STANFORD

Supplementary Lecture 15A

Duality Theory

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Announcements & Agenda

- Announcements
  - Show details of BC duality

- Agenda
  - MAC/BC Duality Basics
    - Input deflection
    - Mappings
  - Vector MAC/BC Duality
  - MAC-dual Design

MAC / BC Duality Basics

Section 5.5

Order Reversal & The Dual

- Reverse MAC order so that BC has user 1 at top (still best position).

- $\mathcal{J}_x$ applies to $\boldsymbol{x}$ so that $\mathcal{J}_x \cdot \boldsymbol{x}$ reverses the input vector $\boldsymbol{x}$'s user order.

$$\mathcal{J}_x \triangleq \begin{bmatrix} 0 & 0 & I_{L_{x,1}} \\ 0 & \iddots & 0 \\ I_{L_{x,U}} & 0 & 0 \end{bmatrix}$$

- Thus, reversing a MAC's input order corresponds to $\tilde{H} \cdot \mathcal{J}_x$.
  - Can also have a $\mathcal{J}_y$ on BC input side (but since BC has one joint input, less important).

$$H_{dual} = \mathcal{J}_x \cdot \tilde{H}^* \cdot \mathcal{J}_y$$

- The channel $\tilde{H}$'s dual is its order-reversed transpose.

- SVDs:

$$\tilde{H} = F \cdot \Lambda \cdot M^* \qquad H_{dual} = \mathcal{J}_x \cdot M \cdot \Lambda \cdot F^* \cdot \mathcal{J}_y$$

Still an SVD (singular values the same); L_y **inputs**, L_x **outputs**

Specific to MAC and BC

- MAC Channel has normal notation:

$$\tilde{H}_{MAC} = [\tilde{H}_U \quad \cdots \quad \tilde{H}_1]$$

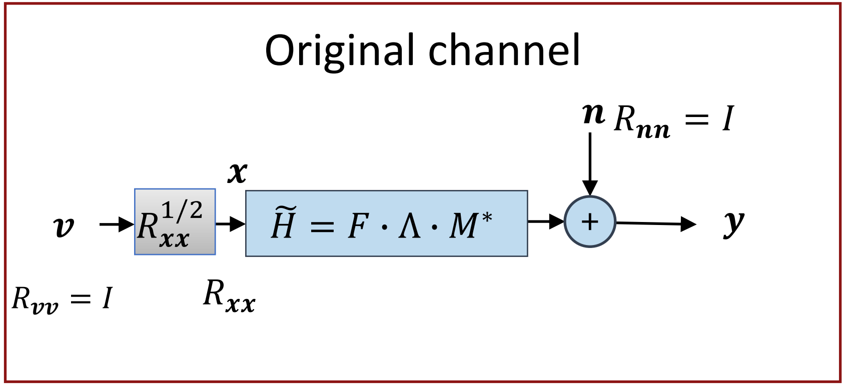
- Dual BC Channel transposes each user channel and reorders outputs and inputs so 1 is at top/left:

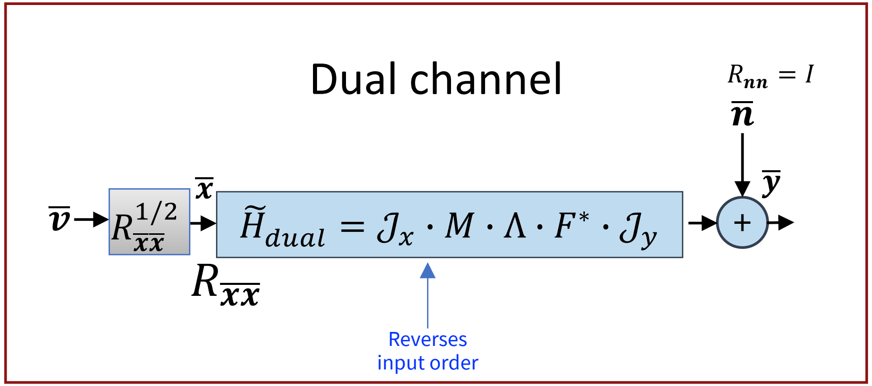
$$\tilde{H}_{BC} = \mathcal{J}_x \cdot \tilde{H}^*_{MAC} \cdot \mathcal{J}_y = \mathcal{J}_x \cdot \begin{bmatrix} \tilde{H}^*_U \\ \vdots \\ \tilde{H}^*_1 \end{bmatrix} \cdot \mathcal{J}_y = \begin{bmatrix} \tilde{H}^*_1 \\ \vdots \\ \tilde{H}^*_U \end{bmatrix} \cdot \mathcal{J}_y$$

- The reversal allows some simplification of notation (its worse without the reversal).

