{(D.281)} \\ \mathcal{G}_{\boldsymbol{x}}(D) &= e^{\mathcal{A}(D)} \cdot S^{-1/2}_{\boldsymbol{x},0} \quad . & \text{(D.282)} \end{aligned}$$

There is a second step that recognizes that $\mathcal{S}_{\boldsymbol{x},0}$ is not diagonal, just positive definite constant. This matrix itself is factored as a constant-matrix Cholesky factorization to

$$\mathcal{S}_{\boldsymbol{x},0} = G_{\boldsymbol{x},0} \cdot S_{\boldsymbol{x},0} \cdot G^*_{\boldsymbol{x},0} \quad . \tag{D.283}$$

with $G_{\boldsymbol{x},0}$ as monic upper triangular and $S_{\boldsymbol{x},0}$ positive definite diagonal. Then the canonical factor is

$$G_{\boldsymbol{x}}(D) = e^{\mathcal{A}(D)} \cdot G^{-1}_{\boldsymbol{x},0} \cdot S^{-1/2}_{\boldsymbol{x},0} \quad . \tag{D.284}$$

The corresponding continuous-time spectrum factorization then would be found with $R_{x_c x_c}(s) = R_{xx}\left(\frac{1-s}{1+s}\right)$ and thus $A_c(s) = A\left(\frac{1-s}{1+s}\right)$. Then, with $s \to \jmath\omega_c$

$$\begin{aligned} \mathcal{S}_{\boldsymbol{x}_c,0} &= e^{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\ln\left[R_{\boldsymbol{x}_c\boldsymbol{x}_c}(\omega_c)\right]}{1+\omega_c^2}\cdot d\omega_c} & \text{(D.285)} \\ \mathcal{G}_{\boldsymbol{x}_c}(s) &= e^{\mathcal{A}_c(s)} \cdot S^{-1/2}_{\boldsymbol{x}_c,0} \quad . & \text{(D.286)} \end{aligned}$$

The continuous-time second step recognizes that $\mathcal{S}_{\boldsymbol{x}_c,0}$ is not diagonal, just positive definite constant. This matrix itself is factored as a constant-matrix Cholesky factorization to

$$\mathcal{S}_{\boldsymbol{x}_c,0} = G_{\boldsymbol{x}_c,0} \cdot S_{\boldsymbol{x}_c,0} \cdot G^*_{\boldsymbol{x}_c,0} \quad . \tag{D.287}$$

with $G_{\boldsymbol{x}_c,0}$ as monic upper triangular and $S_{\boldsymbol{x}_c,0}$ positive definite diagonal. Then the canonical factor is

$$G_{\boldsymbol{x}_c}(D) = e^{\mathcal{A}_c(D)} \cdot G^{-1}_{\boldsymbol{x}_c,0} \cdot S^{-1/2}_{\boldsymbol{x}_c,0} \quad . \tag{D.288}$$

If the original desired spectra were defined in continuous time, then it could be mapped into discrete time through $\omega_c \to \tan(\frac{\omega}{2})$ and then proceeding with that discrete-time mapped equivalent through the process above, ultimately leading to Equations (D.285) and (D.286). Sufficiency has thus been established in both discrete- and continuous-time. **QED.**

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