The (single-user & overall-channel) Dual

- Duality design imposes that the original and dual have the same mutual information equal:





$$\mathcal{I}(\boldsymbol{x},\boldsymbol{y}) = \log_2\left|\tilde{H} \cdot R_{\boldsymbol{xx}} \cdot \tilde{H}^* + I\right| = \log_2\left|\mathcal{J}_x \cdot \tilde{H}^* \cdot R_{\overline{\boldsymbol{xx}}} \cdot \tilde{H} \cdot \mathcal{J}_x + I\right|$$

- A solution is $\bar{\boldsymbol{x}} = \mathcal{J}_y \cdot F \cdot M^* \cdot \mathcal{J}_x \cdot \boldsymbol{x}$, which yields

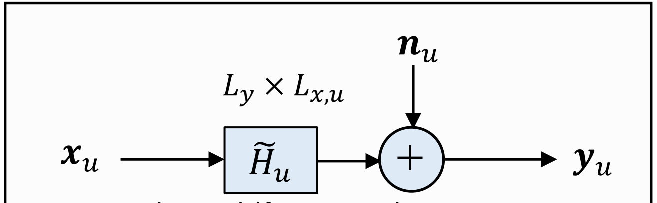
- $R_{\overline{\boldsymbol{xx}}} = \mathcal{J}_y \cdot F \cdot M^* \cdot \mathcal{J}_x \cdot R_{\boldsymbol{xx}} \cdot \mathcal{J}_x \cdot M \cdot F^* \cdot \mathcal{J}_y$

  The $\mathcal{J}_{x,y}$'s don't change rate sums/determinants

- SNR's also equal because $R_{\boldsymbol{vv}} = I = R_{\overline{\boldsymbol{vv}}}$

Proof in Sec 5.5

Each user's dual with crosstalk as noise:



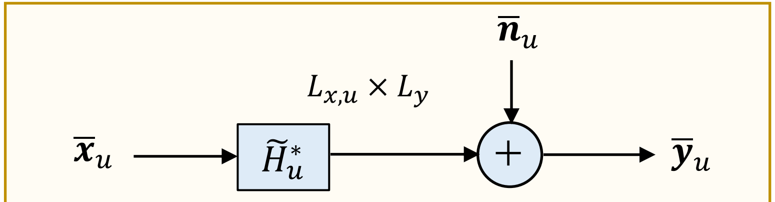
Per-user channel (from MAC)

$$\mathcal{R}_{\boldsymbol{nn}}(u) = I + \sum_{i=u+1}^{U} \widetilde{H}_i \cdot R_{\boldsymbol{xx}}(i) \cdot \widetilde{H}_i^* \quad \text{implies a user order.}$$

$L_y \times L_y$

Noise + Xtalk not white

$$\widetilde{H} = R_{\boldsymbol{nn}}^{-1/2} \cdot H = [\widetilde{H}_U \quad \ldots \quad \widetilde{H}_1]$$



Per-user dual channel for BC

$$\mathcal{R}_{\overline{\boldsymbol{nn}}}(u) = I + \sum_{i=1}^{u-1} \widetilde{H}_u^* \cdot R_{\overline{\boldsymbol{xx}}}(i) \cdot \widetilde{H}_u \quad \text{reverses implied order.}$$

$L_x \times L_x$

- For equal individual-user mutual information:

$$2^{\mathcal{I}_u} = \frac{|\widetilde{H}_u \cdot R_{\boldsymbol{xx}}(u) \cdot \widetilde{H}_u^* + \mathcal{R}_{\boldsymbol{nn}}(\boldsymbol{u})|}{|\mathcal{R}_{\boldsymbol{nn}}(\boldsymbol{u})|} = \frac{|\widetilde{H}_u^* \cdot R_{\overline{\boldsymbol{xx}}}(u) \cdot \widetilde{H}_u + \mathcal{R}_{\overline{\boldsymbol{nn}}}(\boldsymbol{u})|}{|\mathcal{R}_{\overline{\boldsymbol{nn}}}(\boldsymbol{u})|}$$

Suggests an input adjustment (so dimensionality is consistent).

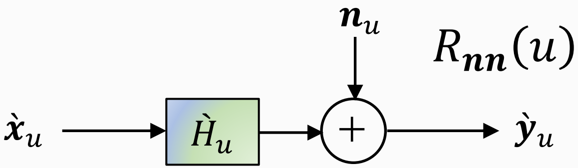
- Duality design follows MAC's independent ($\boldsymbol{x}_u$ and $\boldsymbol{x}_{i \neq u}$) & causes BC's ($\overline{\boldsymbol{x}}_u$ and $\overline{\boldsymbol{x}}_{i \neq u}$) 's to be independent.

Input Deflection

- Duality **deflects input** so that it offsets the channel shaping (same dim as "dual's xtalk):

$$\grave{\boldsymbol{x}}_u = \mathcal{R}_{\boldsymbol{nn}}^{1/2}(u) \cdot \boldsymbol{x}_u$$

$$\grave{\bar{\boldsymbol{x}}}_u = \mathcal{R}_{\boldsymbol{nn}}^{*/2}(u) \cdot \bar{\boldsymbol{x}}_u$$

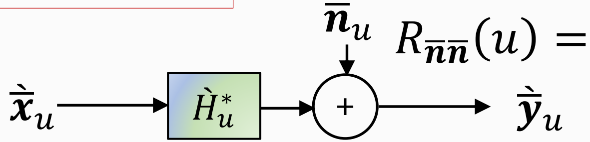


$R_{\boldsymbol{nn}}(u) = I$

$$R_{\boldsymbol{xx}}(u) = \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-1/2}(u) \cdot R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-*/2}(u)$$

$$\grave{H}_u = \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot \tilde{H}_u \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{-1/2}(u)$$

$$R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) = \mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u) \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u)$$



$R_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u) = I$

- This then **whitens both channels' noises** and is 1-to-1 on input deflection, so $\mathcal{I}(\grave{\boldsymbol{x}}_u; \grave{\boldsymbol{y}}_u) = \mathcal{I}(\grave{\bar{\boldsymbol{x}}}_u; \grave{\bar{\boldsymbol{y}}}_u) = \mathcal{I}(\boldsymbol{x}_u; \boldsymbol{y}_u)$.

$$2^{\mathcal{I}_u} = \frac{\left|\tilde{H}_u \cdot R_{\boldsymbol{xx}}(u) \cdot \tilde{H}_u^* + \mathcal{R}_{\boldsymbol{nn}}(u)\right|}{|\mathcal{R}_{\boldsymbol{nn}}(u)|} = \frac{\left|\tilde{H}_u^* \cdot R_{\bar{\boldsymbol{x}}\bar{\boldsymbol{x}}}(u) \cdot \tilde{H}_u + \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)\right|}{|\mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}(u)|}$$

$$= \left|\grave{H}_u \cdot R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) \cdot \grave{H}_u^* + I\right| = \left|\grave{H}_u^* \cdot R_{\grave{\bar{\boldsymbol{x}}}\grave{\bar{\boldsymbol{x}}}}(u) \cdot \grave{H}_u + I\right|$$

Scalar duality revist and example

- Rewrite the scalar-duality input-deflection equations (follow from L10 scalar-energy duality):

$$
\begin{aligned}
x_1^{BC} \cdot \sqrt{1 + \mathcal{E}_2^{MAC} \cdot g_2 + ... + \mathcal{E}_U^{MAC} \cdot g_U} &= x_1^{MAC} \\
x_2^{BC} \cdot \sqrt{1 + \mathcal{E}_3^{MAC} \cdot g_3 + ... + \mathcal{E}_U^{MAC} \cdot g_U} &= x_2^{MAC} \cdot \sqrt{1 + \mathcal{E}_1^{BC} \cdot g_2} \\
\vdots &= \vdots \\
x_U^{BC} &= x_U^{MAC} \cdot \sqrt{(1 + [\mathcal{E}_1^{BC} + ... + \mathcal{E}_{U-1}^{BC}] \cdot g_U)}
\end{aligned}
$$

$\mathcal{R}_{\bar{n}\bar{n}}^{-1/2}(u)$

- Recognize these as a simple form of input deflection.
- $g_u = \frac{|h_u|^2}{\sigma_u^2}$, above equations do not depend on (MAC's) g_1, which is BC's g_2.
- L9's duality example was for

$$H_{MAC} = [50 \quad 80]$$

user 2 user 1

$$H_{BC} = \begin{bmatrix} 80 \\ 50 \end{bmatrix}$$

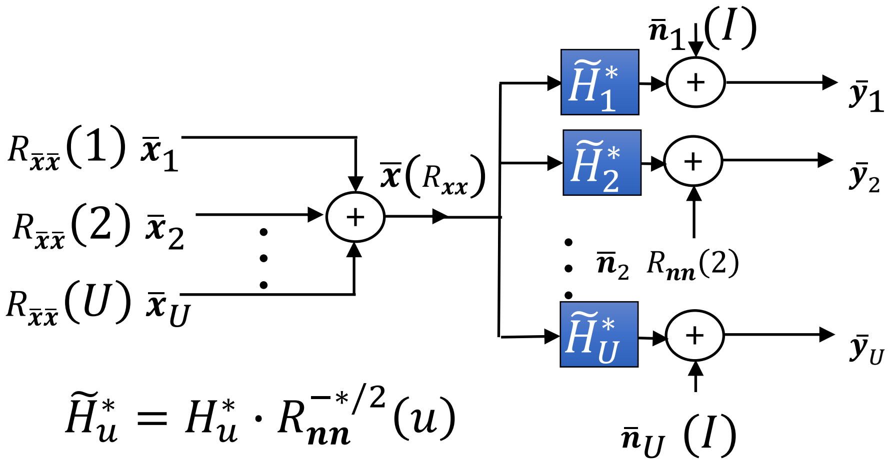
user 1
user 2

$$\mathcal{E}^{BC} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1/7504 \\ 7503/7504 \end{bmatrix} = \mathcal{E}^{MAC}$$

Vector MAC / BC Duality

Section 5.5.2

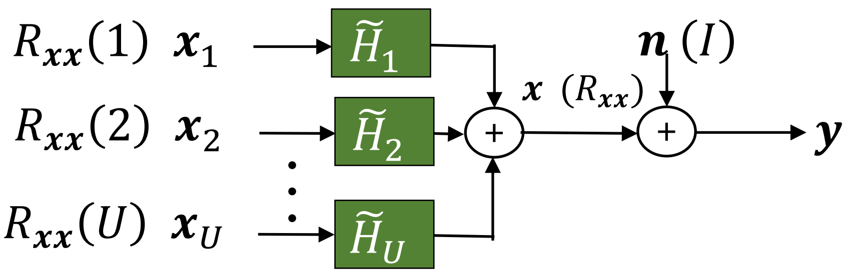
Vector MAC/BC Duals



$$\tilde{H}_u^* = H_u^* \cdot R_{nn}^{-*/2}(u)$$

Broadcast – input is $\bar{x}$

$$\mathcal{R}_{nn}(u+1) = \sum_{i=1}^{u} \tilde{H}_i^* \cdot R_{xx}(i) \cdot \tilde{H}_i + I$$



Multiple Access input is x

Order reversed

$$\mathcal{R}_{nn}(u-1) = \sum_{i=u}^{U} \tilde{H}_i \cdot R_{xx}(i) \cdot \tilde{H}_i^* + I$$

- Esum-MAC $\mathcal{E}_x = \sum_{u=1}^{U} \mathcal{E}_u = \sum_{u=1}^{U} trace\{R_{xx}(u)\} = \sum_{u=1}^{U} trace\{R_{\bar{x}\bar{x}}(u)\}$

Finish Vector Duality

- Write equality for deflected with actual autocorrelation matrices explicitly:

$$\left|\grave{H}_u \cdot \mathcal{R}_{\overline{nn}}^{*/2}(u) \cdot R_{xx}(u) \cdot \mathcal{R}_{\overline{nn}}^{1/2}(u) \cdot \grave{H}_u^* + I\right| = \left|\grave{H}_u^* \cdot \mathcal{R}_{nn}^{*/2}(u) \cdot R_{\overline{xx}}(u) \cdot \mathcal{R}_{nn}^{1/2}(u) \cdot \grave{H}_u + I\right|$$

- SVD: $F_u \cdot \Lambda_u \cdot M_u^* = svd\left(\grave{H}_u\right)$
  - use "economy mode SVD" so that F_u and M_u may be non-square and the multiplication $F_u \cdot M_u^*$ is dimensionally ok.
- Inside determinant above is $F_u \cdot \Lambda_u \cdot M_u^* \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{*/2} \cdot R_{\boldsymbol{xx}}(u) \cdot \mathcal{R}_{\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}}^{1/2} \cdot M_u \cdot \Lambda_u \cdot F_u^* + I$.

- Pre/post multiply by F's causes no change (nor by M's in 2nd determinant).
- Data Rate Equality occurs if $R_{\grave{\boldsymbol{x}}\grave{\boldsymbol{x}}}(u) = M_u \cdot F_u^* \cdot R_{\dot{\grave{\boldsymbol{x}}}\dot{\grave{\boldsymbol{x}}}}(u) \cdot F_u \cdot M_u^*$.

- **MAC2BC:** $R_{\overline{xx}}(u) = \mathcal{R}_{nn}^{-1/2}(u) \cdot F_u \cdot M_u^* \cdot \mathcal{R}_{\overline{nn}}^{-*/2}(u) \cdot R_{xx}(u) \cdot \mathcal{R}_{\overline{nn}}^{-1/2}(u) \cdot M_u \cdot F_u^* \cdot \mathcal{R}_{nn}^{-*/2}(u)$

- **BC2MAC:** $R_{xx}(u) = \mathcal{R}_{\overline{nn}}^{-*/2}(u) \cdot M_u \cdot F_u^* \cdot \mathcal{R}_{nn}^{-1/2}(u) \cdot R_{\overline{xx}}(u) \cdot \mathcal{R}_{nn}^{-*/2}(u) \cdot F_u \cdot M_u^* \cdot \mathcal{R}_{\overline{nn}}^{-1/2}(u)$.

MAC2BC Full Algorithm

$R_{\boldsymbol{xx}}(u)$ from minPMAC

Given:

$$R_{\boldsymbol{xx}}(u) \text{ for } u = 1, \ldots, U \;\; ; \;\; R_{\overline{\boldsymbol{xx}}}(u) = R_{\overline{\boldsymbol{xx}}} = 0$$

$$\mathcal{R}_{\boldsymbol{nn}}(U) = \mathcal{R}_{\overline{\boldsymbol{nn}}}(1) = I$$

$$\text{BC: } \widetilde{H}_u^*, R_{\boldsymbol{nn}}(u)$$

$$\text{MAC: } \widetilde{H}_u = R_{\boldsymbol{nn}}^{-1/2}(u) \cdot H_u$$

$$\text{For } u = U, \ldots, 2 \; ; \; \mathcal{R}_{\boldsymbol{nn}}(u-1) = \mathcal{R}_{\boldsymbol{nn}}(u) + \widetilde{H}_u \cdot R_{\boldsymbol{xx}}(u) \cdot \widetilde{H}_u^*$$

$$\text{For } u = 1, \ldots, U$$

$$\grave{H}_u = \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot \widetilde{H}_u \cdot \mathcal{R}_{\overline{\boldsymbol{nn}}}^{-1/2}(u)$$

$$F_u \cdot \Lambda_u \cdot M_u^* = svd\left(\grave{H}_u\right)$$

$$R_{\overline{\boldsymbol{xx}}}(u) = \mathcal{R}_{\boldsymbol{nn}}^{-1/2}(u) \cdot F_u \cdot M_u^* \cdot \mathcal{R}_{\overline{\boldsymbol{nn}}}^{*/2}(u) \cdot R_{\boldsymbol{xx}}(u) \cdot \mathcal{R}_{\overline{\boldsymbol{nn}}}^{1/2}(u) \cdot M_u \cdot F_u^* \cdot \mathcal{R}_{\boldsymbol{nn}}^{-*/2}(u)$$

$$R_{\overline{\boldsymbol{xx}}} = R_{\overline{\boldsymbol{xx}}} + R_{\overline{\boldsymbol{xx}}}(u)$$

$$\mathcal{R}_{\overline{\boldsymbol{nn}}}(u+1) = \mathcal{R}_{\overline{\boldsymbol{nn}}}(u) + \widetilde{H}_u^* \cdot R_{\overline{\boldsymbol{xx}}} \cdot \widetilde{H}_u \;\; ; \;\; \text{skip } u = U$$

- So, find the $R_{\boldsymbol{xx}}(u)$ for the dual-BC's original MAC that has necessary data rate/energy.

BC2MAC Full Algorithm

Given:

$$R_{\overline{xx}}(u) \text{ for } u = 1, \ldots, U \;;\; \mathcal{R}_{nn}(u) = \mathcal{R}_{\overline{nn}}(u) = 0$$
$$\mathcal{R}_{nn}(U) = \mathcal{R}_{\overline{nn}}(1) = I \;;\; R_{\overline{xx}} = 0$$
BC: $\widetilde{H}_u^*, R_{nn}(u)$
MAC: $\widetilde{H}_u = R_{nn}^{-1/2}(u) \cdot H_u$

For $u = 1, \ldots, U-1$;
$$R_{\overline{xx}} = R_{\overline{xx}} + R_{\overline{xx}}(u)$$
$$\mathcal{R}_{\overline{nn}}(u+1) = \mathcal{R}_{\overline{nn}}(u) + \widetilde{H}_u^* \cdot R_{\overline{xx}} \cdot \widetilde{H}_u$$

For $u = U, \ldots, 1$
$$\grave{H}_u = \mathcal{R}_{nn}^{-1/2}(u) \cdot \widetilde{H}_u \cdot \mathcal{R}_{\overline{nn}}^{-1/2}(u)$$
$$F_u \cdot \Lambda_u \cdot M_u^* = svd\left(\grave{H}_u\right)$$
$$R_{xx}(u) = \mathcal{R}_{\overline{nn}}^{-*/2}(u) \cdot M_u F_u^* \cdot \mathcal{R}_{nn}^{1/2}(u) \cdot R_{\overline{xx}}(u) \cdot \mathcal{R}_{nn}^{*/2}(u) \cdot F_u M_u^* \cdot \mathcal{R}_{\overline{nn}}^{-1/2}(u)$$
$$\mathcal{R}_{nn}(u-1) = \mathcal{R}_{nn}(u) + \widetilde{H}_u \cdot R_{xx}(u) \cdot \widetilde{H}_u^* \;;\; \text{skip } u = 1$$

- Reverse is less interesting, but provided for completeness.

Order Reversal

- Semantics – alternate dual definition $\tilde{H}_{dual} = \left(\mathcal{J}_y \cdot \tilde{H} \cdot \mathcal{J}_x\right)^* = \mathcal{J}_x \cdot \tilde{H}^* \cdot \mathcal{J}_y$.

$$\mathcal{J}_y \triangleq \begin{bmatrix} 0 & 0 & I_{L_{y,U}} \\ 0 & \iddots & 0 \\ I_{L_{y,1}} & 0 & 0 \end{bmatrix}$$

- All information, SVD, energy, etc are preserved as without $\mathcal{J}_y$:
  - Single output on MAC corresponds to single input on BC.
  - $\mathcal{J}_y$ just re-indexes dimensions (not users) – but this is matlab's usual indexing.
  - But the definition said reverse order (so can do it explicitly with $\mathcal{J}_y$).
- The mac2bc and bc2mac programs basically do this tacitly in finding inputs:
  - L16's Hbc=conj(**permute(** Hmac(:,:,end:-1:1) **, [2 1 3])**) for $N = 1$ presumes the channel input has dimension 1 at top
  - So, this is essentially multiplying by $\mathcal{J}_y$ **tacitly**.
- Example has: $$H_{dual}(D) = \left(\mathcal{J}_y \cdot \underbrace{\begin{bmatrix} 1 - .9D & .8 - .7D \\ -1 & 1 + D \end{bmatrix}}_{H(D)} \cdot \mathcal{J}_x\right)^* = \begin{bmatrix} 1 + D^* & .8 - .7D^* \\ -1 & 1 - .9D^* \end{bmatrix}$$

$D^* \to e^{j\omega}$ on unit circle (FFT)

  - Essentially dual here causes:
    - User direct (magnitudes, - phase) to be the same, but priority is reversed.
    - Crosstalk flow to flip from transfer of $u \to u'$ on original to $u \leftarrow u'$.

End Lecture 15A